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FRANZ LEMMERMEYER

1. The Projective Line

Suppose you want to describe the lines through the origin $O = (0,0)$ in the Euclidean plane $\mathbb{R}^2$. The first thing you might think of is to write down the equation $y = mx$, but then you are told that this does not cover the line $x = 0$. The second idea is to consider all the equations $ax + by = 0$ with $(a,b) \neq (0,0)$, and these do indeed describe all lines through O; on the other hand, one and the same line like $y = x$ is described by infinitely many different equations, namely $ax - ay = 0$ for any $a \neq 0$. More generally, two lines $ax + by = 0$ and $a'x + b'y = 0$ will represent the same lines if $(a', b') = (\lambda a, \lambda b)$ for some nonzero λ.

In order to get the best of both worlds, we define an equivalence relation on the set of all points $(a,b) \neq (0,0)$ by saying that $(a,b) \sim (a',b')$ if $(a',b') = (\lambda a, \lambda b)$ for some nonzero λ. The equivalence class of (a,b) is denoted by $[a:b]$, and the set of all equivalence classes is called the real projective line $\mathbb{P}^1\mathbb{R}$.

The same construction works for general fields: we put $\mathbb{P}^1K = (K \times K \setminus \{(0,0)\})/\sim$, where the equivalence relation $\sim$ is defined exactly as above. The space $\mathbb{P}^1K$ (occasionally also denoted by $K\mathbb{P}^1$) is called the projective line over K.

Note that $\mathbb{P}^1K$ is called the projective *line* even though each point is represented by two coordinates: this is because the projective line is, up to some mysterious "point at infinity", the same as the affine line $\mathbb{A}^1K = K$. In fact, $a \longmapsto [a:1]$ defines a map $\iota : \mathbb{A}^1K \longrightarrow \mathbb{P}^1K$ which is clearly injective: $\iota(a) = \iota(b)$ means that $[a:1] = [b:1]$, hence there exists a nonzero $\lambda \in K$ with $a' = \lambda a$, $1 = \lambda 1$, which implies $\lambda = 1$ and then $a = a'$, $b = b'$. Moreover, ι is almost surjective: the only points on the projective line not in the image are those of the form $[a:0]$ with $a \neq 0$ (recall that there is no such thing as $[0:0]$). But if $a \neq 0$, then $[a:0] = [1:0]$, which means $\mathbb{P}^1K = \mathbb{A}^1K \cup \{[1:0]\}$. Thus we can identify the points $\neq [1:0]$ on the projective line with the usual affine points, and we call the additional point $[1:0]$ the point at infinity on $\mathbb{P}^1K$.

In our example with lines through the origin, we now can identify the line $ax + by = 0$ with the point $[a:b]$, since this is a bijection: equations describing the same line correspond to the same point on the projective line. The line $ax + by = 0$ has slope $m = -a/b$ if $b \neq 0$; if you let $b \longrightarrow 0$, the slope will tend to infinity, and in the limit you get the line $x = 0$ and the point $[1:0] \in \mathbb{P}^1K$.

2. Lifting Maps from the Affine to the Projective Line

In calculus you studied real-valued functions on the real line $\mathbb{R} = \mathbb{A}^1R$. Now we can study rational functions $\mathbb{P}^1K \longrightarrow \mathbb{P}^1K$. In fact, consider the rational function $f(x) = \frac{2x-1}{x-2}$. Clearly $f(3) = 5$; since f has a pole at $x = 2$, we are tempted to put $f(2) = \infty$. Can we make this precise?

Yes we can. The function $f : x \longmapsto \frac{2x-1}{x-2}$ is defined on the real line except for $x = 2$; we can map $f(x)$ to the projective line with ι and then get $\iota(f(x)) = [\frac{2x-1}{x-2} : 1]$. For $x \neq 2$, however, we have $[\frac{2x-1}{x-2} : 1] = [2x-1 : x-2]$, so we might just as well put $\iota(f(x)) = [2x-1 : x-2]$. But this does even make sense for $x = 2$, so $f(2)$ gets mapped to $[3 : 0] = [1 : 0] \in \mathbb{P}^1K$, that is, to the point at infinity.

This makes f into a map $\mathbb{A}^1K \longrightarrow \mathbb{P}^1K$. Can we also extend the domain of f to the projective line? In other words, can we make some sense of $f(\infty)$? If you remember your calculus, you probably will guess that we should have $f(\infty) = \lim_{x\to\infty} f(x) = 2$. The only problem is that, in fields $K \neq \mathbb{R}$, we don't have the notion of a limit. What we can do is the following: first observe that we can identify x with $[x : 1]$, hence we can put $f([x : 1]) = f(x)$. In order to get a good definition for $f([x : 0])$ we substitute $x = \frac{s}{t}$ and observe $[x : 1] = [\frac{s}{t} : 1] = [s : t]$; this gets mapped to $[2x-1 : x-2] = [2\frac{s}{t} - 1 : \frac{s}{t} - 2] = [2s-t : s-2t]$. Thus we finally arrive at the map

$$\mathbb{P}^1K \longrightarrow \mathbb{P}^1K : [s : t] \longmapsto [2s-t : s-2t].$$

In fact we have $f([1 : 0]) = [2 : 1]$, that is, $f(\infty) = 2$.

Is it possible to "extend" the domain and codomain of functions in general? The answer is yes for rational functions. In order to be able to state this concisely, let us introduce the homogenization of a polynomial $f \in K[x]$: if

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$$

and $a_n \neq 0$, then the polynomial

$$F(X,Y) = a_n X^n + a_{n-1}X^{n-1}Y \ldots + a_1 XY^{n-1} + a_0 Y^n$$

is called the homogenization of f. In other words: we put

$$F(X,Y) = Y^{\deg f} f\big(XY^{-1}\big).$$

Lemma 2.1. *If G is the homogenization of g, then*

- $G(\lambda X, \lambda Y) = \lambda^{\deg g} G(X,Y)$;
- $G(x,1) = g(x)$.

In our example above, the extension of the rational map $f = \frac{g}{h}$ to the projective line was given by $F[s : t] = [G(s,t) : H(s,t)]$, where $G(X,Y) = 2X - Y$ and $H(X,Y) = X - 2Y$ were the homogenizations of g and h. In general, however, we cannot simply define F like this, as the example $f(x) = \frac{x-1}{x^2}$ shows: the map $[s : t] \longmapsto [s-t : t^2]$ is not well defined! The solution is to define the projective extension here as $[s : t] \longmapsto [st - t^2 : t^2]$.

Proposition 2.2. *Assume that $f : \mathbb{A}^1K \longrightarrow \mathbb{A}^1K$ is a rational map (possibly undefined at finitely many points), that is, $f(x) = \frac{g(x)}{h(x)}$ for coprime polynomials $g, h \in K[t]$. Then there is a polynomial map $F : \mathbb{P}^1K \longrightarrow \mathbb{P}^1K$ that extends f in the sense that $f(x) \in K$ can be identified with $F([x : 1]) \in \mathbb{P}^1K$; in fact, we put $a = \deg g - \deg h$ and have*

$$F[s : t] = \begin{cases} [G(s,t) : t^a H(s,t)] & \text{if } a \geq 0, \\ [t^{-a}G(s,t) : H(s,t)] & \text{if } a < 0, \end{cases}$$

where G and H are the homogenizations of G and H.

Proof. The first thing that we should check is whether the map is well defined. What happens if we replace (s,t) by $(\lambda s, \lambda t)$? The left hand side of course does not change, but on the right hand side we get, in the case $a = \deg g - \deg h \geq 0$,

$$\begin{aligned}[G(\lambda s, \lambda t) : \lambda^a t^a H(\lambda s, \lambda t)] &= [\lambda^{\deg g} G(s,t) : \lambda^{a+\deg h} t^a H(s,t)] \\ &= [G(s,t) : t^a H(s,t)]\end{aligned}$$

as desired. The case $a < 0$ is handled similarly.

We also have to check that $F([s:t]) \in \mathbb{P}^1 K$, i.e., that (in the case $a \geq 0$) $G(s,t)$ and $t^a H(s,t)$ cannot simultaneously vanish. To this end, assume that there is a point $[s:t] \in \mathbb{P}^1 K$ with $G(s,t) = t^a H(s,t) = 0$. If $t \neq 0$, then we may assume that $t = 1$, and we find $G(s,1) = g(s)$ and $H(s,1) = h(s)$. If these values are both 0, then $f - \frac{g}{h}$ was not in lowest terms since $g(x)$ and $h(x)$ share the common factor $x - s$. If $t = 0$, then $G(s,0) = 0$, which implies that $s = 0$, and this is a contradiction since $[0:0] \notin \mathbb{P}^1 K$.

Next we have to check that $F([x:1]) = [f(x):1]$ for all $x \in K$ at which f is defined. In fact, we get (again in the case $a \geq 0$)

$$F([x:1]) = [G(x,1) : H(x,1)] = [g(x) : h(x)] = [f(x) : 1].$$

Thus F coincides with f on the affine part of the projective line. □

I expect that the extended map is unique, but didn't work out the details.

3. Projective Planes

We now can define the projective plane in a similar way: on the set $K^3 \setminus \{(0,0,0)\}$ of all nonzero 3-tuples with entries from K introduce an equivalence relation via $(a,b,c) \sim (a',b',c')$ if there is a $\lambda \in K^\times$ such that $a' = \lambda a$, $b' = \lambda b$, $c' = \lambda c$. The equivalence class of (x,y,z) is denoted by $[x:y:z]$, and the set of all equivalence classes is called the projective plane $\mathbb{P}^2 K$.

Just as the affine line can be embedded in to the projective line, the affine plane $\mathbb{A}^2 K = K \times K$ can be viewed as being a part of the projective plane: the map

$$\iota : \mathbb{A}^2 K \longrightarrow \mathbb{P}^2 K; (a,b) \longmapsto [a:b:1]$$

is injective. In fact, if $[a:b:1] = [a':b':1]$, then by definition of equality in $\mathbb{P}^2 K$ there is a $\lambda \in K^\times$ such that $a' = \lambda a$, $b' = \lambda b$, and $1 = \lambda \cdot 1$. The last equation gives $\lambda = 1$, hence $(a,b) = (a',b')$.

As for the line, the map ι is not surjective: there are points in the projective plane that cannot be seen in the affine picture. These 'points at infinity' are the points $[a:b:0]$ for $a,b \in K$ not both 0. These consist of the set $\{[a:1:0]; a \in K\}$ and the point $[1:0:0]$, hence we can write

$$\mathbb{P}^2 K = \iota(\mathbb{A}^2 K) \cup \{[a:1:0]; a \in K\} \cup \{[1:0:0]\}.$$