ALGEBRAIC GEOMETRY

FINAL

NAME ______________________________

(1) (10) Consider the equation $X^2 - DY^2 = 1$, where $D \in \mathbb{C}[T]$ is a nonconstant polynomial of degree $\deg D > 0$. Let $n(D)$ denote the number of distinct zeros of D. Show that the equation does not have any solutions $X, Y \in \mathbb{C}[T]$ except $(\pm 1, 0)$ if $2n(D) \leq \deg D$.

(2) (10) State Bezout's Theorem.

(3) (10) Let m, n be natural numbers and $f(X) = X^m + a_{m-1}X^{m-1} + \ldots + a_0 \in \mathbb{C}[X]$ a polynomial of degree m. Consider the curve $\mathcal{C} : Y^n = f(X)$.
   (a) Compute how many points at infinity the curve $\mathcal{C}$ has.
   (b) Determine which of these points are singular.

(4) (10) Consider the curve $\mathcal{C} : Y^3 = X^4$. Find a smooth curve $\mathcal{D}$ and a birational map $\phi : \mathcal{C} \longrightarrow \mathcal{D}$.

(5) (10) Compute the intersection multiplicities of the lines through the origin with the cubic $C : Y^2 = X^3 + X^2$.

(6) (10) Show that the cubic surface $X^2 + Y^3 - Y^2 + Z^2 = 0$ has a singular point. Parametrize the surface by using the pencil of lines through it.

(7) (10) Let $f \in K[X]$ be a polynomial of degree ≥ 1, and let $\mathcal{C} : Y = f(X)$ be a curve in $\mathbb{P}^1 K$. Find a parametrization $\phi : K \longrightarrow \mathcal{C}(K)$, show that ϕ is a polynomial map, and show that $K[\mathcal{C}] \simeq K[X]$.

(8) (10) Let $\mathcal{C} : Y^2 = X^3 + X$ be a curve in $\mathbb{P}^2\mathbb{C}$. Why is the maximal ideal (x, y) of $\mathcal{O}_P(\mathcal{C})$ principal (Refer to a theorem; no proof)? Compute a generator.

(9) (10) Compute the divisor of $f(X) = X(X-1)^2$ in the function field $\mathbb{C}(X)$ of $\mathbb{P}^1\mathbb{C}$. Is there a function with divisor $2(1) - (0)$ (why/why not)?

(10) (10) For the function field $\mathbb{C}(X)$ of the projective line $\mathbb{P}^1\mathbb{C}$, Riemann's theorem predicts that the dimension of

$$H^0(D) = \{f \in \mathbb{C}(X) : (f) + D \geq 0\}$$

is equal to $h^0(D) = \deg(D) + 1$ for all divisors D with $\deg(D) \geq 0$.

(a) Compute $h^0(D)$ for $D = (2) - (0)$ and for $D = (2) + (0)$.

(b) Give a basis for $H^0(D)$ for $D = (2) - (0)$ and for $D = (2) + (0)$.