

# ALGEBRAIC GEOMETRY

## HOMEWORK 1

Due Th 19.02.04

- (1) Find all points on the following curves with coordinates in the fields  $\mathbb{F}_2$ ,  $\mathbb{F}_3$ ,  $\mathbb{F}_4$  and  $\mathbb{F}_5$ :
- the line  $2x - y + 1 = 0$ ;
  - the unit circle  $x^2 + y^2 = 1$ ;
  - the curve  $x^2y + xy^2 + 1 = 0$ .

- (2) Determine the rational points on the hyperbola  $X^2 - 2Y^2 = 1$  with as many methods as possible. Do the same for the circle  $X^2 + Y^2 = 2$ .

- (3) The parametrization

$$x = \frac{1 - t^2}{1 + t^2}, \quad y = \frac{2t}{1 + t^2}$$

of the unit circle  $X^2 + Y^2 = 1$  over the field  $\mathbb{Q}$  can be interpreted as a  $\mathbb{Q}(t)$ -rational point on the unit circle itself.

Use the group law to compute  $2P$  and  $P+Q$ , where  $P$  is the parametrization above and where  $Q = (\frac{3}{5}, \frac{4}{5})$ . Do the points  $2P$  and  $P+Q$  also give parametrizations of the unit circle? For example, do they give the point  $(3/5, 4/5)$ ?

- (4) Use sing surf to sketch the following curves:
- (a)  $y^2 - x^3 - x^2 = 0$ ;
  - (b)  $y^3 + y^2x - x^2 = 0$ ;
  - (c) Folium of Descartes:  $x^3 + y^3 - 3xy = 0$  (make sure you leave a blank space between  $x$  and  $y$ ).
  - (d) 5-leaved rose:  $(x^2 + y^2)^3 - 5x^4y + 10x^2y^3 - y^5$ .

All these curves have the property that lines through  $(0,0)$  intersect the curve in exactly one other point. Use the sweeping lines technique to find all rational points on these curves.