LECTURE 22, THURSDAY MAY 13, 2004

FRANZ LEMMERMEYER

1. What we have done so far

Our goal is to sketch a generalization of algebraic geometry from curves with coordinate rings to more general objects whose coordinate rings are arbitrary rings; the coordinate rings $K[\mathcal{C}]$ of curves $\mathcal{C}$ in the affine plane over K all contain the field K, so rings like $\mathbb{Z}$ or $\mathbb{Z}/m\mathbb{Z}$ cannot occur. Our motivation is drawn from the fact that the rings $\mathbb{Z}_{(p)}$ have a lot of properties in common with the local rings $\mathcal{O}_P(\mathcal{C})$.

We started out with a general ring R and defined the spectrum of R, $\operatorname{Spec} R$, as the set of prime ideals in R. These prime ideals will be our "points", and R itself will play the role of the coordinate ring. In fact, ring homomorphisms $R \longrightarrow S$ induce maps $\operatorname{Spec} S \longrightarrow \operatorname{Spec} R$, and this corresponds to the fact that ring homomorphisms $K[\mathcal{D}] \longrightarrow K[cC]$ induce polynomial maps $\mathcal{C} \longrightarrow \mathcal{D}$.

Next we made $\operatorname{Spec} R$ into a topological space: its closed sets are, by definitions, sets of the form

$$V(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Spec} R : \mathfrak{a} \subseteq \mathfrak{p}\},$$

where $\mathfrak{a}$ is an ideal in R. This topology is called the Zariski topology.

For example, $V((6)) = \{(2),(3)\}$ is closed in $\operatorname{Spec}\mathbb{Z}$, therefore the set $(\operatorname{Spec}\mathbb{Z}) \setminus \{(2),(3)\}$ is an open set.

We also found a way to interpret elements $r \in R$ as functions on $\operatorname{Spec} R$ by defining $r(\mathfrak{p}) = r + \mathfrak{p} \in R/\mathfrak{p}$; note that these 'functions' have different domains at different points!

Let me also mention the following

Proposition 1.1. *If R is a domain, then the topological space* $\operatorname{Spec} R$ *is irreducible.*

Recall that a topological space X is called irreducible it cannot be written as the union of two proper closed subsets.

Proof. Assume that R is a domain, and suppose that $\operatorname{Spec} R = X \cup Y$ for closed sets X, Y. Since R is a domain, (0) is a prime ideal, hence $(0) \in \operatorname{Spec} R$ and therefore $(0) \in X$ or $(0) inY$. But if $(0) \in X$, then $X = V(\mathfrak{a})$ for some ideal $\mathfrak{a}$ in R since X is closed, and $(0) \in V(\mathfrak{a})$ means that $\mathfrak{a} \subseteq (0)$. Thus $\mathfrak{a} = (0)$ and therefore $X = V(\mathfrak{a}) = \operatorname{Spec} R$. Thus $X \cup Y$ is not a union of proper subsets, and $\operatorname{Spec} R$ is irreducible. □

The converse is not true: consider $R = \mathbb{Z}/4\mathbb{Z}$. Its only ideals are (0), (1) and (2), and of these only (2) is prime. Thus $\operatorname{Spec} R = \{(2)\}$ has only one point, hence does not have any proper decomposition.

2. Distinguished Open Sets

In the Zariski topology, the complements of $V(\mathfrak{a})$ are open sets. A special class of open sets are the distinguished open sets $D(f)$, where $f \in R$ is an arbitrary element of R:

$$D(f) = \{\mathfrak{p} \in \operatorname{Spec} R : f \notin \mathfrak{p}\}.$$

Why is this set open? Well, its complement consists of all $\mathfrak{p} \in \operatorname{Spec} R$ with $f \in \mathfrak{p}$, in other words, the complement of $D(f)$ has the form $V(\mathfrak{a})$ for $\mathfrak{a} = (f)$.

With the interpretation of elements as functions, the set $D(f)$ is the set of all points $\mathfrak{p} \in \operatorname{Spec} R$ on which f is nonzero. Its complement $V(f) = \operatorname{Spec} R \setminus D(f)$ is therefore the set of all points on which f vanishes: this finally explains the choice of the letter V. In particular, we have $V(0) = \operatorname{Spec} R$ because the function 0 vanishes everywhere.

Since $D(\mathfrak{a}) = \bigcup_{a \in \mathfrak{a}} D(a)$, we see that $D(\mathfrak{a})$ is the set of all points on which some $a \in \mathfrak{a}$ does not vanish; its complement $V(\mathfrak{a})$ is the set of all points $\mathfrak{p} \in \operatorname{Spec}$ on which every $a \in \mathfrak{a}$ vanishes.

3. Localization

The technique of localization generalizes the construction of the rational numbers from the integers, or, more generally, the construction of the quotient field of an integral domain.

In fact, for general rings R we do not have an analog of the function field $K(\mathcal{C})$; in particular, we cannot define the analog of the local rings $\mathcal{O}_P(\mathcal{C})$ as subrings of the quotient field of R since the latter does not exist in general.

Let R be a ring and S a multiplicatively closed subset of R. For an R-module M, we define its localization $S^{-1}M$ as follows. Define an equivalence relation on $M \times S$ by putting $(m, s) \sim (m', s')$ if there is an $s'' \in S$ such that $s''(s'm - sm') = 0$. Let $\frac{m}{s}$ denote the equivalence class of (m, s). Let $S^{-1}M$ denote the set of equivalence classes $\frac{m}{s}$; note that $S^{-1}M = 0$ if $0 \in S$. We can make $S^{-1}M$ into an abelian group by putting $\frac{m}{s} + \frac{m'}{s'} = \frac{s'm+sm'}{ss'}$.

In the special case $M = R$ we can define a ring structure on $S^{-1}R$ by putting $\frac{r}{s} \cdot \frac{r'}{s'} = \frac{rr'}{ss'}$. Note that the unit element is given by $\frac{s}{s}$, the zero by $\frac{0}{s}$ for any nonzero $s \in S$. Fix any $s \in S$; the map $r \longmapsto \frac{rs}{s}$ does not depend on the choice of s and defines a homomorphism $\phi_S : R \longrightarrow S^{-1}R$, making $S^{-1}R$ into an R-algebra.

Example 1. Consider the case $R = \mathbb{Z}$ and $S = \mathbb{N} = \{1, 2, 3, \ldots\}$. Then $S^{-1}R$ consists of all fractions $\frac{m}{n}$ with $m \in \mathbb{Z}$ and $n \in \mathbb{N}$, in other words: $S^{-1}R = \mathbb{Q}$. Note that the set $S' = \mathbb{Z} \setminus \{0\}$ produces the same result: $S'^{-1}\mathbb{Z} = \mathbb{Q}$.

Example 2. Whenever R is an integral domain, then $S = R \setminus \{0\}$ is multiplicative, and $\operatorname{quot}(R) := S^{-1}R$ turns out to be a field called the quotient field of R. For example, $\operatorname{quot}(\mathbb{Z}) = \mathbb{Q}$, $\operatorname{quot}(\mathbb{Z}[i]) = \mathbb{Q}(i)$, $\operatorname{quot}(k[X]) = k(X)$, etc.

Example 3. Let $S = \{p, p^2, p^3, \ldots\}$ for some prime p. Then S is multiplicative, and $Z_S = \mathbb{Z}[\frac{1}{p}]$ is the set of all rational numbers whose denominator is a power of p.

Example 4. Let $S = \mathbb{Z} \setminus p\mathbb{Z}$; Then S is multiplicative and $\mathbb{Z}_S = \{\frac{r}{s} : p \nmid s\} = \mathbb{Z}_{(p)}$.

Example 5. Let $R = \mathbb{Z}/6\mathbb{Z}$ and $S = \{1, 2, 4, 5\} = R \setminus (3)$; then $\frac{3}{1} = \frac{0}{1}$ since $2 \cdot 3 \equiv 2 \cdot 0 \bmod 6$. Similarly, $\frac{4}{1} = \frac{1}{1}$ because $2 \cdot 4 \equiv 2 \cdot 1 \bmod 6$. In fact, we find that

$S^{-1}R \simeq \mathbb{Z}/3\mathbb{Z}$. Thus the localization is a field, but the ring R is not contained in it as a subring.

Proposition 3.1. *Let S be a multiplicative set in a ring R, and let $\phi : R \longrightarrow R_S$ denote the localization homomorphism. Then* $\operatorname{Spec}\phi$ *is injective, and induces a bijection*

$$\operatorname{Spec}\phi : \operatorname{Spec} R_S \longrightarrow \{\mathfrak{p} \in \operatorname{Spec} R : \mathfrak{p} \cap S = \varnothing\}.$$

Proof. Let $\mathfrak{p}$ be a prime ideal in R such that $\mathfrak{p} \cap S = \varnothing$. Then $\mathfrak{q} := S^{-1}\mathfrak{p} = \{\frac{r}{s} : r \in \mathfrak{p}, s \in S\}$ is a prime ideal in $S^{-1}R$ with the property that $\phi^{-1}(\mathfrak{q}) = \mathfrak{p}$. Conversely, if $\mathfrak{q}$ is a prime ideal in $S^{-1}R$, then $\mathfrak{p} = \phi^{-1}(\mathfrak{q})$ is a prime ideal in R with the property that $\mathfrak{p} \cap S = \varnothing$. □

Example 6. For any ring R and any $f \in R$, the set $S = \{f, f^2, f^3, \ldots\}$ is multiplicative. Let $R_f = S^{-1}R$. Then the natural homomorphism $\pi : R \longrightarrow R_f : r \longmapsto \frac{rf}{f}$ induces a homeomorphism $\operatorname{Spec} R_f \longrightarrow D(f)$.

We know that π induces a continuous map $\operatorname{Spec}\pi : \operatorname{Spec} R_f \longrightarrow \operatorname{Spec} R$. The prime ideals of R_f correspond to the prime ideals $\mathfrak{p}$ in R with $\mathfrak{p} \cap S = \varnothing$. Since $\mathfrak{p}$ is prime, this is equivalent to $f \notin \mathfrak{p}$.

4. Presheaves and Sheaves

Let X be a topological space. A presheaf of abelian groups is a contravariant functor from the category of open subsets of X (with inclusions of open sets as morphisms) into the category of abelian groups. Presheaves of rings, vector spaces, R-modules etc. are defined similarly.

In concrete terms, this means the following: a presheaf $\mathcal{F}$ assigns an abelian group $\mathcal{F}(U)$ to every open subset $U \subseteq X$, and a group homomorphism $\rho_{V,U} : \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$ to every inclusion $V \subseteq U$, in such a way that the following two conditions are satisfied:

1. $\rho_{U,U} : \mathcal{F}(U) \longrightarrow \mathcal{F}(U)$ is the identity map;
2. if $W \subseteq V \subseteq U$ are open, then $\rho_{W,V} \circ \rho_{V,U} = \rho_{W,U}$.

We agree to put $\mathcal{F}(\varnothing) = 0$.

We think of $\mathcal{F}(U)$ as a group of functions defined on U, and of the homomorphisms $\rho_{V,U} : \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$ as restriction maps.

Example 1. Let $X = \mathbb{R}$; for any open set $U \subset X$ let $\mathcal{F}(U)$ be the ring of continuous functions $f : U \longrightarrow \mathbb{R}$. This is a presheaf of rings. We also get presheafs if we demand that the functions $f : U \longrightarrow \mathbb{R}$ are differentiable, or bounded, or constant.

A presheaf $\mathcal{F}$ is called a sheaf if it has the following properties: if $U = \bigcup_{i\in I} U_i$ is a union of open sets, then

1. if $\rho_{U_i,U}(a) = 0$ for all $i \in I$ and some $a \in \mathcal{F}(U)$, then $a = 0$.
2. if the elements $a_i \in \mathcal{F}(U_i)$ $(i \in I)$ satisfy $\rho_{U_i\cap U_j,U_j}(a_j) = \rho_{U_i\cap U_j,U_i}(a_i)$ for all $i \in I$, then there is an $a \in \mathcal{F}(U)$ such that $a_i = \rho_{U_i,U}(a)$.

The first property says that if a is locally zero, then it is globally zero (as long as the U_i cover U), the second says that if we have compatible functions defined on an open cover, then then we can patch them together to a global function.

Example 2. Let $X = \mathbb{R}$; for any open set $U \subset X$ let $\mathcal{F}(U)$ be the ring of continuous functions $f : U \longrightarrow \mathbb{R}$. This presheaf is also a sheaf: assume that U is an open subset of $\mathbb{R}$ covered by the open sets U_i; if f vanishes on every U_i, then f vanishes on U, so the first sheaf axiom is satisfied. Now assume that we are given continuous functions $f_i : U_i \longrightarrow \mathbb{R}$ that agree on overlaps. We can define a function $f : U \longrightarrow \mathbb{R}$ by observing that every $u \in U$ must be in some U_i, and then we set $f(u) = f_i(u)$. This is well defined: if $u \in U_i \cap U_j$, then $f_i(u) = f_j(u)$ since the f_i agree on overlaps. Finally, f is continuous: in fact, f is continuous at $u \in U$ because $u \in U_i$ for some i, and the restriction of f to U_i is the continuous function f_i.

Example 3. The same argument shows that the presheaf of differentiable functions is a sheaf: like continuity, differentiability is a local property, that is, you know that a function is differentiable on a union of open sets U_i if it is differentiable in every U_i.

Example 4. The presheaf $\mathcal{F}$ of bounded functions is not a sheaf.

In fact, consider the functions $f_i(x) = x$ on $U_i = (i-1, i+1)$. Then $\mathbb{R} = \bigcup U_i$, and the f_i agree on the overlaps. Moreover, there is a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ whose restriction to the U_i is f_i, namely $f(x) = x$. Finally, the f_i are bounded on U_i because $|f_i(x)| < |i+2|$ for all $f_i \in \mathcal{F}(U_i)$. But of course f is not bounded on $\mathbb{R}$.

The presheaf of bounded functions fails to be a sheaf because boundedness is not a local property.

5. The Structure Sheaf

Let R be an integral domain; we shall now define a presheaf of functions on $\operatorname{Spec} R$. Let K be the quotient field of R. Then an element $f \in K$ is said to be defined at $\mathfrak{p} \in \operatorname{Spec} R$ if there are elements $a, b \in R$ such that $f = \frac{a}{b}$ with $b(\mathfrak{p}) \neq 0$, that is, $b \notin \mathfrak{p}$. Now we set

$$\widetilde{O}(U) = \{f \in K : \ f \text{ is defined at } \mathfrak{p} \text{ for all } \mathfrak{p} \in U\}.$$

It is clear that if f and g are defined at $\mathfrak{p}$, then so are $f+g$ and fg; thus the $\widetilde{O}(U)$ form rings. Moreover, whenever $U \subseteq V$ we have restriction maps $\widetilde{O}(V) \longrightarrow \widetilde{O}(U)$, turning the system of the $\widetilde{O}(U)$ into a presheaf of rings.

Clearly $R \subseteq \widetilde{O}(U) \subseteq K$ for all $U \neq \varnothing$. Moreover, $\widetilde{O}((0)) = K$, and the bigger the open set U, the closer to R we expect $\widetilde{O}(U)$ to be. Let us determine $\widetilde{O}(\operatorname{Spec} R)$. To this end, consider an $f \in \widetilde{O}(\operatorname{Spec} R)$, and let $\mathfrak{a} = \{r \in R : rf \in R\}$ (you can think of this ideal as the set of possible denominators of f, including 0). Assume that $\mathfrak{a} \neq R$: then there is a prime ideal $\mathfrak{p} \in \operatorname{Spec} R$ (and actually we can choose $\mathfrak{p}$ maximal) such that $\mathfrak{a} \subseteq \mathfrak{p}$. Since f is defined at $\mathfrak{p}$, there are $r, s \in R$ with $f = \frac{r}{s}$ and $s(\mathfrak{p}) \neq 0$, that is, with $s \in \mathfrak{a}$ and $s \notin \mathfrak{p}$: contradiction. Therefore $\mathfrak{a} = R$, hence $1 \in \mathfrak{a}$ and thus $f \in R$. We have proved

Proposition 5.1. *If R is an integral domain, then $\widetilde{O}(\operatorname{Spec} R) = R$.*

In fact, we have more:

Proposition 5.2. *If R is an integral domain, then the presheaf $\widetilde{O}$ is a sheaf on $\operatorname{Spec} R$.*

Proof. Consider a nonempty open subset U of $\operatorname{Spec} R$; we know that $R \subseteq \widetilde{O}(U) \subseteq K$, where K is the quotient field of R. If some $f \in K$ lies in $\widetilde{O}(U)$, that is, if f is defined at all $\mathfrak{p} \in U$, then $f \in \widetilde{O}(V)$ for all nonempty $V \subseteq U$. If $f = 0$ as an element of $\widetilde{O}(V)$, then clearly $f = 0$ in K, hence in $\widetilde{O}(U)$. This implies the first property of a sheaf.

Now let $U = \bigcup_{i\in I} U_i$ be an open covering with nonempty sets U_i, and consider elements $f_i \in \widetilde{O}(U_i)$ such that $f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j}$. We can consider the f_i to be elements of K. Since $\operatorname{Spec} R$ is irreducible, any two nonempty open subsets have a nonempty intersection. This implies that $f_i = f_j$ as elements of K for all $i, j \in I$; thus all the f_i coincide and give an element $f \in \widetilde{O}(U)$ with the desired properties. □

Let R be a ring; then we can define $\mathcal{O}(D(f)) = R_f$. It can be shown with a little work that this defines a sheaf on $X = \operatorname{Spec} R$ called the structure sheaf, which, for integral domains R, coincides with the sheaf $\widetilde{O}$ defined above.

The pair $(X, \mathcal{O})$ is called an affine scheme.

6. An Example: $R = \mathbb{Z}/12\mathbb{Z}$

Consider the ring $R = \mathbb{Z}/12\mathbb{Z}$. Its ideals are

- $(0) = \{0\}$;
- $(6) = \{0, 6\}$;
- $(4) = \{0, 4, 8\}$;
- $(3) = \{0, 3, 6, 9\}$;
- $(2) = \{0, 2, 4, 6, 8, 10\}$;
- $(1) = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$.

How do we compute $R/(4)$? Its elements are cosets $[r] = r + 4R$, and clearly $[0] = [4] = [8]$, $[1] = [5] = [9]$ etc; this easily shows that $R/(4) \simeq \mathbb{Z}/4\mathbb{Z}$. We easily find

$\mathfrak{a}$	(0)	(6)	(4)	(3)	(2)	(1)
$R/\mathfrak{a}$	$\mathbb{Z}/12$	$\mathbb{Z}/6$	$\mathbb{Z}/4$	$\mathbb{Z}/3$	$\mathbb{Z}/2$	0

Among these quotients, $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$ are the only integral domains, and also the only fields; this implies that $X = \operatorname{Spec} R = \operatorname{Specm} R = \{(2), (3)\}$.

Let us next compute the closed subsets of X. We find

$\mathfrak{a}$	(0)	(6)	(4)	(3)	(2)	(1)
$V(\mathfrak{a})$	X	X	$\{(2)\}$	$\{(3)\}$	$\{(2)\}$	$\varnothing$

Thus every subset of X is closed (and therefore open); in particular, X has the discrete topology.

The distinguished open sets are $X_2 = \{\mathfrak{p} : 2 \notin \mathfrak{p}\} = \{(3)\}$, $X_3 = \{(2)\}$, $X_1 = X$ and $X_0 = \varnothing$. Thus every open set happens to be distinguished.

Now let us compute the localizations R_f for $f \in R$. We have $R_0 = 0$ and $R_u = R$ for all $u \in R^\times$. Moreover we find $R_2 = R_4 \simeq \mathbb{Z}/3\mathbb{Z}$, $R_3 = \mathbb{Z}/4\mathbb{Z}$ and $R_6 = 0$.

Let us check that $R_3 = \mathbb{Z}/4\mathbb{Z}$. The set of powers of $3 + 12\mathbb{Z}$ is $S = \{3 + 12\mathbb{Z}, 9 + 12\mathbb{Z}\}$; the ring $S^{-1}R$ consists of equivalence classes $\frac{r}{s}$ with $r \in R$ and $s \in S$ such that $\frac{r}{s} = \frac{r'}{s'}$ if and only if $s''(rs' - r's) = 0$. In our case, we can choose $s'' = 3$ and then the fractions are equal if and only if $rs' - r's$ is divisible by 4. It is now straightforward to check that there are only four equivalence classes (represented by $0, 1, 2, 3 \bmod 12$), and that these classes form a ring isomorphic to $\mathbb{Z}/4\mathbb{Z}$.

Now define a presheaf $\mathcal{O}$ sending the open set X_f to R_f. This implies $\mathcal{O}(\varnothing) = 0$, $\mathcal{O}(X_2) = \mathbb{Z}/3\mathbb{Z}$, $\mathcal{O}(X_3) = \mathbb{Z}/4\mathbb{Z}$ and $\mathcal{O}(X) = \mathbb{Z}/12\mathbb{Z}$. Whenever $X_g \subseteq X_f$ we get well behaved restriction maps $R_f \longrightarrow R_g$ as above. Thus $\mathcal{O}$ is a presheaf.

Is $\mathcal{O}$ a sheaf? Consider the open covering $X = X_2 \cup X_3$. Let $a \in \mathcal{O}(X) = \mathbb{Z}/12\mathbb{Z}$ be an element whose restriction to X_2 and X_3 vanishes. Note that $a|_{X_2} = 4a + 3\mathbb{Z}$ and $a|_{X_3} = 3a + 4\mathbb{Z}$. These elements vanish if and only if $3 \mid 4a$ and $4 \mid 3a$, that is, if and only if $12 \mid a$, and this means that $a = 0$ in $\mathbb{Z}/12\mathbb{Z}$. Since all other open coverings of open sets are trivial, this implies that $\mathcal{O}$ is a sheaf. You might already guess that the sheaf property for the general case $R = \mathbb{Z}/m\mathbb{Z}$ is essentially equivalent to the Chinese Remainder Theorem.