LECTURE 15, WEDNESDAY APRIL 8, 2004

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1. Polynomial Maps

We now prove a few modest results about polynomial maps.

Proposition 1.1. *Affine curves with polynomial maps form a category.*

This means that there is a category whose objects are affine algebraic curves defined over some field K, and whose morphisms are polynomial maps. The axioms are all obvious, the most difficult one being that the composition of polynomial maps is polynomial again.

Note that in an arbitrary category, a morphism $f : A \longrightarrow B$ is called an isomorphism if there exists a morphism $g : B \longrightarrow A$ such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$. Do not confuse isomorphism with bijective morphisms! In the category of groups, a bijective group homomorphism is automatically an isomorphism since the inverse map is a group homomorphism; but in the category of topological spaces, a bijective continuous map need not be an isomorphism, since the inverse map is not necessarily continuous! Moreover, the property of being 'bijective' is in general not defined, since the objects of a category need not be sets.

On the other side we have the category of coordinate rings $K[\mathcal{C}_f] = K[X,Y]/(f)$. Coordinate rings are not only rings, they are K-algebras: this means that we have a scalar multiplication, i.e. given $a \in K$ and $g + (f) \in K[\mathcal{C}]$, the element $a[g+(f)] = ag+(f)$ is again in $K[\mathcal{C}]$. The morphisms of this category are K-algebra homomorphisms, that is, ring homomorphisms that also respect scalar multiplication.

We now define a map Φ between these two categories. To each curve $\mathcal{C}_f$ we associate the K-algebra $\Phi(\mathcal{C}_f) = K[\mathcal{C}_f]$; given a morphism $F : \mathcal{C}_f \longrightarrow \mathcal{C}_g$ between two objects, we map it to the morphism $\Phi(F) = F^* : K[\mathcal{C}_g] \longrightarrow K[\mathcal{C}_f]$.

Functors. A map $\Phi : C \longrightarrow D$ between categories C and D is called a **covariant functor** if it maps objects X in C to objects $\Phi(X)$ in D and morphisms $f : X \longrightarrow Y$ in C to morphisms $\Phi(f) : \Phi(X) \longrightarrow \Phi(Y)$ in such a way that

- Φ maps identities to identites: $\Phi(\mathrm{id}_A) = \mathrm{id}_{\Phi(X)}$;
- Φ respects composition of morphisms: $\Phi(f \circ g) = \Phi(f) \circ \Phi(g)$.

A contravariant functor differs from a covariant functor in that it reverses arrows: $f : A \longrightarrow B$ gets mapped to $\Phi(f) : \Phi(B) \longrightarrow \Phi(A)$; similarly, $\Phi(f \circ g) = \Phi(g) \circ \Phi(f)$.

Examples Simple examples of functors are the forgetful functors: these map groups (rings, topological spaces, or more generally any set with an additional structure) to sets and the homomorphisms (ring homomorphisms, continuous mappings, etc.) to maps. A less trivial example is the functor that sends every ring to its unit group: this is a functor from the category of rings to the category of groups, and every ring homomorphism $R \longrightarrow S$ induces a group homomorphism $R^\times \longrightarrow S^\times$. More generally, there is the functor GL_n sending each ring to the group $\mathrm{GL}_n(R)$ of invertible $n \times n$-matrices with entries from R. Nontrivial examples can be found in algebraic topology, where you have functors from the category **Top** of topological spaces to abelian groups, attaching homology groups to a space and group homomorphisms to continuous maps between them.

Φ is a Functor. This suggests that our $\Phi : F \longmapsto F^*$ may actually be a contravariant functor from the category of affine curves to the category of K-algebras, and indeed it is: the verification of this claim is a purely formal exercise.

Now assume that $\phi : K[\mathcal{C}_g] \longrightarrow K[\mathcal{C}_f]$ is a K-algebra homomorphism between two coordinate rings of curves. Does there exist a morphism $F : \mathcal{C}_f \longrightarrow \mathcal{C}_g$ such that $\phi = F^*$? As a matter of fact, there is. In order to construct F, put $F_1 + (f) = \phi(X + (g))$ and $F_2 + (f) = \phi(Y + (g))$. Now consider the map $F : \mathcal{C}_f \longrightarrow \mathbb{A}^2K$ defined by $F(x,y) = (F_1(x,y), F_2(x,y))$. This is clearly a polynomial map, and it is well defined since changing F_j by a multiple of f does not change the image. We claim that F maps $\mathcal{C}_f$ into the curve $\mathcal{C}_g$.

For a proof, consider a polynomial $h \in K[X,Y]$; if we plug in the elements $X + (g), Y + (g) \in K[\mathcal{C}_g]$ and evaluate, we get $h(X + (g), Y + (g)) = h(x,y) + (g)$ since $K[\mathcal{C}_g]$ is a ring. In particular, $g(X + (g), Y + (g)) = 0 + (g)$. Next, since ϕ is a K-algebra homomorphism, we have $0 + (f) = \phi[g(X + (g), Y + (g))] = g[\phi(X + (g)), \phi(Y + (g))]$, and this implies that $g(F_1 + (f), F_2 + (f)) = 0 + (f)$. Plugging in values $(x,y) \in \mathcal{C}_f$ then finally shows that $g(F_1(x,y), F_2(x,y)) = 0$, that is, $(F_1(x,y), F_2(x,y)) \in \mathcal{C}_g$.

Finally we have to check that $F^* = \phi$. For some $h + (g) \in K[\mathcal{C}_g]$ we have, by definition, $F^*(h(X,Y) + (g)) = h(F_1, F_2) + (f) = h(\phi(X + (g)), \phi(Y + (g))) + (f) = \phi(h(X,Y) + (g))$, which is exactly what we wanted to prove.

We have shown:

Theorem 1.2. *The contravariant functor $\Phi : F \longrightarrow F^*$ induces an equivalence of categories between the category of affine curves with polynomial maps on the one hand, and the category of coordinate rings with K-algebra homomorphisms.*

A functor Φ is said to induce an equivalence of cateogries (you should think of such categories as being 'isomorphic') if there is a functir Ψ in the other direction such that the composition of these functors is the identity functor. Think this through until it begins to make sense.

This result has an important corollary:

Corollary 1.3. *A polynomial map $F : \mathcal{C}_f \longrightarrow \mathcal{C}_g$ is an isomorphism if and only if $F^* : K[\mathcal{C}_g] \longrightarrow K[\mathcal{C}_f]$ is an isomorphism.*

Proof. Assume that F is an isomorphism; then there is a polynomial map $G : \mathcal{C}_g \longrightarrow \mathcal{C}_f$ such that $F \circ G$ and $G \circ F$ are identity maps. Applying the functor Φ shows that $G^* \circ F^*$ and $F^* \circ G^*$ are identity maps on the coordinate rings. The converse follows the same way. □

Now we can show that the polynomial map F from the line $\mathcal{C}_f : f(X,Y) = Y = 0$ to the cubic $\mathcal{C}_g : g(X,Y) = Y^2 - X^3 = 0$ given by $(x,0) \longmapsto (x^2, x^3)$ does not have an inverse although it is a bijection: the induced ring homomorphism $F^* : K[\mathcal{C}_g] \longrightarrow K[\mathcal{C}_f]$ is not an isomorphism since the image of $K[\mathcal{C}_g]$ in $K[\mathcal{C}_f] = K[X]$ is $K[X^2, X^3]$.

Maybe even more important is the following observation: since an affine transformation $X = aX' + bY' + e$, $Y = cX' + dY' + f$ with $ad - bc \neq 0$ is a polynomial map $\mathbb{A}^2 K \longrightarrow \mathbb{A}^2 K$, and since its inverse is also polynomial, affine transformations induce *isomorphisms* of the associated coordinate rings. This means that any invariant of an algebraic curve that we can define in terms of its coordinate ring is automatically invariant under affine transformations!