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Zero: the Symbol, the Concept, the Number

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In the history of mathematics considerable confusion exists as to the origin of zero. To the extent that this is the inevitable consequence of lacunæ in the historical data, only the disclosure of further evidence can alter the situation. Nevertheless, there are aspects of the difficulty which can be clarified by the simple expedient of stating the problem unequivocally. There can be no unique answer to the query, "Who first discovered the zero?", for this may refer to any one of several related but distinct historical issues. One of these would seek the earliest use of a *symbol* or mark (cipher) to indicate an empty place in a so-called positional notation for integers and fractions, such as is evident in the number 102.03. One special aspect of this first form of the question would concern specifically the oldest use of zero in a *decimal* positional system; another might refer in particular to the origin of that *type* of symbol which now is employed in all Western civilizations; still another aspect would concern the origin of the *word* zero. A second more general form of the problem would call for the first reference, in connection with the idea of number, to the *concept* of a null class or an absence of magnitude. An answer to this inquiry would depend upon nice distinctions between the philosophical void and the mathematical zero, or between concrete objects and abstract ideas. A third comprehensive form of the query would seek the earliest recognition of zero as itself a *number* subject in general to the ordinary arithmetic operations. To satisfy this question one must know whether an elementary or a more sophisticated concept of number is called for. With the problem stated in these several forms one may well hesitate nonetheless to hazard anything beyond a tentative answer.

The facility with which we now carry out computations is ascribable largely to two principles*—cipherization and local value—both of which were known perhaps 4,000 years ago. In the Egyptian hieratic ciphered numeration, as in all non-positional systems, there was no need for a symbol corresponding to zero. In the positional notation of the Babylonians a symbol for empty places was desirable to avoid

*See my paper, "Fundamental steps in the development of numeration," scheduled to appear shortly in *Isis*.

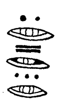
difficulties. However, in this sexagesimal numeration empty positions occur far less frequently than in the decimal system (not at all for integers less than sixty; and in only 59 cases for integers less than 3600, as compared to 917 in the ten-scale), and contextual ambiguities seldom arose. For this reason the early Babylonians who first developed the positional principle felt no urgent need for a symbol to cover this situation. At some time of about the Persian period, however, a conventional sign,  , was adopted to mark empty places in their notation.* In accordance with this notation, the number 162,032 would be written as

—that is, as $45(60)^2+0(60)+32$. This represents probably the earliest appearance of a *symbol* for zero, corresponding to our own use of the "goose-egg" in ordinary calculation.

Symbols for zero or for empty places appear also in inscriptions of the ancient Mayan civilization in which a calendrical notation based on local value had been adopted. These lenticular symbols were variously embellished—as

 or  or  or  , etc.

—but the variants were easily recognized as in all cases indicating empty positions. In the Mayan bar-and-dot scheme the symbolism



represented $2\cdot 20^4\cdot 18+0\cdot 20^3\cdot 18+10\cdot 20^2\cdot 18+0\cdot 20\cdot 18+3\cdot 20+0$ or 5,832,060 days.† However, there is no indication that this system, introduced by the Mayas as early as the beginning of of the Christian era, was used in general computation.

Babylonian influence in Hellenistic astronomy was so strong that although ordinary Greek numeration for integers was decimal, a sexagesimal scheme of subdivision into fractional parts was adopted in astronomy. In terms of the Ionian alphabetic notation, such a number as 321.3892 would take the form $\tau\kappa\alpha\ \kappa\delta'\ \kappa\alpha''$—that is,

$$321+\frac{23}{60}+\frac{21}{60^2}\ .$$

*For Babylonian notation see Otto Neugebauer, *Vorlesungen über Geschichte der antiken mathematischen Wissenschaften*, Vol. I, *Vorgriechische Mathematik* (Berlin, 1934); also F. Thureau-Dangin, "Sketch of a history of the sexagesimal system," *Osiris*, VII, 85-141.

†See S. G. Morley, *An introduction to the study of the Maya hieroglyphs* (Washington, 1915), pp. 129-133; cf. p. 8ff. See also D. E. Smith, *History of mathematics* (2 Vols. Boston and New York, 1923-1925), II, 43-45; Karl Menninger, *Zahlwart und Ziffer* (Breslau, 1934), pp. 39f, 314f; Florian Cajori, "The zero and principle of local value used by the Maya of Central America," *Science* (N. S.), XLIV, (1916) 714-717.

When in such a sexagesimal fractional form an empty position occurred, Greek astronomers indicated this fact by using the letter omikron (not to be confused with the use of this letter for 70 in the decimal alphabetic notation for integers). This symbol may have been suggested by the first letter of the word οὐδὲν (empty or void). The number

$$321 + \frac{21}{60^2}$$

would thus be written as τκα ο′ κα″. Here, just as in Babylonian and Mayan numeration, one sees clearly the use of zero which corresponds in all respects to our own except that the scale of notation was not decimal.

From the Babylonian, Mayan, and Greek representations the answer to the first general query above appears reasonably clear: a symbol for zero appears to have been used first by the Babylonians, somewhat later by the Greeks and the Mayas. It is to be noted, however, that in these systems the symbol  or  or *o* represented an empty position and nothing more. There is at the present time no evidence that any one of these forms appeared alone, independent of other digits in positional notation. Hence, one may not, without further assurance, interpret these symbols as representing the *concept* or the *number* zero.

Early philosophical references to the "void" may be regarded as implying a similar mathematical conception, and so are related in a broad sense to the notion of zero; but specific statements in this respect were lacking. From the infinitesimal concepts of the early Greek mathematicians one can similarly infer the tacit recognition of an idea corresponding to zero; but here too there was no explicit formulation. To Plato has been ascribed the conception of zero (as well as of negative quantity),* but this attribution is based on somewhat devious arguments derived from the philosopher's recondite terminology. Possibly the earliest clear and explicit reference to the mathematical *concept* of zero is found in the *Physics* of Aristotle. Here the Stagirite had propounded the doctrine that the speed of a body is inversely proportional to the resistance of the medium in which it moves. In considering motion in a vacuum he concluded that a thing would move through the void with a speed beyond any ratio (and hence that the existence of a void is impossible), for "there is no ratio in which the void is exceeded by body, as there is no ratio of zero to a number .... For this reason, too, a line does not exceed a point—unless it is com-

*See John Burnet, *Greek philosophy. Thales to Plato* (London, 1932), pp. 320-322, 330; also R. E. Taylor, *Plato. The man and his work* (new ed., New York, 1936), pp. 505-506.

posed of points."* In this quotation it is evident that Aristotle had the arithmetical zero in mind, for it is regarded as bearing to number the same relationship as does a point to a line. Moreoever, the impossibility of division by zero is here definitely stated almost fifteen hundred years before the time of Bhaskara.†

Unfortunately, the Greek interpretation of the word number was very limited. From the time of Thales "number" had been generally accepted as a collection or system of units—"a plurality of 'ones' and a certain quantity of them."‡ This definition included only the natural numbers. In fact, Aristotle went so far as to say that "the smallest number in the strict sense of the word 'number' is two."§ Although the Greeks constructed a sound and extensive theory of commensurable and incommensurable ratios, these were excluded from the realm of number. Needless to say, so also was zero. The *symbol* and the *concept* were familiar to them, but they never achieved the full status of *number*.

The Arabic notation is now recognized as misnamed. The Arabs were not its originators, but simply adopted the system and transmitted it to Europe during the Middle Ages. The source conventionally has been placed in India during the early centuries of our era, so that now the notation customarily is referred to as the Hindu-Arabic. However, the evidence in this connection is far from clear and the possibility of some other origin—perhaps in the Greek world—must be admitted. In any case, the so-called Hindu-Arabic system of numeration involves no principle not familiar to the world several thousand years ago. The ten-scale, local value, symbol for zero, and cipherization were widely used in antiquity. Notwithstanding ubiquitous categorical assertions to the contrary, the Hindus definitely were not the first inventors of any one of these fundamental aspects of numeration, although they may have been independent rediscoverers of one or more of them.

Unqualified claims⁑ for the Indian origin of zero must necessarily be rejected. Nevertheless, it may well be that the Hindus first adapted

Physica IV. 8.215^b. The translation given is that of W. D. Ross and J. A. Smith (*The works of Aristotle,* 11 Vols., Oxford, 1908-1931, Vol. II) except that I have substituted the word zero for the symbol 0, inasmuch as there is no evidence that Aristotle here used a symbol.

†See my paper, "An early reference to division by zero," *American Mathematical Monthly,* L (1943), 487-491.

‡Aristotle, *Physica III.* 7.207^b.

§*Physica IV.* 12.220^a.

⁑Bibhutibuhusan Datta has most erroneously said of zero that "the world is gradually adopting the view that the credit of the invention is entirely due to the Hindus." "Early literary evidence of the use of the zero in India," *American Mathematical Monthly,* XXXIII (1926), 449-454). In another place he has incorrectly added, this to a *decimal* ciphered positional type of numeration. The Babylonians and Greeks had used zero only in the sexagesimal system, and the Mayas only in their quasi-vigesimal notation. Its adaptation to the ten-scale constituted a significant advance on the part of someone, probably during the early Christian centuries. The evidence in this respect still appears to point to the Hindus; but the situation is far from clear. The earliest undoubted occurrence of zero in India dates from 876, but its use may go back many centuries before this. The origin of this use is as uncertain as is the source of our numerals.* Considerable doubt remains also with respect to another point, the origin of our characteristic *form* of symbol for zero. It may have been a Hindu invention (as Hindu writers polemically maintain) or it may have been suggested by the Greek use of omicron (*o*) for zero;† or it could have arisen in some other manner. The earliest form of the Hindu symbol for zero would appear to have been a dot, the present characteristic form being adopted later.‡ With respect to the source of our word "zero" the evidence is far more clear. The Hindus called it sunya (void), and this term passed into the Arabic as *sifr*. Fibonacci spoke of this as *zephiram*, which in the Italian of the following century took on various forms, including *cifra* and *zeron*, from which our words cipher and zero arose.§

At first the symbol for zero played the same role in Hindu numeration as it had earlier among the Babylonians, Mayas, and Greeks: it was a mark indicating an empty place in the positional notation. This is evidenced by the fact that, even in seventeenth-century Europe, serial representations of the ten Hindu-Arabic numerals generally placed the symbol for zero after the nine, in the *tenth* place, rather than before the one, as the *first* number. However, it was to be expected that, with the development of arithmetic, rules should be established for operating on the symbol zero as part of a number; and such rules ultimately led in turn to the recognition of zero as itself a number.

"The arithmetic of zero is entirely the Hindu contribution to the development of the mathematical science. With no other early nations do we find any treatment of zero." "Early history of the arithmetic of zero and infinity in India," *Calcutta Mathematical Society Bulletin*, XVIII (1927), 165-176. As late as 1935 the claim of zero for the Hindus *alone* was reiterated. B. Datta and A. N. Singh, *History of Hindu Mathematics.* A source book. Part I. Numeral notation and arithmetic (Lahore, 1935). See p. 27.

*See Smith, *op.cit.*, II, 64-77; cf., however, Moritz Cantor, *Vorlesungen über Geschichte der Mathematik*, Vol. I (2nd ed., Leipzig, 1894), p. 563.

†Jules Sageret, "La genèse du zéro," *La Revue des Idées*, VII (1910), 320-340. See p. 339.

‡Florian Cajori, *A history of mathematics* (2nd ed., New York, 1931), pp. 88-89.

§Smith, *op. cit.*, II, 71-72. See also Gustav Oppert, "Ueber die Entstehung der Aera Dionysiana und den Ursprung der Null," *Berliner Gesellschaft für Anthropologie, Ethnologie und Urgeschichte*, Verhandlungen, 1900, pp. 102-136.

Aristotle had been familiar with the arithmetical aspect of zero, but the restrictive Greek definition of number excluded all but the positive integers. Precise definitions were not given by the Hindus, so that positive and negative, rational and irrational, quantities were included indiscriminately in the realm of number. Under such a situation there was nothing to stand in the way of accepting zero as a full-fledged number. The transition from the *symbol* zero to the *number* zero is not always clear, and the situation has been further obscured by inadvertencies on the part of some expositors. One of the foremost historians of Hindu mathematics has interpreted the reduction of

$$\frac{407150}{483920} \text{ to } \frac{40715}{48392}$$

as an operation on the *number* zero.* Although the *symbol* zero is indeed incidentally involved, the operation here concerns the integer ten, not the *number* zero. However, it appears that by 505 the Hindus may have looked upon zero as a number, for Varahamihira stated that the value of a quantity is unchanged if zero is added to or subtracted from it.† The evidence for the number zero becomes still more definite in 628 when Brahmagupta correctly stated that $0\times(\pm a)=0$, $0\times 0=0$, and $\sqrt{0}=0$. Brahmagupta expressed doubt as to $a\div 0$, the impossibility of which had been noted by Aristotle almost a thousand years before; and $0\div 0$ he mistakenly thought was necessarily zero.‡ Bhaskara in 1114 characterized division by zero as infinity, although he gave no formal definition of this. The lack among the Hindus of that clear-cut reasoning which characterized Greek work is evident in Bhaskara's statements§ that $(a\times 0)\div 0=a$ and

$$\frac{a}{0}+b=\frac{a}{0}\,.$$

Attempts have been made to justify such work in terms of zero as an infinitesimal and of infinity as a limiting value of an increasing sequence;⁂ but they are unwarranted and misdirected. The fact that the Hindus calculated—whether correctly or incorrectly—with zero as

*Datta, "Early literary evidence of the use of the zero in India," *American Mathematical Month.y*, XXXVIII (1931), 566-572. See p. 567. Cf. pp. 568-569; also p. 570.

†Datta (1926), pp. 451-452.

‡Datta (1927), pp. 169-170. See also H. T. Colebrooke, *Algebra, with arithmetic and mensuration, from the Sanscrit of Brahmegupta and Bhascara* (London, 1817), pp. 339-340; and S. R. Das, "The origin and development of numerals," *Indian Historical Quarterly*, III (1927), 97-120, 356-375, especially p. 119.

§Datta (1927), pp. 170-175. Cf. also Cantor, *op. cit.*, p. 576.

⁂See Datta and Singh (1935), pp. 238-243.

with other numbers, indicates clearly however, that they looked upon zero as itself a number.

The Hindus did not define the domain of number, but it is fairly evident that it included what are now known as the real numbers, positive, negative, and zero. In this respect the Arabic and Latin medieval civilization did not follow them, although Al-Khowarizmi, Fibonacci (Leonardo of Pisa), Sacrobosco, Villedieu, Jordanus, and others popularized the Hindu numerals. Only positive roots of equations were recognized, and of these Al-Khowarizmi accepted only the rational. Roots of numbers which were not rational were referred to as surd, absurd, irregular, irrational, or inexplicable; negative numbers were sometimes called fictitious. Zero was not regarded as a root of an equation, and there is no evidence that it was thought of as a number by itself. In serial representation of the first ten digits it again habitually follows the 9 rather than precedes the 1. However, the early modern development of algebra seems to have led to the recognition once more of the *number* zero. This change is usually placed in the sixteenth century,* or even in the seventeenth.† However, one sees the rise of this attitude as early as 1484 in the *Triparty* of Chuquet. In mentioning zero he wrote that it "ne vault ou signifie rien ... et pour ce est appellée chiffre ou nulle ou figure de nulle valeur."‡ This language does not necessarily imply that zero is to be regarded as a number, but this understanding is evident from Chuquet's use of it. He wrote .0. *m* . 12 for $0-12$ and .0. *p* . 12. for $0+12$. Moreoever, he appears to have been the first person to consider zero (as well as positive and negative integers) as an exponent. He wrote .9.° for $9x^0$, or 9; and in a table of powers (integral and zero) of two he has the number one corresponding to the exponent zero. Although Chuquet seems in general not to have admitted zero as a root of an equation, nevertheless in one such case he said the number sought was zero.§

Chuquet's work was not published until almost four centuries later, but it may have influenced his contemporaries through the circulation of his manuscripts. At any rate, the use of zero as a number was continued in the algebra of the sixteenth century. Stifel in 1544 wrote the polynomial x^3+1 in a form corresponding to x^3+0x^2+0x+1;

*See, for example, G. Eneström, "Über die Anfänge der Benutzung von Null als eine werkliche Grösse," *Bibliotheca Mathematica* (3), VII (1906-1907), 309.

†J. Tropfke, *Geschichte der Elementar-Mathematik,* Vol. II, (2nd ed., Leipzig, 1921), pp. 7-9, 56.

‡Ch. Lambo, "Une algèbre française de 1484. Nicolas Chuquet," *Revue des Questions Scientifiques* (3), II (1902), 442-472. See p. 446. Or see Aristide Marre, "Notice sur Nicolas Chuquet et son Triparty en la science des nombres," *Bullettino di. Bibliografia e di Storia delle Scienze Mathematiche e Fische,* XIII (1880), 555-659, 693-814.

§Lambo, *op. cit.*, pp. 366-467.

Tartaglia in 1556 wrote numeral forms such as $\sqrt{45}+0$ and $\sqrt{45}-0$; Cardan in 1570 wrote the equivalents of $x^3=0+x$ and $x^3=216+0x$.* These men appear not to have recognized zero as a root of an equation, but this step was taken in 1629 by Girard.† From this time on—two thousand years after the symbol and concept had found a place in numeration—there was general acceptance of zero as a full-fledged number. From the point of view of elementary mathematics the development may therefore be regarded as completed; but from a more advanced standpoint the history of zero might better be carried somewhat further. The critical reexamination of mathematical concepts carried out in the nineteenth century led Frege in 1884 to what may be regarded as the first satisfactory definition of cardinal number—the class of all equivalent classes. One is easily tempted to regard the class of all null classes as the final step in the rise of zero; but this brief survey will be concluded with a reminder—strengthened by the paradoxes of Russell and others—that the story of number in general, and of zero in particular, probably never will end. The endlessness of mathematical research is one of the brightest facets of civilization.

*See Eneström, *loc. cit.*
†See *Encyclopédie des sciences mathématiques*, I, 1^1 (1904), p. 33, note 147.

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