ALGEBRAIC GEOMETRY

MIDTERM 1, MARCH 22, 2004

(1) (5) Find all points on the projective closure of the curve $y^2 = x^3 + x$ over $\mathbb{F}_3$.

The projective closure has equation $Y^2Z - X^3 - xZ^2 = 0$. The points at infinity is $[0:1:0]$; the affine points are $[0:0:1]$, $[2:1:1]$ and $[2:2:1]$.

(2) (10) Find all singular points on $x^3 + y^3 + 1 + 3axy = 0$, where $a \in \mathbb{C}$.

The projectivation has equation $X^3 + Y^3 + Z^3 + 3aXYZ = 0$, and the derivatives are

$$F_X = 3X^2 + 3aYZ,$$
$$F_Y = 3Y^2 + 3aXZ,$$
$$F_Z = 3Z^2 + 3aXY.$$

If $a = 0$, then the curve is not singular; assume therefore that $a \neq 0$. This easily implies $XYZ \neq 0$; in particular, the point at infinity is smooth, and we may assume that $Z = 1$.

From $X^2 = -aY$ we get $X^4 = a^2Y^2 = -a^3X$, hence $X^3 = -a^3$. Similarly we find $Y^3 = -a^3$. Thus $X = -a\zeta$, where $\zeta^3 = 1$, and then $-aY = X^2 = a^2\zeta^2$ gives $Y = -a\zeta^2$. The point $[-a\zeta : -a\zeta^2 : 1]$ satisfies the equations if and only if $a^3 = -1$.

Thus the curve is smooth if $a^3 \neq -1$. If $a^3 = -1$, on the other hand, then the points $[-a\zeta : -a\zeta^2 : 1]$ are singular. In particular, the curve is a triple of lines in this case.

(3) (20) Parametrize the conic $\mathcal{C} : x^2 + xy + y^2 = 3$ over $\mathbb{Q}$. Extend the corresponding map $\phi : \overline{A}^1\mathbb{Q} \longrightarrow \mathcal{C}(\mathbb{Q})$ to a polynomial map $\phi^\# : \mathbb{P}^1\mathbb{Q} \longrightarrow \mathcal{C}^\#(\mathbb{Q})$. Is ϕ injective, surjective, bijective? What about $\phi^\#$?

A standard calculation using the known point $(1,1)$ gives

$$x = \frac{t^2 - 2t - 2}{t^2 + t + 1}, y = \frac{-2t^2 - 2t + 1}{t^2 + t + 1}.$$

This map is defined on the whole affine line, injective by geometry, and surjective except for $(1, -2)$, which would correspond to $t = \infty$.

The projectivation is

$$\phi^\# : [m : n] \longmapsto [m^2 - 2mn - 2n^2 : -2m^2 - 2mn + n^2 : m^2 + mn + n^2].$$

Since $\phi^\#([1:0]) = [1:-2:0]$, the map $\phi^\#$ becomes surjective. Since ϕ is injective and since $\phi^\#([1:0]) = [-1:2:0]$ does not correspond to a point in the image of ϕ, the map $\phi^\#$ remains injective, hence is bijective.

(4) (10) Compute the tangent of the real curve $x^3 + y^3 + 1 = 0$ at infinity.

The point at infinity is $[1 : -1 : 0]$, and the equation of the tangent is $X + Y = 0$.

(5) (15) Show that $X^3 + Y^3 = Z^4$ does not have nonconstant solutions in the polynomial ring $\mathbb{C}[T]$.

Put $A = X^3$, $B = Y^3$ and $C = -Z^4$; since $\deg \operatorname{rad} ABC \le \deg X + \deg Y + \deg Z$, Mason's theorem gives

$$\begin{aligned} 3\deg X = \deg A &\le \deg X + \deg Y + \deg Z - 1, \\ 3\deg Y = \deg A &\le \deg X + \deg Y + \deg Z - 1, \\ 4\deg Z = \deg A &\le \deg X + \deg Y + \deg Z - 1, \end{aligned}$$

and adding these inequalities shows that $\deg Z \le -3$, a contradiction. Thus one of X, Y, Z must be a constant.

(6) (20) Let C be a cubic with three double points. Show that C consists of three lines. Are there cubics with exactly two double points?

If P, Q, R are the three double points, then Bezout's theorem shows that C contains the lines PQ, QR, and RP. If two lines would coincide, the cubic would have infinitely many double (or even triple) points. Thus the cubic consists of three lines.

The $(y - 1)(y - x^2) = 0$, has exactly two double points.

(7) (20) Consider the curve $C : (x^2 + y^2)^2 - x^2 + y^2 = 0$; for all lines L through $(0, 0)$, compute the intersection multiplicity of C and L.

First consider $y = tx$; then $g(x) = x^2[(1 + t^2)^2 - 1 + t^2] = 0$, so the multiplicity is 2 except when $t = \pm 1$, when it is 4.

If $= 0$, we get $g(y) = y^2(y^2 + 1)$, hence this line intersects the lemniscate with multiplicity 2.