LECTURE 18, MONDAY APRIL 19, 2004

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1. Examples

Consider e.g. the parabola $\mathcal{C}_f$ defined by $f(X,Y) = Y - X^2 = 0$ at $P = (0,0)$. A uniformizer is given by the image x of the line defined by $X = 0$. Consider the line $L : Y - mX = 0$; what is $\operatorname{ord}_P(y - mx)$? Since $f = Y - X^2$, we have $y - x^2 = 0$, hence $\operatorname{ord}_P(y) = 2\operatorname{ord}_P(x)$. Now $\operatorname{ord}_P(y - mx) = \operatorname{ord}_P(x^2 - mx) = \operatorname{ord}_P(x) + \operatorname{ord}_P(x - m) = 1 + \operatorname{ord}_P(x - m)$, and $x - m$ is a unit in $\mathcal{O}_P$ unless $m = 0$. Thus $\operatorname{ord}_P(y - mx) = 1$ or 2 according as $m \neq 0$ or $m = 0$.

As another example take the unit circle $X^2 + Y^2 = 1$; by changing coordinate system we get the equation $X^2 + (Y - 1)^2 = 1$, i.e. $f(X,Y) = X^2 + Y^2 - 2Y$. Here we have $f(X,Y) = Y(Y - 2) + X^2$, hence $g(X,Y) = Y - 2$ and $h(X) = -1$ in the proof of the theorem above. Thus $\operatorname{ord}_P(y) = \operatorname{ord}_P(x^2h/g) = 2$ since both $y - 2$ and -1 are units in $\mathcal{O}_P$.

Now consider $f(X,Y) = Y^2 - X^3$. Here $g = Y$ and $h = X$, hence g is not a unit in $\mathcal{C}_P$ at $P = (0,0)$. The maximal ideal $\mathfrak{m}_P = (x,y)$ is not principal: assume that $(x,y) = (g)$; then $\operatorname{ord}_P(g) = 1$ and $2\operatorname{ord}_P(y) = 3\operatorname{ord}_P(x)$. But $g \in (x,y)$ implies $g = h_1x + h_2y$, and then $\operatorname{ord}_P(g) = \operatorname{ord}_P(h_1x + h_2y) \geq \min\{\operatorname{ord}_P(h_1x), \operatorname{ord}_P(h_2y)\} \geq 2$, which is a contradiction.

2. Exact Sequences

Lemma 2.1. *Let M be an ideal in some ring R. Then the sequence*

$$0 \longrightarrow M^n/M^{n+1} \xrightarrow{\ f\ } R/M^{n+1} \xrightarrow{\ g\ } R/M^n \longrightarrow 0$$

is exact. Here f is the inclusion map and g the projection $r + M^{n+1} \longmapsto r + M^n$.

Note that a sequence

$$0 \longrightarrow A \xrightarrow{\ f\ } B \xrightarrow{\ g\ } C \longrightarrow 0$$

of groups, rings etc. is exact if the image of the ingoing morphism is equal to the kernel of the outgoing morphism at each point. Thus $0 \longrightarrow A \longrightarrow B$ is exact at A if and only if the image of the map $0 \longrightarrow A$ (which is the neutral element of A) is equal to the kernel of $f : A \longrightarrow B$, i.e., if and only if f is injective. Similarly, $B \longrightarrow C \longrightarrow 0$ is exact if and only if $g : B \longrightarrow C$ is surjective. Finally, the sequence is exact at B if and only if $\operatorname{im} f = \ker g$.

Note that a sequence of abelian groups as above is exact if and only if $C \simeq B/A$.

Proof. First we have to make clear what the involved maps are. Clearly f sends $m + M^{n+1} \in M^n/M^{n+1}$ to $m + M^{n+1} \in R/M^{n+1}$, hence is injective.

Next $g(r + M^{n+1}) = r + M^n$ is well defined: changing r by some element in M^{n+1} does not change the coset $r + M^n$ since $M^{n+1} \subseteq M^n$. Moreover, g is clearly surjective since $r + M^n$ is the image of $r + M^{n+1}$.

Now $\operatorname{im} f \subseteq \ker g$: in fact, the image of f comsists of elements $m + M^{n+1}$ with $m \in M^n$, and they get mapped to $m + M^n = 0 + M^n$ by g. Conversely, assume $r + M^{n+1} \in \ker g$. This means that $g(r + M^{n+1}) = r + M^n = 0 + M^n$, i.e., that $r \in M^n$. But then $r + M^{n+1} \in \operatorname{im} f$. □

Once you are familiar with the abstract language, such proofs become pretty mindless.

Lemma 2.2. *Assume that* U, V, W *are finite dimensional* K*-vector spaces. If the sequence*

$$0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0$$

is exact, then $\dim V = \dim U + \dim W$.

Proof. Let $u_1, \ldots, u_r$ be a basis of U and $w_1, \ldots, w_s$ a basis of W. Since $g : V \longrightarrow W$ is onto, there exist $v_1, \ldots v_s \in V$ such that $g(v_i) = w_i$. We claim that $u_1, \ldots, u_r, v_1, \ldots v_s$ is a basis of V.

First, these vectors are independent: if $\lambda_1 u_1 + \ldots + \lambda_r u_r + \mu_1 v_1 + \ldots \mu_s v_s = 0$, then applying g to this relation and observing that the u_i are in the kernel gives $\mu_1 w_1 + \ldots \mu_s w_s = 0$. Since the w_i form a basis of W, this means that $\mu_1 = \ldots = \mu_s = 0$. But then $\lambda_1 u_1 + \ldots + \lambda_r u_r = 0$, and since the u_i form a basis of U, we conclude that $\lambda_1 = \ldots = \lambda_r = 0$.

Now we have to show that any vector $v \in V$ can be written as a linear combination of the u_i and v_i. Clearly $g(v) = \mu_1 w_1 + \ldots \mu_s w_s$ for some $\mu_i \in K$. Put $u = v - \mu_1 v_1 + \ldots \mu_s v_s$; then $g(u) = 0$, hence $u \in \ker g = \operatorname{im} f = U$, and this shows that $u = \lambda_1 u_1 + \ldots + \lambda_r u_r$. □

3. Miscellanea

In this section we will discuss various properties of the ideal $\mathfrak{m} = (x, y)$ in $\mathcal{O}_P$, as well as its relatives in the polynomial or the coordinate ring.

Lemma 3.1. *Let* $R = K[x, y]$ *be a polynomial ring in two variables over some field* K*. Then, for any ideal* I *in* R*, the quotient ring* R/I *is a* K*-vector space.*

Proof. This is because R is a vector space (it has basis $\{1, x, y, x^2, xy, y^2, \ldots\}$ and because I is a subspace of R: it is closed under addition of vectors and under scalar multiplication by $K \subset R$. □

Lemma 3.2. *Consider the ideal* $I = (x, y)$ *in* $R = K[x, y]$*. Then* $\dim_K K[x, y]/I^n = \frac{n(n+1)}{2}$.

Proof. We claim that $B = \{1, x, y, x^2, xy, y^2, \ldots, x^{n-1}, x^{n-2}y, \ldots, y^{n-1}\}$ is a K-basis for $V = K[x, y]/I^n$. First let us show that these elements generate V. Given any polynomial f, we can reduce it modulo I^n by omitting any term $x^a y^b$ with $a + b \geq n$. The reduced polynomial is then a K-linear combination of elements in B.

Next we have to show that the elements in B are linearly independent. Assume therefore that $\sum_{i+j<n} a_{ij} x^i y^j \in I^n$. This means that $\sum_{i+j<n} a_{ij} x^i y^j = \sum_{i+j \geq n} b_{ij} x^i y^j$ for suitable $b_{ij} \in K$. But then the difference of both sides is the zero polynomial, and since there cannot occur any cancellation, we must have $a_{ij} = 0$. □

Let $\mathcal{O}_P(\mathbb{A}^2)$ be the ring of all rational functions $g \in K(X,Y)$ such that $g = \frac{a}{b}$ with $b(P) \neq 0$. We clearly have $K[X,Y] \subseteq \mathcal{O}_P(\mathbb{A}^2)$. The map $\mathcal{O}_P(\mathbb{A}^2) \longrightarrow \mathcal{O}_P(\mathcal{C}_f); g \longmapsto g + (f)$ is well defined with kernel $(f)\mathcal{O}_P(\mathbb{A}^2)$.

Lemma 3.3. *Let $I = (X,Y)$ be the ideal generated by X and Y in $K[X,Y]$. Then $K[X,Y]/I^n \simeq \mathcal{O}_P(\mathbb{A}^2)/I^n$, where $P = (0,0)$ is the point with the property $g(P) = 0$ for all $g \in I$.*

Proof. Consider the ring homomorphism

$$\phi : K[X,Y]/I^n \longrightarrow \mathcal{O}_P(\mathbb{A}^2)/I^n\mathcal{O}_P(\mathbb{A}^2) : g + I^n \longmapsto g + I^n\mathcal{O}_P(\mathbb{A}^2).$$

We claim that ϕ is an isomorphism.

Now consider a function $b \in K[X,Y]$ with $b(0,0) = 1$. Then $c = 1 - b \in I$, and $(1-c)(1+c+c^2+\ldots+c^{n-1}) = 1 - c^n \in 1 + I^n$. Thus $bh - 1 \in I^n$ for $h = 1 + c + c^2 + \ldots + c^{n-1}$.

First we claim that ϕ is injective. In fact, if $g + I^n \in \ker\phi$, then $g \in I^n\mathcal{O}_P(\mathbb{A}^2)$, i.e. $g = \frac{a}{b} \cdot \sum_{i+j\geq n} a_{ij}x^iy^j$ for some polynomials a, b with $b(0,0) \neq 0$. But then $bg \in I^n$. Multiplying through by h shows that $bhg \in I^n$, and this implies $g \in I^n$. Thus $\ker\phi = 0$, and ϕ is injective.

In order to show that ϕ is surjective, take some $g = \frac{a}{b} \in K(X,Y)$ with $b(0,0) \neq 1$. Replacing a and b by $a/b(0,0)$ and $b/b(0,0)$, respectively, we may assume that $b(0,0) = 1$. But then $\frac{a}{b} + I^n\mathcal{O}_P(\mathbb{A}^2) = \frac{ah}{bh} + I^n\mathcal{O}_P(\mathbb{A}^2) = ah + I^n\mathcal{O}_P(\mathbb{A}^2)$, where h is chosen as above. Thus $g + I^n\mathcal{O}_P(\mathbb{A}^2) = \phi(ah)$. □

Corollary 3.4. *We have $K[X,Y]/(I^n, f) \simeq \mathcal{O}_P(\mathbb{A}^2)/(I^n, f)$ for any polynomial $f \in K[X,Y]$.*

Using these preparations, we will prove (next time)

Theorem 3.5. *Let P be a point on the irreducible curve $\mathcal{C}_f : f(X,Y) = 0$. Then*

$$m_P(\mathcal{C}_f) = \dim_K \mathfrak{m}_P^n/\mathfrak{m}_P^{n+1}$$

for all sufficiently large n. In particular, the multiplicity of P only depends on the local ring $\mathcal{O}_P(\mathcal{C}_F)$.

The proofs given today are not too interesting; what is important are, first of all, the notion of exact sequences, and second, the observation that you need to know a lot of algebra to do algebraic geometry properly.