ALGEBRAIC GEOMETRY

HOMEWORK 2

Due Th 04.02.04

(1) Consider the unit circle $\mathcal{C} : X^2 + Y^2 = 1$ and the group $\mathcal{C}(\mathbb{Q})$. Show that $P = (x, y) \in \mathcal{C}(\mathbb{Q})$ with $x \neq -1$ is in $2\mathcal{C}(\mathbb{Q})$ (i.e., can be written as $P = 2Q$ for some $Q \in \mathcal{C}(\mathbb{Q})$) if and only if $2(x + 1)$ is a rational square.

(2) Find all $\mathbb{Q}(T)$-rational points on the conic $X^2 - (T^4 + T^3)Y^2 = 1$.

(3) Show that $X^2 - (T^4 + T^3)Y^2 = 1$ does not have any nontrivial solutions in $\mathbb{Q}[T]$. Hint: Mason's theorem.

(4) Find a solution of $X^2 - (T^4 + T^3)Y^2 = 1$ in $\mathbb{F}_5[T]$.
Hint: solve $X^2 - (T^2 + T)Y^2 = 1$ first and then compute the powers of the corresponding unit $X + Y\sqrt{T^2 + T}$ in $\mathbb{F}_q(X)[\sqrt{X^2 + X}\,]$.

(5) Describe all solutions $X, Y, Z \in \mathbb{F}_p[T]$ of the Fermat equation $X^p + Y^p = Z^p$.

(6) Does $x^4 + y^2 = z^2$ have any nontrivial solutions in $\mathbb{C}[T]$?

(7) Let $x, y \in \mathbb{C}(t)$ be polynomials. Show that $y^2 - x^3$ is either 0 or has degree $> \frac{1}{2} \deg x$.

(8) Find all singular points on the projective closures of the following curves:
(a) $x^3 + y^3 - 3xy = 0$;
(b) $y^2 = x^4 + 1$;
(c) $(x^2 + y^2)^3 - 5x^4y + 10x^2y^3 - y^5$.