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EARLY LITERARY EVIDENCE OF THE USE OF THE ZERO IN INDIA[1]

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In a previous article in this Monthly,[2] the writer collected from the earlier literature of the Hindus certain passages proving the existence of the zero as a distinct symbol and its use as a numeral.[3] Those evidence were mostly from the *Pañcasiddhāntikā* (505 A.D.) of Varāhamihira and works anterior to it. In this second article we propose to collect similar evidence from the posterior Hindu works.

To the earlier period also belongs the Bakhshālī Manuscript.[4] It contains epigraphical as well as literary evidence of the use of the modern decimal place value numerals. For example,

". . . become $\left|\begin{smallmatrix}120\\60\end{smallmatrix}\middle|\begin{smallmatrix}90\\60\end{smallmatrix}\middle|\begin{smallmatrix}80\\60\end{smallmatrix}\middle|\begin{smallmatrix}75\\60\end{smallmatrix}\middle|\begin{smallmatrix}72\\60\end{smallmatrix}\right|$. On leaving out (the denominators) of them, result $120\,|\,90\,|\,80\,|\,75\,|\,72\,|$. The sum of these being taken, becomes 437 . . ."[5]

" $\left|\begin{smallmatrix}880\\84\end{smallmatrix}\middle|\begin{smallmatrix}964\\168\end{smallmatrix}\right|$ multiplied become $\left|\begin{smallmatrix}848320\\14112\end{smallmatrix}\right|$. The square of forty different places is $|1600|$. On subtracting this from the number above, the remainder is $\left|\begin{smallmatrix}846720\\14112\end{smallmatrix}\right|$. On removal of the common factor, becomes $|60|$."[6]

There are innumerable passages of this kind in the work. It will be noticed that in each of these instances, without the figures, the sentences would be grammatically incomplete and the whole composition incoherent. So the figures must have been put there at the time of the original composition of the text, and can not be suspected of being later interpolations For an explicit reference to the zero and an operation with it, we take the following instance:

". . . $\left|\begin{smallmatrix}\cdot\\1\end{smallmatrix}\middle|\begin{smallmatrix}2\\1\end{smallmatrix}\middle|\begin{smallmatrix}3\\1\end{smallmatrix}\middle|\begin{smallmatrix}4\\1\end{smallmatrix}\middle|\begin{smallmatrix}\text{visible } 200\\1\end{smallmatrix}\right|$. *Adding unity to the zero* $1\,|\,2\,|\,3$ . . . ."[7]

This will dispel the doubts of even those critics who, while admitting the figures to be authentic, would still doubt whether they were originally given in this form. The Bakhshālī treatise on mathematics was written in the early centuries of the Christian era.[8] Hence the decimal place value numerals with the zero must have been well known to the Hindu scholars of the time.

One of the earliest known writers of the second period to bear witness to the use of zero, is a contemporary of Varāhamihira, Jinabhadra Gaṇi by name. He lived

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[1] This is the writer's second article with this title. The first was published in this Monthly, vol. 33 (1926), pp. 449–454.

[2] Loc. cit.

[3] The writer is gratified to see that his paper inspired the distinguished German savant, Professor Julius Ruska, to collect similar evidence from the Arabic literature (*Zahl und Null bei Jabir ibn Haiyan, Archiv f. Ges. Math. Natur. Tech.*, Bd. ii, pp. 256–264). Jabir wrote about 770 A.D.

[4] *The Bakshālī Manuscript—Parts I & II*, ed. G. R. Kaye, Calcutta, 1927.

[5] *Ibid.*, folio 1, verso. Portions in italics in this passage have been restored.

[6] *Ibid.*, folio 56, verso.

[7] *Ibid.*, folio 22, verso.

[8] Bibhutibhusan Datta, *The Bakhshālī Mathematics*, *Bull. Cal. Math. Soc.*, vol. 21 (1929), pp. 1–60; R. Hoernle, *The Bakhshālī Manuscript*, *Ind. Ant.*, vol. 17, pp. 33ff, 275ff. [But c.f. the Kaye ed. Editor.]

in 529–589 A.D. While mentioning large numbers with several zeros, Jinabhadra Gaṇi often enumerates, obviously for abridgement, the number of zeros contained. For instance 224400000000 is mentioned as "twenty-two, forty-four, eight zeros;"[1] 3200400000000 as "thirty-two, two zeros, four, eight zeros."[2] There are several instances of this kind in his work.[3] In certain calculations relating to the mensuration of the northern Bhāratavarsa, which is of the form of a segment of a circle bounded by two parallel chords, it is necessary to extract the square root of 58545048750. Now

$$\sqrt{58545048750} = 241960\frac{407150}{483920} = 241960\frac{40715}{48392}.$$

This result has been described by Jinabhadra Gaṇi thus: "Two hundred thousand forty-one thousand nine hundred and sixty, *removing the zero*, the numerator is four-zero-seven-one-five, and the denominator four-eight-three-nine-two."[4] It should be noted that the term *apavartana* ("removal") means in Hindu mathematics what is called in modern mathematics, the reduction of a fraction to lowest terms by removing the common factors from the numerator and the denominator. Hence the zero of Jinabhadra Gaṇi is certainly not a mere concept of nothingness but is a specific symbol.

Another contemporary mathematician, the elder Bhāskara (not the one living in the 12th century), refers like Varāhamihira, to the subtraction of zero.[5] This writer should not be confused with his celebrated namesake, the author of the *Siddhānta-śiromani* (1150). The former, on his own admission, is a direct disciple of Âryabhaṭa (born 476). We also learn from Pṛthudakasvāmī[6] (860) that he was anterior to Brahmagupta (*b.* 598). So the elder Bhāskara must belong to the sixth century of the Christian era.[7]

Here are two typical extracts from the commentary of Siddhasena Gaṇi on the gloss of Umāsvāti on his *Tattvārthādhigama-sūtra*[8]

"The number of *yojaṇa* in the circumference multiplied by twenty-five thousand becomes this 7905675000. Three *gavyuti* (in the circumference) multiplied by twenty-five thousand becomes this 75000. The number of *gavyuti* are

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[1] *Brhat Ksetra-samāsa* of Jinabhadra Gani, edited with the commentary of Malayagiri, Bombay, i. 69.

[2] *Ibid.*, i. 71.

[3] *Ibid.*, i. 90, 97, 102, 108, 113, 119, etc.

[4] *Ibid.*, i. 83.

[5] *Mahābhāskarīya*, i. 34. This work is still unpublished. Mss. of it and of the other work of the same author *Laghubhāskarīya* are found in the Government Oriental Mss. Library, Madras. The present writer has transcripts of these two works.

[6] *Brāhma-sphuṭa-siddhānta*, xi. 26 (*com*).

[7] See the writer's article, "The Two Bhâskaras," in the *Indian Historical Quarterly*, vol. 6 (1930), pp. 727–736.

[8] *Tattvārthādhigama-sūtra* of Umāsvāti, with his own gloss, elucidated by Siddhasena Gani, edited by H. R. Kapadia, Bombay, 1926, iii. 11 (*com*).

to be changed to *yojaṇa*, hence on dividing by four, the quotient is 18750. These are the *yojaṇas*. The number of *dhanu* being also multiplied by twenty-five thousand, become this 3200000, . . ."

"Now the square of the chord is this 75600000000; the square of the diameter multiplied by three hundred and sixty-one is 3610000000000. On subtracting the square of the chord from that, the remainder is this 3534400000000. The square root of this is extracted; *half of the eight zeros are four zeros*; the root of the remaining portion is one-eight-eight; hence the resulting root is this 1880000."

It will be clear from these extracts that Siddhasena Gaṇi employed in his calculation the decimal place-value numerals. In the second extract there is an enumeration of the number of zeros. Elsewhere the writer says, "On removing the four zeros, the quotient obtained after that is this 100000." According to Professor Hermann Jacobi, Siddhasena lived at the middle or end of the sixth century.[1]

The evidence of another distinguished writer, who flourished about the close of the century, is equally conclusive. It is still more noteworthy inasmuch as it comes not from a mathematician but from a romance writer, Subandhu by name. In the *Vāsavadattā* of Subhandhu,[2] we meet with the following metaphor:

"And at the time of the rising of the moon with its blackness of night, bowing low, as it were, with folded hands under the guise of closing blue lotuses, immediately the stars shone forth, . . . like ciphers [*śūnya-bindu*] because of the nullity of metempsychosis, scattered in the sky as if on the ink-blue skin rug of the Creator who reckoneth the sum total with a bit of the moon for chalk."

In the beginning of the seventh century, it is found that a section in the chapter on algebra of the astronomical treatise, *Brāhma-sphuṭa-siddhānta* (628) of Brahmagupta is devoted to the treatment of all the fundamental operations with zero, including involution and evolution. Similar sections are indeed found in the mathematical treatises of almost all the posterior Hindu writers, such as *Triśatikā* of Śrīdhara (*C.* 750), *Gaṇita-sāra-saṁgraha* of Mahāvīra (850), *Mahāsiddhānta* of Âryabhaṭa II (950), etc.[3] These arithmetical operations with the zero certainly presuppose the existence of the zero as a numeral. Brahmagupta's term *taccheda* for the quotient of a quantity divided by zero is especially noteworthy in this connexion. That term literally means "having

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[1] *ZDMG*, Vol. 60, 1906, p. 289.

[2] *Vāsavadattā* of Subandhu, edited by F. Hall (Calcutta, 1859, p. 182) and translated into English by Louis H. Gray (New York, 1913, pp. 99f). It may be noted that the romantic love of Queen Vāsavadattā and King Udayana had been the theme of several Sanskrit works. The earliest of them seems to be a drama, called *Vāsavadattā*, by Subhandhu, the minister of King Bindusāra (280 B.C.) of Pātalīputra. The famous Patañjali (150 B.C.) has referred to an epic poem of the same name. The available prose romance *Vāsavadattā* was written after the middle of the sixth century. (See *Proc. 2nd Orient. Conf., Calcutta, 1922*, pp. 197–7, 203–213.

[3] Bibhutibhusan Datta, "Early history of the arithmetic of zero and infinity in India," *Bull Cal. Math. Soc.*, vol. 18 (1927), pp. 165–176.

that for denominator"; having in this instance cipher (*kha*) in the denominator, it is equivalent to and has been obviously meant for *kha-cheda*. Now to describe a certain quantity as a fraction having zero for the denominator, requires a knowledge of the characteristics of the zero. Like Jinabhadra Gaṇi, Brahmagupta is sometimes found to have enumerated the number of zeros contained in a particular large number. For example; 4320000 is spoken of as "four zeros, *rada*, *veda*"[1] (*rada* = 32, *veda* = 4); 4320000000 as "seven zeros *rada*, *veda*;"[2] 57753300000 as "five zeros, *guṇa*, *guṇa*, five, *muni*, *svara*, *śara*"[3] (*guṇa* = 3, *muni* = 7, *svara* = 7, *śara* = 5); etc.[4]

In the arithmetical treatise of Mahāvīra, the use of zero occurs also in other connexions besides in connexion with its arithmetic. For instance in explaining a method of finding the sum of a series in geometrical progression, he says:

"The number of terms in the series is caused to be marked (in a separate column) by *zero* and by *one* (respectively) corresponding to the even (value) which is halved and to the uneven (value from which *one* is subtracted till by continuing these processes *zero* is ultimately reached);"[5]

This passage reappears verbatim on two other occasions in the *Gaṇita-sāra-saṁgraha*.[6] It should be observed that the last of these rules is the same as that of Piṅgala (before 200 B.C.).[7]

In enumerating the various "technical terms" (*saṁjñā*) signifying the "numbers" (*saṁkhyā*), for the purpose of a glossary on word-numerals, Mahāvīra first mentions the terms for 1, 2, · · · , 9 and then gives the terms denoting the zero.[8] Hence it is clear that, according to him, the zero is as much a number as any of 1, 2, · · · , 9 is. So Professor Tropfke is not correct in thinking that the zero was not regarded as a number before the seventeenth century A.D.[9]

In describing certain large numbers, Mahāvīra writes:

"The (figures) 7, 0, 2, 2, 5, and 1 are *put down* (in order from the unit's place upwards); and then this (number) which is to be multiplied by 73 . . ."[10]

"In this (problem) *put down* (from the unit's place upwards) 1, 1, 0, 1, 1, 0, 1, and 1, which (figures so placed) give the measure of a (particular) number; . . ."[11]

". . . write down the figures 4, 0, 6, 0, 5, and 9 in order (from right to left) and work out the cube of the number . . ."[12]

Such large numbers have been described as " numbers occupying two or more notational places."[13]

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[1] *Brāhma-sphuṭa-siddhānta*, i. 7.
[2] *Ibid.*, i. 15.
[3] *Ibid.*, i. 16.
[4] *Ibid.*, i. 22, 51–5, etc.
[5] *Ganita-sāra-samgraha*, ii. 94.
[6] vi. $311\frac{1}{2}$, $333\frac{1}{2}$.
[7] See the first article.
[8] *Ganita-sāra-samgraha*, i. 53–62.
[9] J. Tropfke, *Geschichte d. Elementar-Mathematik*, Bd. II (1921), p. 56. For the Hindu definition of the zero, the reader is referred to the article, *Early History of the arithmetic of zero etc.*, *loc. cit.*
[10] *Ibid.*, ii. 15.
[11] *Ibid.*, ii. 17.
[12] *Ibid.*, ii. 52.
[13] *Ganita-sāra-samgraha*, ii. 30.

In the writings of a contemporary scholar, we find innumerable references to the zero and the decimal place-value numerals. He is no other than the eminent mathematician and commentator Pṛthudakasvāmī. Pṛthudakasvāmī wrote commentaries of the two works of Brahmagupta, viz., *Brāhma-sphuṭa-siddhānta* and *Khaṇḍakhādyaka*. Copies of the commentary on the latter, written in 864 A.D., are available in the Imperial Library, Berlin, and the Deccan College Library, Poona. One copy, collected by Bhau Daji, is now in the library of the Calcutta University, but the earlier parts of it are worn out. We select the following extracts from this commentary:[1]

"That being divided by *dvi-nava-rasa*[2] (namely) by this 692, the quotient in days etc. is 0 | 34 | 56. That being added to the remainder of the intercalary months, (namely) to this 433 | 29 | 13, becomes this 434 | 4 | 9. That being multiplied by thirty becomes this 13022 | 4 | 30. That on division by *rtu-kha-dik*,[3] (that is) by this 1006, the quotient in days etc, called 'the corrected second,' is this 12 | 56 | 39."[4]

"The *ahargaṇa* is unity 1; multiplied by *kha-kha-vasu*[5] becomes | 800 |. Dividing this by *muni-nakha-dvi-nanda-yama*,[6] (we get) as revolution 0. The remainder multiplied by twelve becomes 9600. (Dividing) by that divisor, (we get) as signs 0. The remainder, this 9600, multiplied by thirty becomes 288000. On division of this by the same divisor, degrees are 0. The remainder 288000 being multiplied by sixty 60 and with the same divisor, minutes are 59. The remainder being multiplied by sixty and divided by the same divisor, seconds are 8. Thus the daily motion of the sun in signs etc., is 0 | 0 | 59 | 8."[7]

"Thus the daily motion of the sun is fifty-nine minutes and eight seconds; that of the moon is seven hundred and ninety minutes and thirty-four seconds; that of the moon's perigee is six minutes and forty seconds; and that of moon's apogee is three minutes and eleven seconds. These are again written in figures (*aṅkenāpi likkhyate*):

Sun	Moon	Moon's Perigee	Moon's Apogee
59	790	6	3
8	34	40	11

These minutes etc., are the daily motions of the sun, the moon, the moon's perigee and moon's apogee, respectively."[8]

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[1] A photostat copy of the Berlin Mss. of *Khaṇḍakhādyaka* commentary has recently been secured by the Calcutta University. It is being edited by Professor P. C. Sengupta. Our references here are to the folios of that copy. The *Khaṇḍakhāyaka* with the commentary of Âmarāja has been edited previously by Pandit Babua Misra for the Calcutta University. The numbering of the verses of the original text in this copy is slightly different from the Berlin copy.

[2] *Dvi* = 2, *nava* = 9, *rasa* = 6; *dvi-nava-rasa* = 692.

[3] *Rtu* = 6, *kha* = 0, *dik* = 10; *rtu-kha-dik* = 1006.

[4] Folios 9 verso-10 recto; i. 11–12 (C.U. ed. i. 8–9).

[5] *Kha* = 0, *vasu* = 8.

[6] *Muni* = 7, *nakha* = 20, *dvi* = 2, *nanda* = 9, *yama* = 2.

[7] Folio 12 recto and verso; i. 14 (C.U. ed. ii. 1).

[8] Folio 13, recto & verso; i. 14 (C.U. ed. ii. 1).

"These are again written in figures:

Sun	Moon	Mars	Mer.	Jup.	Ven.	Sat.	Moon's Peri.	Moon's Apo.
59	790	31	245	4	96	2	6	3
8	34	26	32	59	7	0	40	11

These being multiplied by the difference in longitudes, (namely) by this 120, become respectively (beginning with the moon) 94870 | 800 | 380 | 3772 | 29464 | 598 | 11534(?6) | 240."[1]

There are numerous other instances of this kind in the work.[2]

Finally coming to the tenth century of the Christian era, we find a considerable volume of evidence of the use of zero and the decimal place-value numerals. The commentators such as Praśastidhara[3] (964) and Bhaṭṭotpala[4] (966) have freely used them in calculation. For illustration we take the passage, "adding this to the constant quantity *dvi-utkrti-kha*, (namely) to this 2260 . . ."[5] from the former and "by the number *kha-kha-aṣṭa-dik*, (namely) by this 10800 . . ."[6] from the latter. Nemicandra has several times referred to the addition and subtraction of the zero; e.g.,

"One hundred plus seventeen, eleven, *zero*, and four. . . ."[7]

"Deduct *zero*, one, four and five (in each of the four rows, respectively). . . ."[8]

The zero also enters into certain combinations of things described by him:

"One, two, three and five; *zero*, five, ten and fifteen; *zero*, twenty, forty and sixty; putting down (*saṃsthāpya*) these in three rows for three classes of carelessness, find out the elements or number of analysis and synthesis, i.e., combinations."[9]

"One, two, three and four; *zero*, four, eight and twelve; *zero*, sixteen, thirty-two, forty-eight and sixty-four; putting down these in three rows for three classes of carelessness, find out the number of analysis and synthesis."[10]

We find the use of the decimal place value numerals with the zero in the works of Mādhavacandra Traividyadeva, a contemporary disciple of Nemicandra. It may be noted that the latter has defined a very large number, "one-nine-seven-

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[1] Folio 15, verso.

[2] Compare folios 16(verso), 18(recto), etc.

[3] Praśastidhara of Kashmir wrote in 964 A.D. a commentary on the *Laghumānasa* (932) of Mañjula. Ms. of this work is found in the Calcutta University Library, which has been "copied from Ms. No. 583 and compared with other Mss. in the Oriental Library, Mysore."

[4] *Brhat Samhitā* of Varāhamihira, edited, with the commentary of Bhattotpala, by Sudhakara Dvivedi, in two parts, Benares, 1895, chap. ii; *Brhajjātaka* of Varāhamihira, edited, with the commentary of Bhattotpala, by Sitarama Jha, Benares, 1923, vii. 1–2, viii. 4, ix. 1–7, etc.

[5] *Laghumānasa*, ii. 5 (com.); *dvi* = 2, *utkrti* = 26, *kha* = 0.

[6] *Brhajjātaka*, vii. 2 (com.); *kha* = 0, *aṣṭa* = 8, *dik* = 10.

[7] *Gommaṭa-sāra* of Nemicandra, edited with English translation and notes in two parts, by J. L. Jaini, Lucknow, 1927; *Karmakānda*, Gāthā 276, 282.

[8] *Ibid.*, *Karma-kāṇḍa*, Gāthā 383; cf. also Gāthā 94, 322, 376.

[9] *Ibid.*, *Jīva-kāṇḍa*. Gāthā 43.

[10] *Ibid.*, *Jīva-kāṇḍa*, Gāthā 44.

nine-twelve-zero-nine-two-nine-nine-nine-six-eight- and thirty-one zeros"[1] (1979120929996800000000000000000000000000000) as "4294967296 multiplied by the square of sixteen and then multiplied by twice nine, together with thirty-one zeros." Mādhavacandra has expressed the number in figures.

Another very conclusive proof is supplied by the "Kaṭapayādi" variety of the Hindu alphabetic notations. In this system, as is well known, groups of nine letters of the Devanāgarī alphabets beginning with *ka* and *ṭa*, five letters beginning with *pa* and eight letters from *ya*, denote successively the numerals beginning with unity. The letters *ña* and *na*, sometimes also the vowels standing alone, denote the zero.[2] Thus we have on the whole symbols for nine numeral characters and the zero. These are employed, of course, on the principle of place value. Instances of the application of this system of numeral notation are found as early as the beginning of the sixth century of the Christian era in the Laghu-Bhāskarīya[3] (522) of Bhāskara I. According to the Commentator Sūryadeva Yajvā, the system was in use still earlier, before the time of Âryabhaṭa I (*c.* 476).

To recapitulate, in these two articles, we have collected from the earlier literature of the Hindus a volume of substantial evidence proving the existence, as also the use as a numeral, of the zero. They range over a long period beginning before 200 B.C. and ending with the tenth century A.D. It is needless to produce evidence of later writers. On closer analysis, it will be found that the volume of evidence of scholars coming after the fifth century A.D. is more than that of the earlier scholars. Again the evidence of a few, such as Varāhamihira (505), can possibly be explained away as referring to the mere concept of nothingness, but those of many others like Piṅgala (before 200 B.C.), the Bakhshalī Ms., Bhāskara I (*c.* 525), Jinabhadra Gaṇi and others, cannot be so done. With the latter writers the zero is undoubtedly a distinct numeral. There is nothing to prevent us from supposing that the former also used it in the same sense. On the contrary, one will easily be led to this conclusion by the fact that the contemporary writers employed the zero as a numeral.

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[1] *Triloka-sāra* of Nemicandra, edited, with the commentary of Mādhavacandra, by Manoharalal Sastri, Bombay, 1919, Gāthā 20–1.

[2] For further particulars on this point, the reader is referred to the writer's article, *The alphebetic numeral system* (in Bengali) in the *Sāhitya Parisat Patrikā*, Calcutta, vol. 36, pp. 22–50. The article gives a comprehensive history of the subject with special notice of the Hindu systems. Compare also J. F. Fleet, *The Katapayadi system of expressing numbers*, *JRAS*, 1911, pp. 788ff.

[3] i. 18.