HISTORY OF MATHEMATICS

HOMEWORK 3

This is about Archimedes' computation of the area of a segment of a parabola.

(1) Prove Archimedes's result about the area of segments of a parabola using calculus: consider the parabola $y = ax^2$ and compute the area A between the chord defined by $R = (r, ar^2)$ and $S = (s, as^2)$ as well as the area B of the triangle RST, where $T = (\frac{1}{2}(r+s), \frac{a}{4}(r+s)^2)$, and show that $A = \frac{4}{3}B$.

This is standard calculus. The line RS has equation $y = a(r+s)x - ars$, hence the area A of the segment of the parabola is

$$\begin{aligned} A &= \int_r^s (a(r+s)x - ars - ax^2)dx \\ &= \frac{1}{2}a(r+s)x^2 - arsx - \frac{1}{3}ax^3\Big|_r^s \\ &= \frac{1}{6}a(s-r)^3. \end{aligned}$$

The area of a triangle whose vertices have coordinates (x_i, y_i) $(i = 1, 2, 3)$ is given by

$$B = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

In our case we get

$$B = \frac{1}{2}\begin{vmatrix} r & ar^2 & 1 \\ s & as^2 & 1 \\ \frac{r+s}{2} & \frac{a(r+s)^2}{4} & 1 \end{vmatrix} = \frac{a}{8}(r-s)^3$$

(this calculation was done using pari).

Thus $A = \frac{4}{3}B$ as claimed.

(2) Compute the area A of a segment of the unit circle defined by the points $R = (-u, \sqrt{1-u^2})$ and $S = (u, \sqrt{1-u^2})$, and compare it to the area B of the triangle with maximal are inside this segment, namely RST with $T = (0, 1)$. Show that the ratio $A : B$ depends on u; also derive from your calculations that $\sin x \le x$ for $x > 0$.

The Chinese had a formula for the area of a segment with base $s = 2u$ and height h: $A = \frac{1}{2}(sh + h^2)$. The same formula appears in a manuscript from Cairo written in the third century BC, and in the metrica of Heron of Alexandria. Develop your formula for A into a power series and show that $A = \frac{2}{3}sh + O(u^4)$, where $O(u^4)$ is a power series consisting of terms of order 4 and higher.

Will be given later. Observe that the ratio of area and inscribed triangle depends on u and is not constant as in the case of the parabola; this explains why Archimedes couldn't compute the area of a segment of a circle by exhaustion: this really requires calculus.

(3) We have seen that the triangle Archimedes inscribes into a segment of a parabola is the one with maximal area. Describe the quadrilateral with maximal area that can be inscribed into a segment of a parabola.

Assume that the coordinates of the two 'division points' are (u, au^2) and (v, bv^2). Then the points with x-coordinates u, v, s form a triangle; since the quadrilateralt has maximal area, so does the triangle, hence $v = (u+s)/2$. Considering the triangle formed by the points with x-coordinates r, u, v we find $u = (r+v)/2$. Eliminating v from the last two equations gives $u = (2r+s)/3$ and $v = (r+2s)/3$. In other words: the points divide the interval $[r, s]$ into three equal parts.

The same idea shows that, for inscribed n-gons, one gets the maximal area by dividing $[r, s]$ into n equal parts.

Extra credit: compute the ratio of the areas of the segment and the quadrilateral with maximal area.

The answer was $A = \frac{9}{8}B$ (instead of $\frac{4}{3}B$ for the case of triangles).

(4) Reading assignment: read the three articles about zero on the web page.
- **Briefly** summarize the main claims made by the authors;
- Discuss whether the authors seem biased, and support your claims.

My impression was that Boyer seemed more biased than Datta. His agenda is the defense of Greek mathematics against the contributions by the rest of the world.

His first article starts with the heading 'the Hindu legend' referring to the statement that they were the first to introduce 0 as a number; the text, however, contains no hints as to why this should be a legend. Also, when Aristotle talks about the 'void', it is not clear whether we should translate this as 0, thereby suggesting that Aristotle regarded this 'void' as a number. There is no number 0 in the whole of Greek mathematics, not even in Diophantus. In his second article, Boyer is more careful and acknowledges that, for Aristotle, numbers were multiples of 1, that is, 2, 3, 4, ... Still he claims that 'Aristotle had been familiar with the arithmetical aspect of zero', but he fails to give sources: for me, arithmetical aspects of zero means that $a+0$, $a-0$, and $a \cdot 0$ have a meaning – but there is nothing of such things in Aristotle.

As for Datta, the fact that he doesn't know about Aristotle's 'division by 0' should not be interpreted as bias but as a gap in his knowledge. Also, as his title suggests, he wants to talk about the 0 in Indian mathematics, not in that of the Greeks or the Mayas.