An infinite series of surprises.

C J Sangwin
School of Mathematical and Statistics, University of Birmingham,
Birmingham, B15 2TT, UK
Telephone 0121 414 6197, Fax 0121 414 3389

Email: C.J.Sangwin@bham.ac.uk http://www.mat.bham.ac.uk/C.J.Sangwin/

June 29, 2001

1 Introduction

Infinite series, that is an infinite sum of the form

$$a_1 + a_2 + a_3 + \cdots = \sum_{k=1}^{\infty} a_k,$$

occupy a central and important place in modern mathematics. Much of this topic was developed during the seventeenth century. Leonhard Euler continued this study and in the process solved many important problems.

The purposes of this lecture are (i) to collect together some interesting and intriguing results connected with infinite series, (ii) to present these results in a simple way, perhaps sacrificing absolute standards of mathematical rigour to show sensitivity to the historical origin of the results, (iii) to give examples of genuinely interesting extended arguments and, (iv) to demonstrate the possibilities of elementary arguments – that is to say arguments not involving any advanced theory.

In particular we will review Euler's argument involving one of the most surprising series. I am not claiming this is a *proof* by modern standards – Euler certainly took many audacious steps. However his boldness is refreshing, although the reader may feel uneasy during the course of the supposed proofs. Identifying these false steps and filling in the gaps is a task I believe well worth the effort. Perhaps I might claim that this is an argument a good A-level student could follow.

2 Before Euler

Perhaps the most important series is the *geometric progression.* Given constants a and r we want to sum

$$a + ar + ar^2 + \cdots + ar^N.$$

The proof of the value of this sum is elementary and well known. If $|r| < 1$ we can make sense of the infinite sum – something known by Newton – which is

$$a + ar + ar^2 + \cdots + ar^N + \cdots = \frac{a}{1-r}. \tag{1}$$

This was one of the first, and only, general results. Other series were known during the seventeenth century such as the following one of Bernoulli:

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{2}{k(k+1)} &= 1 + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \frac{1}{15} + \cdots \\ &= 2\left[\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots\right] \qquad (2) \\ &= 2\left[\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \cdots\right] \qquad (3) \\ &= 2. \qquad (4)\end{aligned}$$

Other series summed by similar methods were

$$\sum_{k=1}^{\infty} \frac{k^2}{2^k} = 6 \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{k^3}{2^k} = 26.$$

Now all these series converge – not true of the most famous series – the *harmonic series*

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \sum_{k=1}^{\infty} \frac{1}{k}.$$

2.1 The medieval proof of the divergence of the harmonic series

The following medieval proof that the harmonic series diverges was discovered and published by Orseme around 1350 and relies on grouping the terms in the series as follows

$$\begin{aligned}&1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \cdots \\ &\quad = 1 + \tfrac{1}{2} + \left(\tfrac{1}{3} + \tfrac{1}{4}\right) + \left(\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8}\right) + \left(\tfrac{1}{9} + \tfrac{1}{10} + \tfrac{1}{11} + \tfrac{1}{12} + \tfrac{1}{13} + \tfrac{1}{14} + \tfrac{1}{15} + \tfrac{1}{16}\right) + \cdots \\ &\qquad \geq 1 + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \cdots\end{aligned}$$

It follows that the series can be made arbitrarily large. In fact this series diverges quite slowly. A more accurate estimate of the speed of divergence can be made using the following more modern proof.

2.2 The modern proof of the divergence of the harmonic series

The modern proof that the harmonic series diverges uses the *integral test.* This compares the graph of a function with the terms of the series. By integrating the function we can compare the sum of the series with the integral of the function and draw conclusions from this.

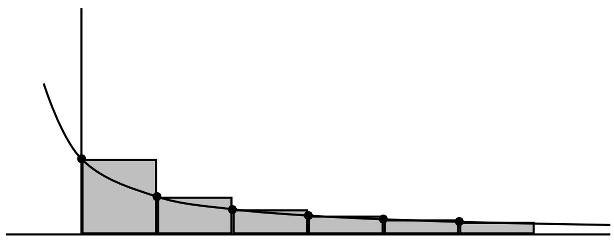

Figure 1: Comparing the series $1/n$ with the function $1/(1+x)$.

In this case we compare terms in the series with the area under the graph of the function $1/(1+x)$. In particular Figure 1 shows that

$$\sum_{k=1}^{n} \frac{1}{k} > \int_0^n \frac{1}{1+x} \mathrm{d}x.$$

Of course the integral on the right is easy. Solving this gives

$$\sum_{k=1}^{n} \frac{1}{k} > \ln(1+n).$$

Now, the function $\ln(1+n)$ is unbounded so that we can make $\sum_{k=1}^{n} \frac{1}{k}$ as large as we please. A similar argument comparing the series to the function $1/x$ shows that

$$1 + \ln(n) > \sum_{k=1}^{n} \frac{1}{k} > \ln(1+n)$$

so that we can estimate how fast the series diverges.

3 The harmonic series generalized

The harmonic series can be described as "the sum of the reciprocals of the natural numbers". Another series that presents itself as being similar is the "the sum of the squares of reciprocals of the natural numbers". That is to say the series

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \sum_{k=1}^{\infty} \frac{1}{k^2}. \tag{5}$$

The first question we ask is "Does this series converge?" Next we ask "What is the sum?". To answer the first we notice that

$$2k^2 \geq k(k+1)$$

and so

$$\frac{1}{k^2} \leq \frac{2}{k(k+1)}$$

and so Bernoulli's series (4) gives that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \leq \sum_{k=1}^{\infty} \frac{2}{k(k+1)} = 2.$$

The series converges[1]. But the exact value of the sum proved hard to find. Jakob Bernoulli considered it and failed to find it. So did Mengoli and Leibniz. Finding the sum became known as the *Basel Problem* and we concentrate for much of the rest of the lecture on Euler's solution.

4 Euler's solution to the Basel Problem

The history of this problem is told in full in [1] and also [2, pp. 284–285]. However, Euler was working on it at the age of 24 in 1731 by calculating a numerical approximation. An arduous task by hand with such a slow converging series. In 1735 he arrived at the following result:

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$

This he showed as follows.

Euler starts with an nth degree polynomial $p(x)$ with the following properties:

(i) $p(x)$ has non-zero roots $a_1, \cdots, a_n$,

(ii) $p(0) = 1$.

Then $p(x)$ may be written as a product in the following form:

$$p(x) = \left(1 - \frac{x}{a_1}\right)\left(1 - \frac{x}{a_2}\right)\cdots\left(1 - \frac{x}{a_n}\right).$$

We paraphrase Euler's next claim as *"what holds for a finite polynomial holds for an infinite polynomial"*. He applies this claim to the polynomial

$$p(x) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$$

which is an infinite polynomial with $p(0) = 1$. Furthermore, as Euler knew, $\sin(x)$ can be written as a series

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^{k+1}\frac{x^{2k+1}}{(2k+1)!} + \cdots$$

Since

$$xp(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sin(x)$$

[1] The method of proving convergence by comparing terms with a convergent series is known as the *comparison test*. We could also use the integral test here to prove convergence by comparing the series with the function $f(x) = 1/x^2$.

this polynomial has zeros at $x = \pm k\pi$ for $k = 1, 2, \cdots$. We can now write $p(x)$ as an infinite product and equate the two as

$$\begin{aligned} 1 - \frac{x^2}{3!} - \frac{x^4}{5!} + \frac{x^6}{7!} - \cdots &= p(x) \\ &= \left(1 - \frac{x}{\pi}\right)\left(1 + \frac{x}{\pi}\right)\left(1 - \frac{x}{2\pi}\right)\left(1 + \frac{x}{2\pi}\right) \times \cdots \\ &= \left[1 - \frac{x^2}{\pi^2}\right]\left[1 - \frac{x^2}{4\pi^2}\right]\left[1 - \frac{x^2}{9\pi^2}\right] \times \cdots \end{aligned}$$

The second line pairs the positive and negative roots – the last line uses the difference of two squares to combine these.

Euler's trick is to write $p(x)$ in two different ways. He exploits this by expanding the right hand side. This infinite product will be very complicated but there will be a constant term 1 and one can collect the x^2 term without too much effort as follows:

$$1 - \frac{x^2}{3!} - \frac{x^4}{5!} + \frac{x^6}{7!} - \cdots = 1 - \left(\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \frac{1}{16\pi^2} + \cdots\right) x^2 + \cdots$$

Now Euler equates the coefficients of x^2 to conclude that

$$-\frac{1}{3!} = -\left(\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \frac{1}{16\pi^2} + \cdots\right)$$

which gives the result

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}.$$

Now Euler didn't stop here – he expanded the product further and equated other coefficients to sum other series. In this way he obtained

$$\sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{1}{k^6} = \frac{\pi^6}{945}.$$

In 1744 he obtained

$$\sum_{k=1}^{\infty} \frac{1}{k^{26}} = \frac{2^{24} 76977927 \pi^{26}}{27!}$$

by this method and in principle his method solves

$$\sum_{k=1}^{\infty} \frac{1}{k^{2n}}$$

for all natural numbers n.

5 Extensions of the Basel problem

In a style typical of Euler, he not only solved the problem in hand but also used the method to solve a class of related problems. You will notice that his method only works for *even* powers. What then, about

$$\sum_{k=1}^{\infty} \frac{1}{k^3}?$$

The answer is: *we don't know.* This is an open problem, and quite a famous one.

Euler tried to solve it of course, and failed. The best he could do was

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3} = 1 - \frac{1}{27} + \frac{1}{125} + \cdots = \frac{\pi^3}{32}.$$

More information on the status of this problem can be found in [1].

Since the three series

$$\sum_{k=1}^{\infty} \frac{1}{k}, \quad \sum_{k=1}^{\infty} \frac{1}{k^2} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{1}{k^3}$$

are all so similar, one might try to define a function

$$\zeta(s) := \sum_{k=1}^{\infty} \frac{1}{k^s}.$$

In fact this can be done, and it is known as the Riemann Zeta function. Of course we have seen $\zeta(1)$ – the harmonic series – diverges and that $\zeta(2) = \pi^2/6$. In general evaluation of this function will be extremely difficult.

The function $\zeta(s)$ is generally defined for complex numbers s. We do this and try to solve the equation $\zeta(s) = 0$. This problem is also unsolved and of great contemporary importence. People think that all solutions have real part 1/2 but this, perhaps, is a good place to end!

6 Exercises

1. How many terms does one need to take to approximate $\sum \frac{1}{k^2}$ accurately to 2 decimal places?

2. Apply the formula (1) to the geometric progression

   $$\frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots$$

   What does this tell us about the value that the recurring decimal $0.9\dot{9}\cdots$ represents?

3. Apply Euler's argument to $\cos(x)$ and adapt it to find

   $$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}$$

4. Hence or otherwise find the sum of the squares of reciprocals of the *even* natural numbers.

5. Equate other coefficients in the argument of question 3 above to obtain values for

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^n}$$

for various *even* values of n.

6. Work through Euler's argument and justify each step or identify the flaw.

7. The following problem appeared in [4]. The answer is surprising! Remember that *rational* numbers have decimal representations that terminate or are eventually periodic. For example $1/11 = 0.0909090\cdots$ and $1/4 = 0.2500\cdots$. Conversely all eventually periodic decimals represent rational numbers. So the number $0.001001001001\cdots$ is rational and we notice that the *period* of the decimal is 3. That is to say every 3rd digit repeats. What is the period of $(0.001\cdots)^2$? Guess first then work it out!

References

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[3] Kalman, D., *Six ways to sum a series*, The College Mathematics Journal, Vol. 14, No. 5, 1993, pp. 402–421.

[4] Fowler, D., *400 years of decimal fractions*, Mathematics Teaching, Vol. 110, 1985, pp. 20–21. See also follow up article in Vol. 111, pp. 30–31.

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[5] Euler, L. *Opera Omnia* Ser. I, Vol 14, pp. 84–85.

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