Chapter 1

Numbers

1.1 Prehistory

How to count without numbers

We will start talking about the development of the concept of numbers, from the natural numbers used for counting things up to the various extensions: fractions, negative numbers, real numbers, complex numbers, etc.

How could someone 5000 years ago who couldn't count to ten make sure he had all of the 41 sheep of his flock when he went home in the evening? One simple solution is the following: he kept a bag of 41 pebbles, and for each sheep walking into the sheep-pen in the evening he would put one pebble in the bag; if there were pebbles left over, he had to go looking for missing sheep.

Much later, the Romans used a board with pebbles to do their calculations; the roman word for pebble is 'calcule', which is the etymological source of the word 'calculate'.

Decimal System

The number system we are using today has base 10. This means that we can write arbitrarily large numbers with a set of just ten symbols, the digits 0, 1, . . . , 9. As already the Greeks have seen, however, the fact that we use base 10 has deeper roots; Aristotle writes:

> Why do all men, whether barbarians or Greeks, count up to ten, and not up to any other number, such as 2, 3, 4, or 5, so that, for example, they do not say *one-plus-five* (for 6), *two-plus-five* (for 7), as they say *one-plus-ten* ($\epsilon\nu\delta\epsilon\kappa\alpha$, for 11), *two-plus-ten* ($\delta\omega\delta\epsilon\kappa\alpha$ for 12) [. . .]?

This observation means that the base 10 of the number system we use has left its traces in the way we count. Thus English number words have a base ten because the word for 14, for example, is composed of the words for 4 and for 10. (The words for 11 and 12 are exceptions in most Germanic languages; twelve is formed from one [hardly visible] and two, and the word that in English eventually became 'leave' but originally meant 'take away'; so twelve is the number that gives two when you take away ten]. In some number words you can actually find examples with base 20: the French word for 80 is quatrevingt (four [times] twenty), the Latin word for 18 is duodeviginti (two from twenty). But these are minor exceptions: the best way to determine the base of a number system is by looking at the words for *big* numbers: in base 10, there will be new words for 100, 1000, $10,000$ etc; in base 20, the new words would occur for 400, $8,000$ etc.

In this way we can determine the base of number systems even when we don't know how (or whether) these numbers were represented by symbols; in other words: the basis of number systems is more of a linguistic than of a mathematical nature. An examination of the languages that are or have been used reveals that most cultures used the bases 5, 10 or 20.

Aristotle gives a number of possible reasons why people preferred base 10 systems; his last one is the following:

> Or is it because men were born with ten fingers and so, because they possess the equivalent of pebbles to the number of their own fingers, come to use this number for counting everyhting else as well?

This natural explanation is now accepted; in fact we now know of people who have developed number systems with bases 5 (fingers on one hand) as well as 20 (fingers and toes). The Luli's of Paraguay words for 10, 20, 30 is hands, hands feet, and hands feet hands. There are also people (not many though) with a base 4 system: they didn't use thumbs when they counted with their fingers. Several Australian tribes used a binary system for their numbers (they counted only up to 4 or 6, one even used some kind of a ternary system: they count mal, bulan, guliba, bulan bulan, bulan guliba, guliba guliba.

Exercises.

1. Here are the words for numbers from $1-15$ in an old Celtic language (Welsh).

1	yan	6	sethera	11	yanadik
2	tan	7	lethera	12	tanadik
3	tethera	8	hovera	13	tetheradik
4	pethera	9	covera	14	petheradik
5	pimp	10	dik	15	bumpit

   What do you think are the words for the numbers $16-19$? Describe two plausible possibilities and guess which one they were using.

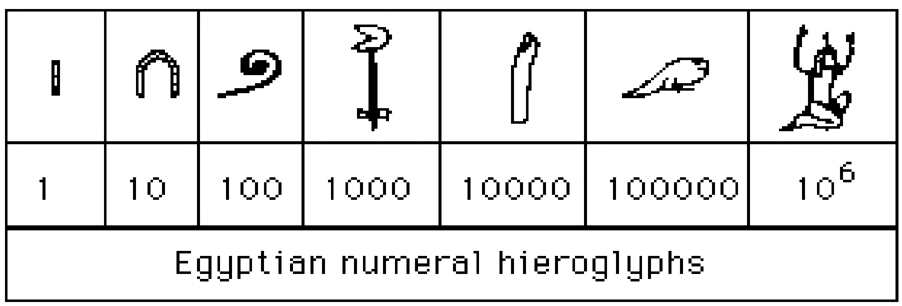
2. Here are the Danish words for some numbers:

1	en	10	ti	60	tre-sinds-tyve
2	to	20	tyve	70	halv-fir-sinds-tyve
3	tre	30	tredive	80	fir-sinds-tyve
4	fire	40	fyrre	90	half-fem-sinds-tyve
5	fem	50	halv-tred-sinds-tyve	100	hundrede

Explain the meaning of the words for 50 – 90.

1.2 Egyptians

Egyptians had symbols for 1, 10, 100, 1000 and $10,000$ (eventually even up to $10,000,000$!); numbers were represented by appropriate repetitions of these symbols. After papyrus had been invented, the scribes introduced symbols for each number from 1 to 9, for each multiple of 10 up to 90, etc. The record number written in Egyptian hieroglyphs is $1,422,000$.



The Egyptians knew fractions; they wrote a fraction $\frac{1}{17}$ by putting a mouth over 17, which means that they had symbols for 1/2, 1/3, 1/4, 1/5 etc, (they're called Egyptian Fractions) but not for, say 2/7, 3/7 etc with the single exeption of 2/3, which had its own symbol. When other fractions were needed, they would express them by summing "Egyptian fractions", but for some reason, repetition of the same fraction was not allowed. Thus they did not write 2/7 as $1/7 + 1/7$, but as $1/4 + 1/28$.

There are rules for writing fractions as continued fractions, but the rules that the Egyptians used are unkown. Today, we can do this by applying the "greedy algorithm": given a fraction $\frac{p}{q}$, look for the largest fraction of the form $\frac{1}{n}$ smaller than $\frac{p}{q}$, compute the difference $\frac{p}{q} - \frac{1}{n} = \frac{r}{s}$, and then repeat the game.

For $\frac{2}{7}$, the greedy algorithm looks for the smallest n with $\frac{2}{7} > \frac{1}{n}$, that is, with $n > \frac{7}{2}$. This is $n = 4$, and $\frac{2}{7} - \frac{1}{4} = \frac{1}{28}$. The last fraction is an Egyptian fraction, and we are done.

In general you will often need more than just one step: $\frac{5}{17} - \frac{1}{4} = \frac{3}{68}$, $\frac{3}{68} - \frac{1}{23} = \frac{1}{1564}$, hence $\frac{5}{17} = \frac{1}{4} + \frac{1}{23} + \frac{1}{1564}$.

Exercise

1. Write $\frac{p}{11}$ as Egyptian fractions for $p = 2, 3, \ldots, 10$.
2. Prove that the greedy algorithm always terminates, i.e., produces a correct answer after finitely many steps.
3. Write the following fractions in the Egyptian system: $\frac{2}{11}$, $\frac{3}{13}$, $\frac{4}{13}$, $\frac{5}{21}$.
4. Write $\frac{3}{3k+2}$ as the sum of two Egyptian fractions.
5. It is conjectured that every fraction $\frac{4}{n}$ with $n \geq 5$ odd is the sum of at most three Egyptian fractions. Prove this.

   Just kidding. Show that $\frac{4}{3n+2}$ can be written as a sum of three Egyptian fractions for all $n \geq 1$.

1.3 Babylonians

The Babylonians had a number system based on the number 60, but there were some traces of base 10. Each number from 1 to 9 was represented by vertical strokes; the number 10 looked a bit like $<$. So 21 would look something like this: $\ll |$. The number 60 was again denoted by a vertical stroke, but the lack of a symbol for 0 made it problematic to distinguish between, say, $64 = 60 + 4$ and $3604 = 60^2 + 4$. Sometimes gaps were used to indicate a missing digit, occasionally we also find 'separation symbols': the Babylonians stopped short of inventing the 0 in its full glory.

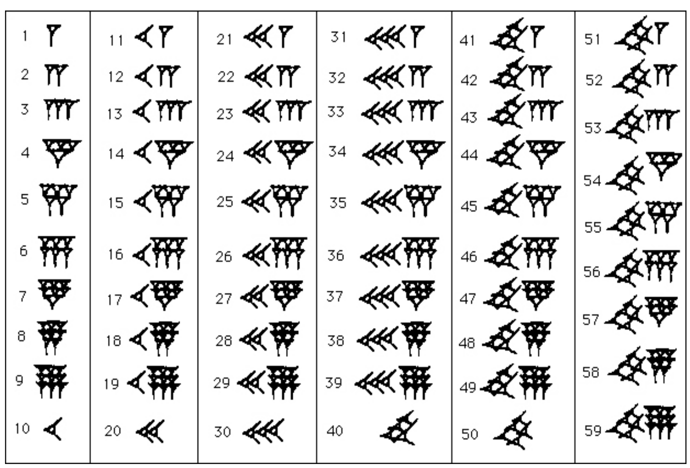


Figure 1.1: Babylonian numerals

The Babylonians divided the circle into 360 parts (degrees), each degree into 60 minutes, and each minute into 60 seconds; this system has survived to this day, but the Babylonians also used it to represent numbers: for example,

$12;34,7$ (or $12\ 34'\ 7''$) represents the rational number $12 + \frac{34}{60} + \frac{7}{3600}$ (of course they were not using our Hindu-Arabic numerals).

Exercise 2. Compute $1;41 + 5;34$ and $1;41' \cdot 5;34'$.

Exercise 3. The number $1;24,51,10$ occurs in a Babylonian manuscript (a tablet, actually). Compute its decimal value – do you recognize the number?

1.4 Greeks

The Greeks used letters for denoting numbers; one of the systems in use was the following: The first nine letters of their (archaic) alphabet, namely α, β, $\ldots$, θ denoted numbers from 1 to 9, the next nine were used for 10 to 90, and the final nine for 100 up to 900. Putting a mark at the lower left of a symbol represented multiples of 1000; thus e.g. $,\beta$ would denote 2000. For multiples of $10,000$, they wrote the letter above a capital M (for 'myriad', meaning ten thousand). For larger numbers, they would use MM for $10,000^2$ and so on.

The Greeks got their alphabet from the Phoenicians, who invented an alphabet consisting only of consonants. When the Greeks adapted this alphabet to their needs, they used a few letters for vowels (α, ϵ, η, ι, o, υ, ω); others, they didn't use at all, and these were revived later to denote some numbers.

1.5 Hindus and the decimals

The roots of the Hindu numerals for 1 through 9 can already be found around 250 BC and are quite possibly older. At that time, they used symbols for tens (10 - 90), hundreds etc. different from those for 1 - 9; around 600 AD, however, they introduced the positional system and used a dot for today's 0. Within a hundred years this system seems to have spread to China and Baghdad, then the center of the Islamic culture. The first textbook on how to compute with these numbers was written by al-Khwarizmi (his name gave rise to our word 'algorithm'; the word 'al-jabr' in the title of one of his books became our 'algebra'; the Arabic word 'cifr' for zero gave rise both to 'zero' as well as 'cipher'), who used a circle to represent the 0.

The Hindu numerals were introduced to Europe by Leonardo di Pisa (the son of a merchant named Bonacci; the French mathematician Libri apparently gave Leonardo the nickname Fibonacci, short for Filius [son of] Bonacci) in his book 'Liber Abaci' from 1202.

Decimal fractions as we know them today were not used before 1429 by al-Kashi, and apparently at that time were introduced independently in Europe.

1.6 Remarks

Despite of its shortcomings (no 0, no 'decimal' point that allows to distingish, say, 61 form $1\frac{1}{60}$), the Babylonian system was much better suited for computing

1	α′	εἷς, μία, ἕν	11	ια′	ἕνδεκα
2	β′	δύο	12	ιβ′	δώδεκα
3	γ′	τρεῖς, τρία	13	ιγ′	τρεῖς (τρία) καὶ δέκα (or τρεισκαίδεκα)
4	δ′	τέτταρες, τέτταρα (τέσσαρες, τέσσαρα)	14	ιδ′	τέτταρες (τέτταρα) καὶ δέκα
5	ε′	πέντε	15	ιε′	πεντεκαίδεκα
6	ϛ′ or ϝ′	ἕξ	16	ιϛ′	ἑκκαίδεκα
7	ζ′	ἑπτά	17	ιζ′	ἑπτακαίδεκα
8	η′	ὀκτώ	18	ιη′	ὀκτωκαίδεκα
9	θ′	ἐννέα	19	ιθ′	ἐννεακαίδεκα
10	ι′	δέκα	20	κ′	εἴκοσι(ν)
21	κα′	εἷς καὶ εἴκοσι(ν) or εἴκοσι καὶ εἷς	100	ρ′	ἑκατόν
			200	σ′	διακόσιοι, -αι,-α
30	λ′	τριάκοντα	300	τ′	τριακόσιοι
40	μ′	τετταράκοντα	400	υ′	τετρακόσιοι
50	ν′	πεντήκοντα	500	φ′	πεντακόσιοι
60	ξ′	ἑξήκοντα	600	χ′	ἑξακόσιοι
70	ο′	ἑβδομήκοντα	700	ψ′	ἑπτακόσιοι
80	π′	ὀγδοήκοντα	800	ω′	ὀκτακόσιοι
90	ϙ′	ἐνενήκοντα	900	ϡ′ or ↑	ἐνακόσιοι

1,000	͵α		χίλιοι, -αι, -α
2,000	͵β		δισχίλιοι
3,000	͵γ		τρισχίλιοι
10,000	͵ι	$\overset{A}{M}$	μύριοι, -αι, -α
20,000	͵κ	$\overset{B}{M}$	δισμύριοι
100,000	͵ρ		δεκακισμύριοι

Figure 1.2: Greek numerals

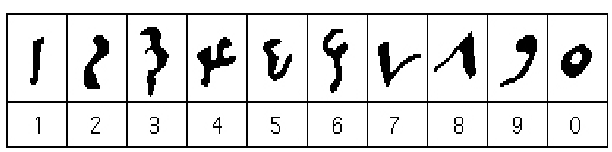


Figure 1.3: Al-Sijzi's numbers from 969AD

than the Egyptian system; accordingly, Egyptian astronomers (such as Prolemy) used the Babylonian sexagesimals.

Early Egyptian and Babylonian writings have survived on temple walls, columns, or clay tablets. We'd probably know a lot more about ancient Chinese mathematics were it not for the fact that the Chinese used to write on material made out of bamboo (which is responsible for the fact that they started writing vertically), so most of the manuscripts (just like the Egyptian and Greek books written on papyrus) simply decayed. We know a lot about classical Greek manuscripts because they have been copied over and over again; for example, there are no original copies of Euclid's elements; all the copies we possess seem to go back to Theon of Alexandria, Hypatia's father.

1.7 Maya Numbers

The Maya civilization had a number system with base 20.

number	word	number	word
1	hun	11	buluc
2	ca	12	lahca
3	ox	13	oxlahun
4	can	14	canlahun
5	ho	15	holahun
6	uac	16	uaclahun
7	uuc	17	uuclahun
8	uaxac	18	uaxaclahun
9	bolon	19	bolonlahun
10	lahun	20	hunkal

So far, the Mayan system looks like a regular base 10 system (with the exception of 11). But look what happens now:

number	word	number	word
21	huntukal	31	buluctukal
22	catukal	32	lahcatukal
23	oxtukal	33	oxlahuntukal
24	cantukal	34	canlahuntukal
25	hocakal	35	holahuncakal
26	uactukal	36	uaclahuntukal
27	uuctukal	37	uuclahuntukal
28	uaxactukal	38	uaxaclahuntukal
29	bolontukal	39	bolonlahuntukal
30	lahuncakal	40	cakal

This is clearly a system with base 20. In fact, here are some more words:

number	word	explanation
41	huntuyoxkal	1 to 60
50	lahunyoxkal	10 to 60
60	oxkal	$3 \cdot 20$
80	cankal	$4 \cdot 20$
100	hokal	$5 \cdot 20$
120	uackal	$6 \cdot 20$
	...	
380	bolonlahunkal	$19 \cdot 20$
400	hunbak	$1 \cdot 400$
500	hotubak	$5 \cdot 20 + 400$
	...	
1000	lahuntuyoxbak	$10 \cdot 20$ to $3 \cdot 400$
8000	hun pic	
160,000	hun kalab	
3200,000	hun kinchil	
64,000,000	hun alau	

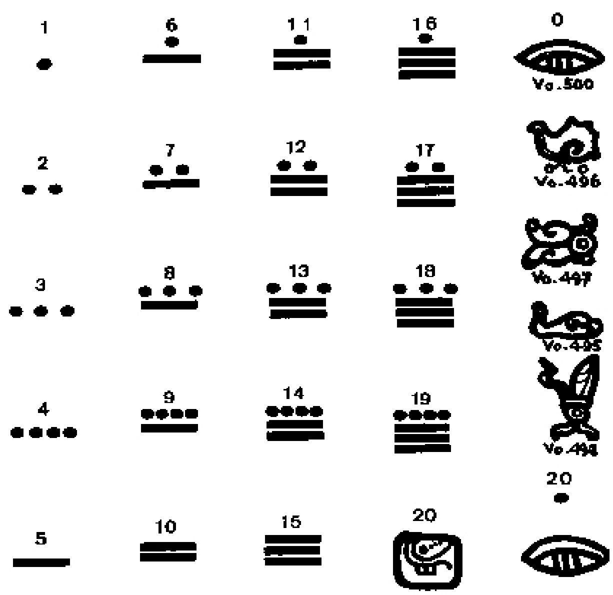


Figure 1.4: Mayan numerals

Remark. All the pictures in this file were stolen from the web.