ALGEBRAIC GEOMETRY

HOMEWORK 3

(1) Consider the curve $Y^2 = X^2(X+1)$.
   (a) Sketch the curve.
   (b) Determine the singular point P on $\mathcal{C}$.
   (c) For all lines through P, determine the intersection multiplicity at P.

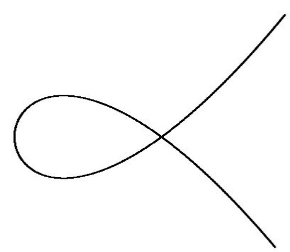

FIGURE 1. $y^2 = x^3 + x^2$

For computing the singularities, consider the projective closure $Y^2Z - X^3 - X^2Z$ and compute the partials:

$$F_X = -3X^2 - 2XZ,$$
$$F_Y = 2YZ,$$
$$F_Z = Y^2 - X^2.$$

The second equation tells us that, over a field of characteristic $\neq 2$, we have $Y = 0$ or $Z = 0$. But if $Z = 0$, then the first equation gives $X = 0$ (over a field of characteristic $\neq 3$), and then the last shows that $Y = 0$: but this does not give a point in the projective plane. Thus $Y = 0$, and then we easily find $X = Y = 0$ and the singular point $[0 : 0 : 1]$.

Now consider the lines $y = mx$; intersecting them with the curve leads to $m^2x^2 - x^3 - x^2 = x^2(m - 1 - x) = 0$. Thus all lines through $(0,0)$ (we have to treat the line $x = 0$ separately: we get $y^2 = 0$ in this case, hence $x = 0$ intersects the curve with multiplicity 2) intersect the curve with multiplicity ≥ 2, and with multiplicity 3 if and only if $m^2 = 1$. Thus there are two tangents at $(0,0)$, and their slopes are $m = +1$ and $m = -1$.

(2) Compute all singular points, along with the tangents at these points and their multiplicities, of the projective curve

$$(X^2 + Y^2)^3 - 4X^2Y^2Z^2 = 0.$$

Sketch the curve. Why does the graph of the curve imply that the degree of the curve is at least 6?

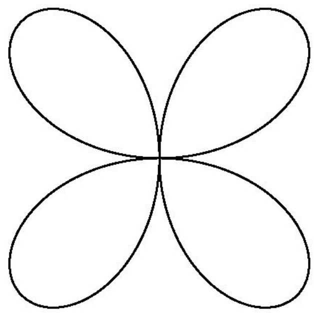

FIGURE 2. $(X^2 + Y^2)^3 - 4X^2Y^2Z^2 = 0$

We start by computing the partials:

$$F_X = 6X(X^2 + Y^2)^2 - 8XY^2Z^2,$$
$$F_Y = 6Y(X^2 + Y^2)^2 - 8X^2YZ^2,$$
$$F_Z = -8X^2Y^2Z.$$

The last equation tells us that, over fields of characteristic $\neq 2$, singular points have at least one 0 coordinate. Assume first that $Z = 0$; then the other two equations show that $X^2 + Y^2 = 0$ or $X = Y = 0$. The last case is impossible, but $[1 : i : 0]$ and $[1 : -i : 0]$ are singular points.

Now consider the case $Z = 1$; then $X = 0$ or $Y = 0$, and this easily implies $X = Y = 0$, hence the only affine singular point is $O = [0 : 0 : 1]$.

In order to compute tangents at O, we have to intersect lines $aX+bY = 0$ with the curve and get

$$\begin{aligned} G(Y, Z) &= F(aX, aY, aZ) = F(-bY, aY, aZ) \\ &= (b^2Y^2 + a^2Y^2)^3 - 4a^2b^2Y^4Z^2 \\ &= Y^4[(a^2 + b^2)^3Y^2 - 4a^2b^2Z^2]. \end{aligned}$$

Thus O is a singularity with multiplicity 4, and we only get a higher multiplicity of $ab = 0$, that is, if the lines are $X = 0$ or $Y = 0$. Thus the curve has two tangents at), namely the coordinate axes, and in these cases, the multiplicity is 6.

Now consider the two points at infinity. It is clear that they cannot have multiplicity higher than 2, since a line through O and one of these points intersects the curve with multiplicity 6. In fact, lines through $[1 : i : 0]$ have the equation $iaX - aY + cZ = 0$. Thus

$$\begin{aligned} G(X, Z) &= F(aX, aY, aZ) = F(aX, iaX + cZ, aZ) \\ &= (a^2X^2 + (iaX + cZ)^2)^3 - 4a^4X^2(iaX + cZ)^2Z^2 \\ &= Z^2[Z(2aciX + cZ)^3 - 4a^4X^2(iaX + cZ)^2] \end{aligned}$$

Thus we have multiplicity 2 as expected. Note that we cannot set $a = 0$ here since we multiplied through by a at the beginning. If $a = 0$, then the line is $Z = 0$, and we get $G(X, Y) = F(X, Y, 0) = (X^2 + Y^2)^3 = (X + iY)^3(X - iY)^3$, hence the line at infinity is a tangent at $[1 : i : 0]$ and intersects the curve with multiplicity 3.

As for the last question: the following picture shows that some lines intersect the curve in 6 points.

(3) Determine the components of the curve

$$F(X, Y, Z) = X^3 - X^2Z + XY^2 - XZ^2 - Y^2Z + Z^3 = 0.$$

A sketch will help.

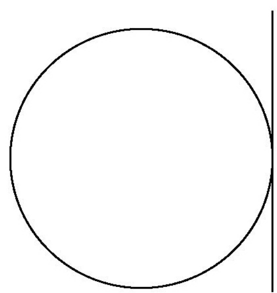

FIGURE 3. $X^3 - X^2Z + XY^2 - XZ^2 - Y^2Z + Z^3 = 0$

The picture suggests that $F(X, Y, Z) = (X - Z)(X^2 + Y^2 - Z^2)$, hence the components of the curve are the line $X - Z = 0$ ($x = 1$ in affine coordinates) and the unit circle.

(4) Compute the points at infinity and their tangents (the asymptotes) for the curve $xy^4 + x^2 + y^2 = 0$. Sketch the curve. Also sketch $xy^4 + x^2 + y^2 - \delta x = 0$ for $\delta = 0.1$ and $\delta = 0.01$.

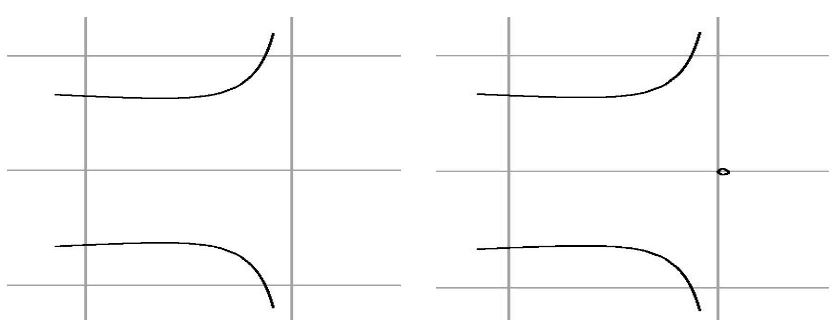

FIGURE 4. $xy^4 + x^2 + y^2 = 0$ and $xy^4 + x^2 + y^2 - 0.1x = 0$

The points at infinity satisfy $XY^4 = 0$, hence $[1 : 0 : 0]$ and $[0 : 1 : 0]$ are the only points at infinity. The partials are

$$F_X = Y^4 + 2XZ^3,$$
$$F_Y = 4XY^3 + 2YZ^3,$$
$$F_Z = 3Z^2(X^2 + Y^2),$$

hence the tangent at $[0 : 1 : 0]$ is $X = 0$, the affine y-axis. The point $[1 : 0 : 0]$ is singular, and intersecting the curve with the lines $Y = tZ$ shows that they all intersect the curve with multiplicity 3; the other line $Z = 0$ intersects the curve with multiplicity 4, hence the point $[0 : 1 : 0]$ has multiplicity 3, and $Z = 0$ is the only tangent at infinity.

(5) Let $f, g \in K[x, y]$ be nonconstant polynomials with coefficients in the algebraically closed field K. Let $\mathcal{C}_f : f(x, y) = 0$, $\mathcal{C}_g : g(x, y) = 0$ and $\mathcal{C}_{fg} : f(x, y)g(x, y) = 0$ be the curves defined by them, and let $\operatorname{Sing}\mathcal{C}_f$ denote the set of singular points on $\mathcal{C}_f$. Prove that

$$\operatorname{Sing}\mathcal{C}_{fg} = \operatorname{Sing}\mathcal{C}_f \cup \operatorname{Sing}\mathcal{C}_g \cup (\mathcal{C}_f \cap \mathcal{C}_g).$$

In the projective plane, we have $H(X, Y, Z) = F(X, Y, Z)G(X, Y, Z)$, and we find

$$H_X = F_X \cdot G + F \cdot G_X,$$
$$H_Y = F_Y \cdot G + F \cdot G_Y,$$
$$H_Z = F_Z \cdot G + F \cdot G_Z.$$

At points $P \in \operatorname{Sing}\mathcal{C}_f \cup \operatorname{Sing}\mathcal{C}_g \cup (\mathcal{C}_f \cap \mathcal{C}_g$, these three partials all vanish, so this proves the easy part of the inclusion. Assume therefore that P is a point on $\mathcal{C}_{fg}$ at which the three partials of H vanish. Since $P \in \mathcal{C}_{fg}$ we have $P \in \mathcal{C}_F$ or $P \in \mathcal{C}_g$. Assume that $P \in \mathcal{C}_f$, and that moreover P is not a singular point on $\mathcal{C}_f$. Then $F(P) = 0$, hence the three equations above show that $F_X G = F_Y G = F_Z G = 0$ at P. Since one of the partials of F is nonzero, this implies $P \in \mathcal{C}_g$, hence $P \in \mathcal{C}_f \cap \mathcal{C}_g$.

(6) Let P be a singular point on the affine curve $\mathcal{C}_f : f(x, y) = 0$. Use the quadratic terms of the Taylor expansion of f to show that P is a node (double point with distinct tangents) if and only if $f_{xy}(P)^2 \neq f_{xx}(P)f_{yy}(P)$, where e.g. $f_{xy} = \frac{\partial^2}{\partial x \partial y}$.

The Taylor expansion of f around a singular point (no linear terms) with coordinates $(0, 0)$ is $f(x, y) = f_{xx}x^2 + 2fxyxy + f_{yy}y^2+$ terms of higher degree. Computing the multiplicity of the line $y = tx$ means plugging this into f; we find $g(x) = f(x, tx) = x^2(f_{xx} + 2tf_{xy} + t^2 f_{yy}$ + terms of higher degree. That the curve have two distinct tangents is equivalent to the fact that there are exactly two values of t that give multiplicity ≥ 3; this is equivalent to the fact that the quadratic equation in t

$$f_{xx}x^2 + 2fxyxy + f_{yy}y^2 = 0$$

has two distinct solutions, which happens (we are working over an algebraically closed field) if and only if its discriminant is nonzero, i.e. if and only if

$$f_{xy}^2 \neq f_{xx} f_{yy}.$$

Note that over arbitrary fields (e.g. over the reals) the tangents will be "visible" if the discriminant is a square.