ALGEBRAIC GEOMETRY

HOMEWORK 2

(1) Consider the unit circle $\mathcal{C} : X^2 + Y^2 = 1$ and the group $\mathcal{C}(\mathbb{Q})$. Show that $P = (x, y) \in \mathcal{C}(\mathbb{Q})$ with $x \neq -1$ is in $2\mathcal{C}(\mathbb{Q})$ (i.e., can be written as $P = 2Q$ for some $Q \in \mathcal{C}(\mathbb{Q})$) if and only if $2(x + 1)$ is a rational square.

Let $Q = (u, v)$; then $2(u, v) = (u^2 - v^2, 2uv)$, hence $2(x + 1) = 2(u^2 - v^2 + 1) = 4u^2$ since $1 - v^2 = u^2$.

Conversely, assume that $2(x + 1) = 4u^2$ for some rational u. Then $y^2 = 1 - x^2 = 1 - (2u^2 - 1)^2 = 4u^2(1 - u^2)$ shows that $1 - u^2 = v^2$ is a square, and now it is easily checked that $2(u, v) = (x, y)$ for either choice of v.

(2) Find all $\mathbb{Q}(T)$-rational points on the conic $X^2 - (T^4 + T^3)Y^2 = 1$.

This should be a standard calculation: take the known point $P = (x, y) = (-1, 0)$; the lines through P have the form $Y = m(X + 1)$, and the second point of intersection satisfies $(X - 1) - (T^4 + T^3)m^2(X + 1) = 0$, which leads to

$$X = \frac{1 - m^2(T^4 + T^3)}{1 - m^2(T^4 + T^3)}, \quad Y = m(X + 1) = \frac{2m}{1 - m^2(T^4 + T^3)}.$$

You can also write the lines as $X = mY - 1$; then you get $m^2Y^2 - 2mY - (T^4 + T^3)Y^2 = Y(m^2Y - 2m - (T^4 + T^3)Y) = 0$, and you can finish similarly.

(3) Show that $X^2 - (T^4 + T^3)Y^2 = 1$ does not have any nontrivial solutions in $\mathbb{Q}[T]$. Hint: Mason's theorem.

Put $A = X^2$, $B = (T^4 + T^3)Y^2$, and $C = 1$. Then $\deg \operatorname{rad} ABC \leq \deg X + \deg Y + 2$. By Mason's Theorem, we have $\deg A = 2 \deg X \leq \deg X + \deg Y + 1$ and $\deg B = 2 \deg Y + 4 \leq \deg X + \deg Y + 1$. Adding these gives a contradiction, hence ABC must be constant, which is only possible if $Y = 0$.

(4) Find a solution of $X^2 - (T^4 + T^3)Y^2 = 1$ in $\mathbb{F}_5[T]$.
Hint: solve $X^2 - (T^2 + T)Y^2 = 1$ first and then compute the powers of the corresponding unit $X + Y\sqrt{T^2 + T}$ in $\mathbb{F}_q(X)[\sqrt{X^2 + X}\,]$.

It is readily seen that $(2T + 1, 2)$ solves $X^2 - (T^2 + T)Y^2 = 1$. Now $(2T + 1 + 2\sqrt{T^2 + T}\,)^5 = (2T + 1)^5 + 2(T^2 + T)^2\sqrt{T^2 + T}$ over $\mathbb{F}_5$, and $(X, Y) = (2T^5 + 1, 2T^3 - T^2 + 2T)$ is a solution of $X^2 - (T^4 + T^3)Y^2 = 1$ in $\mathbb{F}_5[T]$.

(5) Describe all solutions $X, Y, Z \in \mathbb{F}_p[T]$ of the Fermat equation $X^p+Y^p = Z^p$.

Since $(X+Y)^p = X^p + Y^p$ in fields of characteristic p, clearly all triples $(X, Y, X+Y)$ are solutions.

Conversely, assume that $X^p + Y^p = Z^p$; then $Z^p = (X+Y)^p$, hence $Z = u(X+Y)$ for some $u \in \mathbb{F}_p[T]$ with $u^p = 1$. Clearly $\deg u = 0$, hence $u \in \mathbb{F}_p$. But for such u we have $u^p = u$, hence $u = 1$. Thus $(X, Y, X+Y)$ are the only solutions.

(6) Does $x^4 + y^2 = z^2$ have any nontrivial solutions in $\mathbb{C}[T]$?

Yes: $(1-T^2)^2 + (2T)^2 = (1+T^2)^2$, hence $x = 2T$, $y = 2T(1-T^2)$ and $z = 2T(1+T^2)$ does it.

(7) Let $x, y \in \mathbb{C}(t)$ be polynomials. Show that $y^2 - x^3$ is either 0 or has degree $> \frac{1}{2} \deg x$.

This is Mason's theorem at work. If there are nonconstant solutions of $y^2 - x^3 = k$, then $3 \deg x \le \deg x + \deg y + \deg k - 1$ and $2 \deg y \le \deg x + \deg y + \deg k - 1$, hence $2 \deg x \le \deg y + \deg k - 1$ and $\deg y \le \deg x + \deg k - 1$. Thus $2 \deg x \le \deg x + 2 \deg k - 2$, hence $2 \deg k > \deg x$, and this was the claim.

(8) Find all singular points on the projective closures of the following complex curves:
(a) $x^3 + y^3 - 3xy = 0$;
(b) $y^2 = x^4 + 1$;
(c) $(x^2+y^2)^3 - 5x^4y + 10x^2y^3 - y^5$.

For a) we easily see that $(0,0)$ is singular; since irreducible cubics have at most one singular points, this is the only one (or simply check that the point at infinity is smooth).

For b), the partial derivatives vanish only at $(0,0)$, which is not on the curve. The points $[x : y : 0]$ at infinity satisfy $0 = x^4$, hence the only point at infinity is $[0 : 1 : 0]$, which is immediately seen to be singular.

The last curve does not have any points at infinity; clearly $(x, y) = (0, 0)$ is a singularity of multiplicity 5. If the curve had any other singularity, a line through these two would intersect the sextic curve in more than 6 points, counting multiplicity.