ALGEBRAIC GEOMETRY

FINAL

(1) (10) *Consider the equation* $X^2 - DY^2 = 1$, *where* $D \in \mathbb{C}[T]$ *is a nonconstant polynomial of degree* $\deg D > 0$. *Let* $n(D)$ *denote the number of distinct zeros of* D. *Show that the equation does not have any solutions* $X, Y \in \mathbb{C}[T]$ *except* $(\pm 1, 0)$ *if* $2n(D) \le \deg D$.

We have $\deg X^2, \deg DY^2 \le \operatorname{rad} DX^2Y^2 - 1$. Moreover $\operatorname{rad} DX^2Y^2 \le n(D) + \deg X + \deg Y$, hence $2\deg X \le n(D) + \deg X + \deg Y - 1$ and $\deg D + 2\deg Y \le n(D) + \deg X + \deg Y - 1$. Adding these inequalities gives $\deg D + 2 \le 2n(D)$, contradicting the assumption $2n(D) \le \deg D$.

(2) (10) *State Bezout's Theorem.*

The weakest version states that two plane algebraic curves of degree m and n without common component intersect in at most mn points.

The strongest version says that the number of intersection points is equal to mn if the points are counted with multiplicity, and if we work in the projective plane over some algebraically closed field. The multiplicity can be defined and computed using resultants.

(3) (10)*Let* m, n *be natural numbers and* $f(X) = X^m + a_{m-1}X^{m-1} + \ldots + a_0 \in \mathbb{C}[X]$ *a polynomial of degree* m. *Consider the curve* $\mathcal{C} : Y^n = f(X)$.

(a) *Compute how many points at infinity the curve* $\mathcal{C}$ *has.*

(b) *Determine which of these points are singular.*

There are a few cases. The homogenization of $g(X,Y) = Y^n - f(X)$ is given by $G(X,Y,Z) =$

$$\begin{cases} Y^nZ^{m-n} - X^m - a_{m-1}X^{m-1}Z - \ldots - a_0Z^m & \text{if } m \ge n, \\ Y^n - X^mZ^{n-m} - a_{m-1}X^{m-1}Z^{n-m+1} - \ldots - a_0Z^n & \text{if } m < n. \end{cases}$$

Points at infinity have $Z = 0$.

(a) If $m > n$, then $Z = 0$ implies $X = 0$, hence $P = [0 : 1 : 0]$ is the only point at infinity; moreover $F_X(P) = F_Y(P) = 0$, so P is singular if and only if $F_Z(P) = 0$. But $F_Z = (m-n)Y^nZ^{m-n-1} - a_{m-1}X^{m-1} - \ldots - ma_0X^{m-1}$, hence $F_Z(P) = 0$ if and only if $m - n - 1 > 0$. In particular, P is nonsingular if and only if $m = n + 1$.

(b) If $m = n$, then $Z = 0$ implies $Y^n - X^n = 0$, and this equation has exactly n different solutions, namely $Y = \zeta^a X$, where ζ is a primitive nth root of unity and $0 \le a < n$. Thus the points at infinity are $P_0 = [1 : 1 : 0]$, $P_1 = [1 : \zeta : 0]$, …, $P_{n-1} = [1 : \zeta^{n-1} : 0]$. Since $F_Y(P_a) \ne 0$, all these points are smooth.

(c) If $m < n$, then $Z = 0$ implies $Y = 0$, hence $P = [1 : 0 : 0]$ is the only point at infinity on $\mathcal{C}$. Here $F_X(P) = F_Y(P) = 0$ (since $m \ge 1$ and $n \ge 2$), and $F_Z(P) \ne 0$ if and only if $m = n - 1$. Thus P is singular except when $m = n - 1$.

(4) (10) *Consider the curve $\mathcal{C} : Y^3 = X^4$. Find a smooth curve $\mathcal{D}$ and a birational map $\phi : \mathcal{C} \longrightarrow \mathcal{D}$.*

Put $X = u$ and $Y = uv$; then $u^3v^3 = u^4$, hence $u = v^3$ for $u \ne 0$. This curve $\mathcal{D} : u = v^3$ is smooth in the affine plane, and $\psi : \mathcal{D} \longrightarrow \mathcal{C} : (u, v) \longmapsto (X, Y) = (u, uv)$ is a birational map with inverse $\phi : \mathcal{C} \longrightarrow \mathcal{D} : (X, Y) \longmapsto (u, v) = (X, Y/X)$. Observe that if (u, v) is on $\mathcal{D}$, i.e. if $u = v^3$, then $X, Y) = (u, uv)$ satisfies $Y^3 = u^3v^3 = u^4 = X^4$.

Actually, $\mathcal{D}$ is singular at $[1 : 0 : 0]$, so strictly speaking you have to move the point at infinity to the origin (homogenize $UW^2 = V^3$ and dehomogenize $w^2 = v^3$) and then blow up again.

(5) (10) *Compute the intersection multiplicities of the lines through the origin with the cubic $C : Y^2 = X^3 + X^2$.*

Set $Y = tX$; then $X^2(t^2 - 1 - X) = 0$, so the intersection multiplicity at $(0, 0)$ is 2 unless $t = \pm 1$, when it is 3.

If $X = 0$, we get $Y^2 = 0$, hence this line intersects the curve with multiplicity 2.

(6) (10) *Show that the cubic surface $X^2 + Y^3 - Y^2 + Z^2 = 0$ has a singular point. Parametrize the surface by using the pencil of lines through it.*

It is straightforward to check that the only singular point is $(0, 0, 0)$. The lines through it (not contained in the plane $X = 0$) are parametrized by $X = t$, $Y = at$, $Z = bt$. Computing the third point of intersection gives $t = \frac{a^2-b^2-1}{a^3}$ (unless $a = 0$, when $Y = 0$; these points satisfy $X^2 + Z^2 = 0$, so the only rational point we miss is the origin), hence

$$X = \frac{a^2 - b^2 - 1}{a^3}, \quad Y = a\frac{a^2 - b^2 - 1}{a^3}, \quad Z = b\frac{a^2 - b^2 - 1}{a^3}.$$

The points in the plane $X = 0$ are given by $Z^2 = Y^3 - Y^2$. This is a singular cubic, which is easily parametrized.

(7) (10) *Let $f \in K[X]$ be a polynomial of degree ≥ 1, and let $\mathcal{C} : Y = f(X)$ be a curve in $\mathbb{P}^1 K$. Find a parametrization $\phi : K \longrightarrow \mathcal{C}(K)$, show that ϕ is a polynomial map, and show that $K[\mathcal{C}] \simeq K[X]$.*

The points on $F(X, Y) = Y - f(X) = 0$ are parametrized by $\phi(t) = (t, f(t))$. This map is obviously polynomial, and it induces a ring homomorphism $\phi^* : K[\mathcal{C}] \longrightarrow K[X] = K[X, Y]/(Y)$ via $\phi^*(g(X, Y) + (F)) = g(X, f(X))$. Moreover, the map $\psi : \mathcal{C} \longrightarrow L$ from the curve to the line $L : Y = 0$ defined by $\psi(X, Y) = X$ is the inverse map of ϕ since $\psi \circ \phi(t) = (t, f(t)) = t$ and $\phi \circ \psi(X, Y) = \phi(X) = (X, f(X)) = (X, Y)$. Thus ψ^* will be the inverse map of ϕ^*.

(8) (10) *Let $\mathcal{C} : Y^2 = X^3 + X$ be a curve in $\mathbb{P}^2\mathbb{C}$. Why is the maximal ideal (x, y) of $\mathcal{O}_P(\mathcal{C})$ at $P = (0, 0)$ principal (Refer to a theorem; no proof)? Compute a generator.*

The curve $\mathcal{C}$ is smooth, hence $\mathcal{O}_P(\mathcal{C})$ is a discrete valuation ring, and in particular the maximal ideal $\mathfrak{m}_P$ of $\mathcal{O}_P$ is principal. In fact, it is generated by any line through P not a tangent, in particular by y.

This can be seen directly from $x = \frac{y^2}{1+x^2}$ and the fact that $1 + x^2$ is a unit in $\mathcal{O}_P$.

(9) (10) *Compute the divisor of $f(X) = X(X - 1)^2$ in the function field $\mathbb{C}(X)$ of $\mathbb{P}^1\mathbb{C}$. Is there a function with divisor $2(1) - (0)$ (why/why not)?*

We have $\operatorname{div}(f) = (0) + 2(1) - 3(\infty)$. There is no function with divisor $2(1) - (0)$ because divisors of functions have degree 0.

(10) (10) *For the function field $\mathbb{C}(X)$ of the projective line $\mathbb{P}^1\mathbb{C}$, Riemann's theorem predicts that the dimension of*

$$H^0(D) = \{f \in \mathbb{C}(X) : (f) + D \geq 0\}$$

is equal to $h^0(D) = \deg(D) + 1$ for all divisors D with $\deg(D) \geq 0$.

(a) *Compute $h^0(D)$ for $D = (2) - (0)$ and for $D = (2) + (0)$.*

(b) *Give a basis for $H^0(D)$ for $D = (2) - (0)$ and for $D = (2) + (0)$.*

(a) Clearly $h^0(D) = \deg(D) + 1 = 1$ for $D = (2) - (0)$, and $h^0(D) = 3$ for $D = (2) + (0)$.

(b) Since $H^0(D)$ for $D = (2) - (0)$ has dimension 1 and f must have a zero at 0 in order to compensate the $-(0)$, and it cannot have a pole at infinity but is allowed to have a pole at 2. Thus the only functions in $H^0(D)$ are multiples of $\frac{x}{x-2}$.

If $D = (2) + (0)$, on the other hand, the functions in $H^0(D)$ are allowed to have simple poles at 2 or 0, so 1, $\frac{1}{x}$ and $\frac{1}{x-2}$ are in there. Since they are independent over $\mathbb{C}$ ($a + \frac{b}{x} + \frac{c}{x-2} = 0$ implies $a = b = c = 0$), they form a basis.