LECTURE 19, MONDAY APRIL 26, 2004

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1. Preliminaries

Last time, we proved a few lemmas we will need today; here they are:

Lemma 1.1. *Let M be an ideal in some ring R. Then the sequence*

$$0 \longrightarrow M^n/M^{n+1} \xrightarrow{\ f\ } R/M^{n+1} \xrightarrow{\ g\ } R/M^n \longrightarrow 0$$

is exact. Here f is the inclusion map and g the projection $r + M^{n+1} \longmapsto r + M^n$.

Lemma 1.2. *Assume that U, V, W are finite dimensional K-vector spaces. If the sequence*

$$0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0$$

is exact, then $\dim V = \dim U + \dim W$.

Lemma 1.3. *Consider the ideal $I = (x, y)$ in $R = K[x, y]$. Then $\dim_K K[x, y]/I^n = \frac{n(n+1)}{2}$.*

Corollary 1.4. *We have $K[X, Y]/(I^n, f) \simeq \mathcal{O}_P(\mathbb{A}^2)/(I^n, f)$ for any polynomial $f \in K[X, Y]$.*

2. Multiplicity of Points

Theorem 2.1. *Let P be a point on the irreducible curve $\mathcal{C}_f : f(X, Y) = 0$. Then*

$$m_P(\mathcal{C}_f) = \dim_K \mathfrak{m}_P^n/\mathfrak{m}_P^{n+1}$$

for all sufficiently large n. In particular, the multiplicity of P only depends on the local ring $\mathcal{O}_P(\mathcal{C}_F)$.

Proof. Let $\mathcal{O} = \mathcal{O}_P(\mathcal{C}_f)$ and $\mathfrak{m} = \mathfrak{m}_P(\mathcal{C}_f)$. The exact sequence

$$0 \longrightarrow \mathfrak{m}^n/\mathfrak{m}^{n+1} \xrightarrow{\ f\ } \mathcal{O}/\mathfrak{m}^{n+1} \xrightarrow{\ g\ } \mathcal{O}/\mathfrak{m}^n \longrightarrow 0$$

shows that $\dim \mathfrak{m}^n/\mathfrak{m}^{n+1} = \dim \mathcal{O}/\mathfrak{m}^{n+1} - \dim \mathcal{O}/\mathfrak{m}^n$, hence it is sufficient to prove that

$$\dim \mathcal{O}/\mathfrak{m}^n = n \cdot m_P(\mathcal{C}_f) + s$$

for some constant s and all $n \ge m_P(\mathcal{C}_f)$.

We now assume that $P = (0, 0)$; then $\mathfrak{m} = I\mathcal{O}$, where $I = (X, Y)$ is an ideal in $R = K[X, Y]$, and $\mathfrak{m}^n = I^n\mathcal{O}$. Now

$$R/(I^n, f) \simeq \mathcal{O}_P(\mathbb{A}^2)/(I^n, f) \simeq \mathcal{O}_P/I^n\mathcal{O} = \mathcal{O}/\mathfrak{m}^n.$$

In fact, the first isomorphism is Corollary 1.4, and the second is induced by the map $g + (I^n, f) \longmapsto (g + (f)) + I^n$. Thus we have to compute the dimension of $R/(I^n, f)$.

Let $m = m_P(\mathcal{C}_f)$. Then the Taylor expansion of f around $P = (0, 0)$ does not have any terms of degree $< m$, hence $f \in I^m \setminus I^{m+1}$. Moreover we have $fg \in I^n$ whenever $g \in I^{n-m}$. Thus multiplication by f induces a K-linear map

$$\psi : R/I^{n-m} \longrightarrow R/I^m$$

(not a ring homomorphism, obviously, since $gh + I^{n-m}$ gets mapped to $ghf + I^n$, which is not equal to $(gf + I^n)(hf + I^n)$). We also have the projection

$$\pi : R/I^n \longrightarrow R/(I^n, f),$$

which of course is surjective. We now claim that the sequence

$$0 \longrightarrow R/I^{n-m} \xrightarrow{\ \psi\ } R/I^n \xrightarrow{\ \pi\ } R/(I^n, f) \longrightarrow 0$$

is exact. In fact, let $g+I^n \in \ker \pi$. Then $g \in (I^n, f)$, hence there exists a polynomial $h \in I^n$ such that $g + h \in (f)$; this implies $g + h = af$ for some $a \in R$, and now $\psi(a + I^{n-m}) = af + I^n = (g + h) + I^n = g + I^n$. Conversely, $\operatorname{im} \psi \subseteq \ker \pi$ since $\pi(af+I^n) = af+(I^n, f) = 0+(I^n, f)$. Finally, injectivity of ψ is easily established: if $af \in I^n$, then $f \in I^m \setminus I^{m+1}$ shows that $a \in I^{n-m}$. In order to see this, write $a = a_0+a_1+\ldots+a_i$, where a_k is homogeneous of degree k, and $f = f_m+F_m$, where $f_m \neq 0$ is homogeneous of degree m, and where $F_m \in I^{m+1}$. Now $af = a_0f_m+$ terms of degree $> m$; thus if $n > m$, then $a_0 = 0$. Next $af = a_1f_m+$ terms of degree $> m+1$; thus if $n > m+1$, then $a_1f_m = 0$ and therefore $a_1 = 0$. Continuing this way we find that $a_0 = a_1 = \ldots = a_{n-m-1} = 0$.

Now Lemma 1.3 tells us the dimensions of the first two terms in the exact sequence; this gives

$$\dim_K R/(I^n, f) = nm - \frac{m(m-1)}{2}$$

for all $n \geq m$. □

Corollary 2.2. *If $\mathcal{O}_P$ is a discrete valuation ring, then $m_P(\mathcal{C}_f) = 1$.*

In fact, the theorem shows that $m_P(\mathcal{C}_f) = \dim \mathfrak{m}^n/\mathfrak{m}^{n+1}$. Now $\mathfrak{m} = (t)$ for some $t \in \mathcal{O}_P$, hence the elements $g \in \mathfrak{m}^n$ are multiples of t^n, hence $\mathfrak{m}^n = t^nK \oplus \mathfrak{m}^{n+1}$ as K-vector spaces. This implies $\mathfrak{m}^n/\mathfrak{m}^{n+1} \simeq t^nK$, hence $\dim \mathfrak{m}^n/\mathfrak{m}^{n+1} = 1$.