ALGEBRAIC GEOMETRY

MIDTERM 2

(1) (10P) Consider the curve $\mathcal{C} : y^3 = x^5 + x^4$. Find a nonsingular plane curve $\mathcal{D}$ and a birational map $\phi : \mathcal{C} \longrightarrow \mathcal{D}$ by blowing up the origin.

(2) (10P) Find a parametrization of the rational points on the sphere

$$X^2 + Y^2 + Z^2 = 3.$$

(3) (20P) Find infinitely many rational points on the cubic surface

$$S : x^3 + 2y^3 + 4z^3 - 6xyz = 1$$

using the following recipe:

(a) Recall that the tangent to a curve $\mathcal{C} : f(x, y) = 0$ at a point (a, b) on $\mathcal{C}$ is given by $f_x(x - a) + f_y(y - b) = 0$. Write down the equation of the tangent plane at a point (a, b, c) on a surface $F(X, Y, Z) = 0$.

(b) Write down the tangent plane to S at the point $(1, 0, 0)$.

(c) The intersection $T \cap S$ is a singular cubic; find its equation, parametrize it, and show that S has infinitely many rational points.

(4) (10P) Consider the following curves given by a parametrization. Compute the inverse maps. Which of them are polynomial? (No proofs required).
 (a) $x = t^2 + 1$, $y = t^3 - t$;
 (b) $x = t^3 + 1$, $y = t^3 + t$.

(5) Let U, V be subspaces of some K-vector space W.
 (a) (10P) Show that the sequence

$$0 \longrightarrow U \cap V \xrightarrow{\ f\ } U \oplus V \xrightarrow{\ g\ } U + V \longrightarrow 0$$

 is exact. Here $U \oplus V = \{(u, v) : u \in U, v \in V\}$ and $U + V = \{u + v : u \in U, v \in V\}$. Moreover, f and g are given by $f(u) = (u, -u)$ and $g(u, v) = u + v$.
 (b) (5P) Show that $\dim U + V = \dim U + \dim V - \dim U \cap V$.

(6) (10P) Consider $f(X,Y) = Y^2 - X^3 - X^2$ and $P = (-1,0)$. Show that the maximal ideal $\mathfrak{m} = (x+1, y)$ of $\mathcal{O}_P(\mathcal{C}_f)$ is principal.

(7) Consider the polynomial ring $R = K[X]$ over some field K.
  (a) (10P) Show that $v(f) = -\deg f$ is a valuation on R, i.e., that $v(fg) = v(f) + v(g)$ and $v(f+g) \geq \min\{v(f), v(g)\}$.
  (b) (5P) Is R a discrete valuation ring? Justify your answer.

(8) (10P) Let $f \in K[X,Y]$ be an irreducible polynomial over some field K, and let $\mathcal{C}_f : f(X,Y) = 0$ be the associated curve. Give the definitions of
  (a) the coordinate ring $K[\mathcal{C}_f]$,
  (b) the function field $K(\mathcal{C}_f)$,
  (c) the local ring $\mathcal{O}_P(\mathcal{C}_f)$,
  (d) the maximal ideal $\mathfrak{m}_P(\mathcal{C}_f)$ in $\mathcal{O}_P(\mathcal{C}_f)$.