

LECTURE 11, THURSDAY MARCH 18, 2004

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1. LINEAR SYSTEMS

We will now briefly discuss linear systems before we come back to the topic of Bezout's theorem. The simplest linear system is the family of all lines in the projective plane. These lines can be written in the form $aX + bY + cZ = 0$, and there is a bijection between such lines and points $[a : b : c] \in \mathbb{P}^2K$.

Similarly, conics in the projective plane can be written as

$$aX^2 + bXY + cY^2 + dXZ + eYZ + fZ^2 = 0,$$

and there is a bijection between conics and points in \mathbb{P}^5K .

In general, curves of order m are described by a homogeneous polynomial of degree m with exactly $\binom{m+2}{2}$ coefficients: in fact, write $F = \sum a_{ijk}X^iY^jZ^k$; then $i+j+k = m$, so we only have to choose i and j ; for $i = 0$, there are $m+1$ choices for $j \leq m$, for $i = 1$ there are m choices, and for $i = r$ there are exactly $m-r$ choices for j . Thus the number of all possible polynomials is $(m+1) + m + (m-1) + \dots + 1 + 0 = \frac{(m+1)(m+2)}{2}$. Thus the set of such polynomials forms a vector space of dimension $\binom{m+2}{2}$. Regarding two polynomials as equivalent if they differ by a nonzero factor we see that the classes of such polynomials correspond bijectively to points in \mathbb{P}^dK for $d = \binom{m+2}{2} - 1 = \frac{m(m-3)}{2}$.

A subspace of \mathbb{P}^dK is called a linear space if it is the zero set of linear equations; the space \mathbb{P}^mK is linear, as is the hyperplane $a_0X_0 + \dots + a_dX_d$, or intersections of such planes. A linear system of curves of degree m now is a set of curves corresponding to a linear subspace of \mathbb{P}^dK . Linear systems of dimension 1 are traditionally called pencils.

If the linear subspace of a projective space is given by a system of linear equations

$$a_{i0}X_0 + a_{i1}X_1 + \dots + a_{in}X_n$$

for $i = 1, \dots, m$, and if the rank of this system is $n - k$, then the subspace is said to have dimension k . In fact, in this case one can introduce new coordinates in which the subspace is given by the equations $Y_{k+1} = \dots = Y_n = 0$, and there is an obvious bijection between this subspace and \mathbb{P}^kK .

An example is the set of all lines through $P = [-1 : 0 : 1]$ that we used to parametrize the unit circle: an arbitrary line $L : aX + bY + cZ = 0$ corresponds to a point $L = [a : b : c] \in \mathbb{P}^2K$, and this line will go through P if and only if $-a + c = 0$, that is, if and only if L is on the line $Z - X$ (a linear subspace of dimension 1) in \mathbb{P}^2K .

2. PARAMETRIZING THE LEMNISCATE

Now consider the lemniscate $(x^2 + y^2)^2 = x^2 - y^2$. Its projective closure has equation

$$F(X, Y, Z) = (X^2 + Y^2)^2 - (X^2 - Y^2)Z^2 = 0$$

, and we find

$$F_X = 2X[2(X^2 + Y^2) - Z^2],$$

$$F_Y = 2Y[2(X^2 + Y^2) + Z^2],$$

$$F_Z = -2(X^2 - Y^2)Z,$$

If $Z = 0$, then the first two equations imply $X = Y = 0$, which is impossible, or $X^2 + Y^2 = 0$, and in fact the points $[1 : i : 0]$ and $[1 : -i : 0]$ are singular (complex) points at infinity. If $Z = 1$, then $X^2 = Y^2$, and this leads to the third singular point $O = [0 : 0 : 1]$.

A circle has equation $X^2 + Y^2 + dXZ + eYZ + fZ^2 = 0$, hence is parametrized by points in \mathbb{P}^2K . Note that all circles go through the two singular points $[1 : \pm i : 0]$ at infinity of the lemniscate.

The circles going through O form a linear subspace described by $f = 0$, i.e., satisfy $X^2 + Y^2 + dXZ + eYZ = 0$. Now look at the circles whose tangent in O is $Y = X$. The equation of the tangent is $dX + eY = 0$, so the circles through O with tangent $Y = X$ have equation $X^2 + Y^2 + dX - dY = 0$.

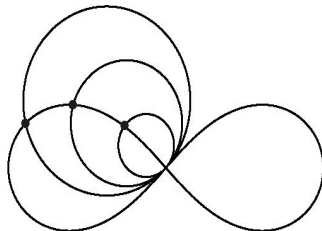


FIGURE 1. Lemniscate and three circles ($d = 0.25, 0.5, 0.75$)

Note that these circles intersect the lemniscate with multiplicity 2 in $[1 : \pm i : 0]$ and with multiplicity ≥ 3 in O , and since there are exactly $8 = 2 \cdot 4$ points of intersection by Bezout, the circles will intersect the lemniscate in exactly one other point, whose coordinates will be rational if d is rational. Thus we can parametrize the lemniscate using the pencil $X^2 + Y^2 + dX - dY = 0$ of circles. Here are the calculations (we use affine coordinates): substitute $x^2 + y^2 = d(y - x)$ in the equation of the lemniscate; this gives

$$0 = d^2(x - y)^2 - (x^2 - y^2) = (x - y)[d^2(x - y) - (x + y)] = 0.$$

The first factor leads to the known point O ; setting the second factor equal to 0 yields $x(d^2 - 1) = y(d^2 + 1)$. Solving for y and plugging this into the equation of the circle gives a quadratic equation in x without constant term; the nonzero solution

is

$$x = \frac{d(d^2 + 1)}{d^4 + 1}, \quad y = \frac{d(d^2 - 1)}{d^4 + 1}.$$

We will later see a somewhat better motivated solution of this problem.