LECTURE 12, MONDAY MARCH 29, 2004

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1. Linear Systems

Linear systems are useful for showing the existence of curves of a given degree with certain properties.

Proposition 1.1. *There is a conic going through any* 5 *points in the projective plane.*

Proof. The space of all conics is described by points $[a_0 : a_1 : a_2 : a_3 : a_4 : a_5] \in \mathbb{P}^5K$. The conic will go through $P_1 = [x_1 : y_1 : z_1]$ if and only if

$$a_0x_1^2 + a_1x_1y_1 + a_2y_1^2 + a_3x_1z_1 + a_4y_1z_1 + a_5z_1^2 = 0,$$

which is a linear condition on the coefficients; thus the set of conics going through P_1 form a 4-dimensional subspace of $\mathbb{P}^5K$. If the conic goes through P_2, another linear condition is added, and unless the condition is linearly dependent, the dimension goes down by 1. Thus the space of all conics going through 5 points is the intersection of 5 projective hyperplanes, hence consists of a point (if the conditions are linearly independent) or is infinite. $\square$

Here is another example:

Proposition 1.2. *The condition for curves of degree m to have a point of multiplicity $\ge s$ at some point P is equivalent to $\frac{s(s+1)}{2}$ linearly independent conditions in $\mathbb{P}^dK$.*

Proof. A curve $\mathcal{C} : F(X_0, X_1, X_2) = 0$ of degree m has a point of multiplicity $\ge s$ if and only if the partial derivatives of F of order s all vanish at P, that is, if all the terms of order $s-1$ in the Taylor expansion vanish. There are exactly $\frac{s(s+1)}{2}$ such terms.

Now we claim that the conditions are linearly independent. In order to prove this we choose a coordinate system in which $P = [0 : 0 : 1]$. Then with $F = \sum a_{ijk}X^iY^jZ^k$ we find that

$$\frac{\partial^{s-1}F}{\partial X_0^i \partial X_1^j \partial X_2^{s-1-i-j}}[0 : 0 : 1] = 0$$

is equivalent to $a_{ijk} = 0$. The conditions $a_{ijk} = 0$ for all i, j, k with $i + j + k = m$ and $i + j \le s - 1$ are linearly independent. $\square$

In the proof we have used

Lemma 1.3. *The point $P \in \mathcal{C}(K)$, where $\mathcal{C} : F(X, Y, Z) = 0$ is a curve of degree m, has multiplicity $m_P(F) \ge s$ if and only if all partial derivatives of order s vanish at P.*

Proof. Euler's formula applied to the partial derivatives of F immediately shows that if all the partials of order s vanish, then the same is true for all partials of order $\le s$. This is equivalent to the Taylor expansion of F at P having no term less than degree s, hence $m_P(F) \ge s$. □

2. The Genus

The next result can be proved using resultants, but the proof is quite technical. I will eventually put a proof here, but will not cover it in class.

Proposition 2.1. *Let $F = 0$ and $G = 0$ be two plane algebraic curves, and let $m_P(F)$ and $m_P(G)$ denote the multiplicity of the point P on F and G. Then*

$$I_P(F, G) \ge m_P(F) \cdot m_P(G).$$

This is clear if $m_P(F) = 0$ or $m_P(G) = 0$, and also if $m_P(F) = m_P(G) = 1$.

Proposition 2.2. *Let $\mathcal{C}$ be a curve of degree d without multiple components. Then*

$$\sum_P m_P(C) \cdot (m_P(C) - 1) \le d(d-1).$$

If $\mathcal{C}$ is irreducible, then we even have

$$\sum_P m_P(C) \cdot (m_P(C) - 1) \le (d-1)(d-2).$$

As a corollary, we observe

Corollary 2.3. *An irreducible curve of degree d over an algebraically closed field of characteristic 0 has at most $\frac{(d-1)(d-2)}{2}$ singular points.*

Make sure you understand why these results have the following corollaries:

- A conic without multiple components has at most one double point.
- An irreducible conic is smooth.
- A cubic without multiple components has at most three double points.
- An irreducible cubic has at most one double point.
- A quartic without multiple components has at most 6 double points.
- An irreducible quartic has at most 3 double points.

You should also be able to construct curves for which these bounds are attained.

Proof of Prop. 2.2. Let $\mathcal{C} : F(X,Y,Z) = 0$, and choose coordinates in such a way that $[0 : 0 : 1]$ is not on the curve and $Z = 0$ is not a component. This implies that F is not divisible by a homogeneous polynomial $G(X,Y)$, because otherwise $G(X,Y)$ would also divide $F(X,Y,1)$, but $G(0,0) = 0$ and $F(0,0,1) \ne 0$. Thus when we interpret F as a polynomial in Z with coefficients in $K[X,Y]$, then each nonconstant factor $G(X,Y,Z)$ of $F(X,Y,Z)$ is also a nonconstant factor $G(Z)$ of $F(Z)$ viewed in $R[Z]$ with $R = K[X,Y]$. If $F(Z)$ and $F'(Z)$ have a common (nonconstant) irreducible factor G, then F has G as a double component: in fact, $F = GH$ implies $F' = G'H + GH'$, hence $G \mid G'H$ and therefore $G \mid H$.

Thus the curves $\mathcal{C} : F = 0$ and $\mathcal{C}' : F' = 0$ do not have a common component; their degrees are d and $d-1$, respectively. By Bezout we know that $\sum_P I_P(F, F') = d(d-1)$. Now if P is a singular point on $\mathcal{C}$, then P is also a point on $\mathcal{C}'$, and we have $m_P(F) \geq m_P(F') - 1$ (multiplicity m means that the $m-1$th, but not all mth derivatives vanish). Thus we get

$$\sum_P m_P(\mathcal{C})(m_P(\mathcal{C}) - 1) \leq \sum_P m_P(\mathcal{C}) \cdot m_P(\mathcal{C}') \leq d(d-1).$$

Now assume that $\mathcal{C}$ is irreducible, and let $P_1, \ldots, P_k$ denote the singular points on $\mathcal{C}$. Put $r_i = m_{P_i}(\mathcal{C})$. Let $\mathcal{L}$ be the linear system of curves of degree $n-1$ with multiplicity $\geq r_i - 1$ at P_i. Then

$$\dim \mathcal{L} \geq \frac{(d+1)(d+2)}{2} - \sum_{i=1}^{k} \frac{r_i(r_i-1)}{2}.$$

We know from the above that $\dim \mathcal{L} > 0$ for $d > 1$. If we choose $\dim \mathcal{L}$ points on $\mathcal{C}$ different from the P_i, then there will be a curve $\mathcal{C}' \in \mathcal{L}$ which intersects $\mathcal{C}$ at these points. By Bezout we see

$$d(d-1) \geq \sum m_P(C) m_P(C') \geq \dim \mathcal{L} + \sum_{i=1}^{k} r_i(r_i - 1).$$

This implies the claim. □

Curves with the maximal number of double points are rational:

Proposition 2.4. *Let $\mathcal{C}$ be a plane irreducible algebraic curve of degree d with $\frac{(d-1)(d-2)}{2}$ double points. Then $\mathcal{C}$ can be parametrized.*

A curve has a parametrization if there is a nonconstant rational map from the affine line to the curve. It can be shown (and we might do so later) that if an irreducible curve can be parametrized at all, then it can be parametrized in such a way that the image is the whole curve except for at most finitely many points (such parametrizations are called proper).

Definition of the Genus. Let $\mathcal{C}$ be an irreducible plane algebraic curve with at most double points as singularities. Let d denote the degree of $\mathcal{C}$ and r the number of double points. Then

$$g = \frac{(d-1)(d-2)}{2} - r$$

is called the genus of $\mathcal{C}$.

Our results above imply that $g \geq 0$; note that $g = 0$ for conics and singular cubics (irreducible, of course) and $g = 1$ for smooth cubics.

The main property of the genus is

Theorem 2.5. *The genus is invariant under birational transformations.*

This is a deep and surprising fact; note that the degree and the number of singular points do change under birational transformations.

For example, the parametrization of the unit circle is a birational transformation changing the degree 2 of the circle into the degree 1 of the line; but both curves have genus 0. Similarly, parametrizations of singular cubics map a curve with degree 3 and one double point into a smooth line of degree 1.

As a corollary we see that curves of genus ≥ 1 cannot be properly parametrized, since there is no birational transformation from a curve of genus ≥ 1 to a line. Hilbert and Hurwitz have shown that curves of genus 1 over algebraically closed fields always can be parametrized.