ALGEBRAIC GEOMETRY

MIDTERM 2

(1) (10P) Consider the curve $\mathcal{C} : y^3 = x^5 + x^4$. Find a nonsingular plane curve $\mathcal{D}$ and a birational map $\phi : \mathcal{C} \longrightarrow \mathcal{D}$ by blowing up the origin.

Put $x = u$, $y = uv$; then $u^3v^3 = u^5 + u^4$, hence $\psi : (u, v) \longmapsto (x, y) = (u, uv)$ is a polynomial map from the nonsingular curve $\mathcal{D} : v^3 = u^2 + u$ to $\mathcal{C}$. In fact, the map $\phi(x, y) = (x, y/x)$ defines the inverse of ψ: $\phi \circ \psi(u, v) = \phi(u, uv) = (u, v)$ for all $(u, v) \in \mathbb{A}^2K$ with $v \neq 0$, and similarly $\psi \circ \phi(x, y) = \psi(x, y/x) = (x, y)$ is the identity map outside $x = 0$. Thus ϕ induces a birational map (defined for all points except the origin) from $\mathcal{C}$ to $\mathcal{D}$.

(2) (10P) Find a parametrization of the rational points on the sphere

$$X^2 + Y^2 + Z^2 = 3.$$

An obvious rational point to start with is $P = (1, 1, 1)$. The lines through P not contained in the plane $X = 1$ are given by $X = 1 + t$, $Y = 1 + at$, $Z = 1 + bt$. Intersecting this line with the surface gives $t = -\frac{2(1+a+b)}{1+a^2+b^2}$, and then $X = 1 + t$, $Y = 1 + at$, $Z = 1 + bt$ is a parametrization of the surface except possibly for points on the plane $X = 1$.

Intersecting this plane with the surface gives the curve $Y^2 + Z^2 = 2$, and this is a conic with a rational point $(1, 1)$, which can be parametrized easily.

(3) (20P) Find infinitely many rational points on the cubic surface

$$S : x^3 + 2y^3 + 4z^3 - 6xyz = 1$$

using the following recipe:

(a) Recall that the tangent to a curve $\mathcal{C} : f(x, y) = 0$ at a point (a, b) on $\mathcal{C}$ is given by $f_x(x - a) + f_y(y - b) = 0$. Write down the equation of the tangent plane at a point (a, b, c) on a surface $F(X, Y, Z) = 0$.

The equation of the tangent plane is

$$F_X(X - a) + F_Y(Y - b) + F_Z(Z - c) = 0.$$

(b) Write down the tangent plane to S at the point $(1, 0, 0)$.

The equation of this tangent plane is $x = 1$.

(c) The intersection $T \cap S$ is a singular cubic; find its equation, parametrize it, and show that S has infinitely many rational points.

Intersecting the plane with the surface gives the cubic $y^3+2z^3-3yz = 0$, which has a singular point at $(0,0)$. The technique of sweeping lines shows that the rational points on this curve are $y(t) = 3t/(1+2t^3)$ and $z(t) = 3t^2/(1+2t^3)$. Thus $(1, y(t), z(t))$ give infinitely many rational points on the cubic surface.

For those with some background in algebraic number theory: the element $\alpha = a+b\omega+c\omega^2 \in \mathbb{Q}(\omega)$, where $\omega^3 = 2$, has norm $N\alpha = a^3+2b^3+4c^3-6abc$. Thus the integral points on the surface correspond to units in the ring $\mathbb{Z}[\omega]$. The point P corresponds to the trivial unit 1; other integral points such as $(1,-3,3)$ (for $t=-1$) or $(1,1,1)$ (for $t=1$) correspond to nontrivial units.

(4) (10P) Consider the following curves given by a parametrization. Compute the inverse maps. Which of them are polynomial? (No proofs required).

(a) $x = t^2+1$, $y = t^3-t$;

The inverse map of $\phi(t) = (t^2+1, t^3-t)$ is $\psi(x,y) = \frac{y}{x-1}$. This is not a polynomial map.

(b) $x = t^3+1$, $y = t^3+t$.

The inverse map of $\phi(t) = (t^3+1, t^3+t)$ is $\psi(x,y) = y-x+1$. This clearly is a polynomial map.

(5) Let U, V be subspaces of some K-vector space W.

(a) (10P) Show that the sequence

$$0 \longrightarrow U \cap V \xrightarrow{\ f\ } U \oplus V \xrightarrow{\ g\ } U+V \longrightarrow 0$$

is exact. Here $U \oplus V = \{(u,v) : u \in U, v \in V\}$ and $U+V = \{u+v : u \in U, v \in V\}$. Moreover, f and g are given by $f(u) = (u,-u)$ and $g(u,v) = u+v$.

(b) (5P) Show that $\dim(U+V) = \dim U + \dim V - \dim U \cap V$.

We have $\ker f = \{u \in U \cap V : (u,-u) = (0,0)\} = 0$. Thus f is injective. Next $\ker g = \{(u,v) \in U \oplus V : u+v = 0\} = \{(u,-u) : u \in U = \operatorname{im} f$. Finally, g is surjective since the element $u+v \in U+V$ is the image of (u,v).

The fact that the alternating sum of dimensions in an exact sequence of vector spaces is 0, plus the fact that $\dim U \oplus V = \dim U + \dim V$, implies the claim.

Note that if $\{u_1, \ldots, u_m\}$ and $\{v_1, \ldots, v_n\}$ are bases of U and V, respectively, then $\{(u_1,0), \ldots, (u_m,0), (0,v_1), \ldots, (0,v_n)\}$ is a basis of $U \oplus V$. If you want a vector space with dimension mn, then you have to look at the tensor product $U \otimes V$.

(6) (10P) Consider $f(X,Y) = Y^2 - X^3 - X^2$ and $P = (-1,0)$. Show that the maximal ideal $\mathfrak{m} = (x+1, y)$ of $\mathcal{O}_P(\mathcal{C}_f)$ is principal.

We know that $x = X + (f)$ and $y = Y + (f)$. Thus $x + 1 = y^2/x^2$, and since x is a unit in $\mathcal{O}_P$ because $x(P) = 1$, we see that $x + 1 \in (y)$. Thus $\mathfrak{m} = (y)$.

(7) Consider the polynomial ring $R = K[X]$ over some field K.
  (a) (10P) Show that $v(f) = -\deg f$ is a valuation on R, i.e., that $v(fg) = v(f) + v(g)$ and $v(f+g) \ge \min\{v(f), v(g)\}$.
  (b) (5P) Is R a discrete valuation ring? Justify your answer.

Let $f = a_m X^m + \ldots + a_0$, $g = b_n X^n + \ldots + b_0$, with $a_m b_n \ne 0$ (here we use that K is a field, i.e., that it has no zero divisors). Then $v(f) = -m$, $v(g) = -n$, and $v(fg) = -m - n$ since $fg = a_m b_n X^{m+n} + \ldots$. Moreover, $\deg(f+g) \le \max\{\deg f, \deg g\}$, hence $v(f+g) \ge -\max\{\deg f, \deg g\} = \min\{v(f), v(g)\}$.

R is not a discrete valuation ring since it is not local: the ideals (X) and $(X+1)$ are two different maximal ideals.

Note that discrete valuation rings are much more than rings with discrete valuations: $\mathbb{Z}$ has many discrete valuations, one for each prime p, but it is not a discrete valuation ring.

(8) (10P) Let $f \in K[X,Y]$ be an irreducible polynomial over some field K, and let $\mathcal{C}_f : f(X,Y) = 0$ be the associated curve. Give the definitions of
  (a) the coordinate ring $K[\mathcal{C}_f]$,
  (b) the function field $K(\mathcal{C}_f)$,
  (c) the local ring $\mathcal{O}_P(\mathcal{C}_f)$,
  (d) the maximal ideal $\mathfrak{m}_P(\mathcal{C}_f)$ in $\mathcal{O}_P(\mathcal{C}_f)$.

We have $K[\mathcal{C}_f] = R/(f)$, $K(\mathcal{C}_f)$ is the quotient field of $K[\mathcal{C}_f]$, the local ring $\mathcal{O}_P(\mathcal{C}_f)$ consists of all elements of $K(\mathcal{C}_f)$ defined at P, and the maximal ideal consists of all elements in the local ring that vanish at P.