ALGEBRAIC GEOMETRY

HOMEWORK 3

(1) Consider the curve $Y^2 = X^2(X+1)$.
   (a) Sketch the curve.
   (b) Determine the singular point P on $\mathcal{C}$.
   (c) For all lines through P, determine the intersection multiplicity at P.

(2) Compute all singular points, along with the tangents at these points and their multiplicities, of the projective curve

$$(X^2 + Y^2)^3 - 4X^2Y^2Z^2 = 0.$$

Sketch the curve. Why does the graph of the curve imply that the degree of the curve is at least 6?

(3) Determine the components of the curve

$$X^3 - X^2Z + XY^2 - XZ^2 - Y^2Z + Z^3 = 0.$$

A sketch will help.

(4) Compute the points at infinity and their tangents (the asymptotes) for the curve $xy^4+x^2+y^2 = 0$. Sketch the curve. Also sketch $xy^4+x^2+y^2-\delta x = 0$ for $\delta = 0.1$ and $\delta = 0.01$.

(5) Let $f, g \in K[x, y]$ be nonconstant polynomials with coefficients in the algebraically closed field K. Let $\mathcal{C}_f : f(x, y) = 0$, $\mathcal{C}_g : g(x, y) = 0$ and $\mathcal{C}_{fg} : f(x, y)g(x, y) = 0$ be the curves defined by them, and let $\operatorname{Sing} \mathcal{C}_f$ denote the set of singular points on $\mathcal{C}_f$. Prove that

$$\operatorname{Sing} \mathcal{C}_{fg} = \operatorname{Sing} \mathcal{C}_f \cup \operatorname{Sing} \mathcal{C}_g \cup (\mathcal{C}_f \cap \mathcal{C}_g).$$

(6) Let P be a singular point on the affine curve $\mathcal{C}_f : f(x, y) = 0$. Use the quadratic terms of the Taylor expansion of f to show that P is a node (double point with distinct tangents) if and only if $f_{xy}(P)^2 \neq f_{xx}(P)f_{yy}(P)$, where e.g. $f_{xy} = \frac{\partial^2}{\partial x \partial y}$.