REVIEW PROBLEMS

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- Find a birational map from the curve $y^3 = x^5$ to some smooth curve by blowing up at the origin.
- Parametrize the quadratic surface $x^2 + 2y^2 + 3z^2 = 1$.
- The parametrization of $\mathcal{C} : x^2 - 2y^2 = 1$ defines a rational map $\mathbb{P}^1\mathbb{Q} \longrightarrow \mathcal{C}(\mathbb{Q})$. Show that the induced map $K(\mathcal{C}) \longrightarrow K(\mathbb{P}^1\mathbb{Q})$ is an isomorphism.
- Show that $x = t^2 + t$, $y = t^2 + 1$ parametrizes the conic $\mathcal{C} : x^2 - 2xy + 2x + y^2 - 3y + 2 = 0$. Show that the inverse map is also polynomial, and that the induced map on the coordinate rings is an isomorphism.
- A valuation on a ring R is a map $v : R \setminus \{0\} \longrightarrow \mathbb{N}$ such that $v(rs) = v(r) + v(s)$ and $v(r + s) \geq \min\{v(r), v(s)\}$.

  For a prime p and nonzero $a \in \mathbb{Z}$, define $v_p(a) = n$ if $a = bp^n$ for some integer n not divisible by p. Show that v_p is a valuation.
- Understand what an exact sequence is. Show that the following sequences are exact:

$$0 \longrightarrow \mathbb{Z}/2 \xrightarrow{\ f\ } \mathbb{Z}/4 \xrightarrow{\ g\ } \mathbb{Z}/2 \longrightarrow 0,$$

where $f(a + 2\mathbb{Z}) = 2a + 4\mathbb{Z}$, and $g(a + 4\mathbb{Z}) = a + 2\mathbb{Z}$.

$$0 \longrightarrow \mathbb{Z}/2 \xrightarrow{\ f\ } \mathbb{Z}/2 \oplus \mathbb{Z}/2 \xrightarrow{\ g\ } \mathbb{Z}/2 \longrightarrow 0,$$

where $f(a + 2\mathbb{Z}) = (a + 2\mathbb{Z}, 0)$ and $g(a + 2\mathbb{Z}, b + 2\mathbb{Z}) = b + 2\mathbb{Z}$.

$$0 \longrightarrow \mathbb{Z}/2 \xrightarrow{\ f\ } \mathbb{Z}/6 \xrightarrow{\ g\ } \mathbb{Z}/3 \longrightarrow 0,$$

where $f(a + 2\mathbb{Z}) = 3a + 6\mathbb{Z}$ and $g(a + 6\mathbb{Z}) = a + 3\mathbb{Z}$.

Let $\mathcal{O}_P(\mathcal{C}_f) \subseteq K(\mathcal{C}_f)$ be the local ring at P with maximal ideal $\mathfrak{m}$. Show that

$$0 \longrightarrow \mathfrak{m} \longrightarrow \mathcal{O}_P \xrightarrow{\ e\ } K \longrightarrow 0$$

is exact, where the first map is the inclusion map, and where e is evaluation at P.

Assume that the sequence

$$0 \longrightarrow A \longrightarrow B \xrightarrow{\ f\ } C \xrightarrow{\ g\ } D \longrightarrow 0$$

of abelian groups (or rings, K-vector spaces) is exact. Show that it induces the following exact sequences:

$$0 \longrightarrow A \longrightarrow B \xrightarrow{\ f\ } \operatorname{im} f \longrightarrow 0$$

$$0 \longrightarrow \operatorname{im} f \longrightarrow C \xrightarrow{\ g\ } D \longrightarrow 0.$$

Be sure to explain all the maps.

Show that if

$$0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow X \longrightarrow 0$$

is an exact sequence of K-vector spaces, then $\dim_K U - \dim_K V + \dim_K W - \dim_K X = 0$.

- Consider the ring $K[[X]]$ of all formal power series $\sum_{i=0}^{\infty} a_i X^i$ with $a_i \in K$. Show that $K[X]$ is a subring of $K[[X]]$. Is it also an ideal? Show that $K[[X]]$ is a discrete valuation ring with maximal ideal (X).
- Consider $f(X,Y) = Y^2 - X^3 - X^2$ and $P = (0,0)$. Show (from first principles, not by simply referring to a theorem) that the maximal ideal $\mathfrak{m} = (x,y)$ of $\mathcal{O}_P(\mathcal{C}_f)$ is not principal.
- Consider the curve $\mathcal{C}_f : f(X,Y) = Y - X^2 = 0$ and let $P = (1,1)$. Show that the maximal ideal $\mathfrak{m}_P$ of $\mathcal{O}_P$ is principal, and write down a uniformizer (generating element). Compute $\operatorname{ord}_P(y - m(x-1) - 1)$.