HISTORY OF MATHEMATICS

HOMEWORK 2

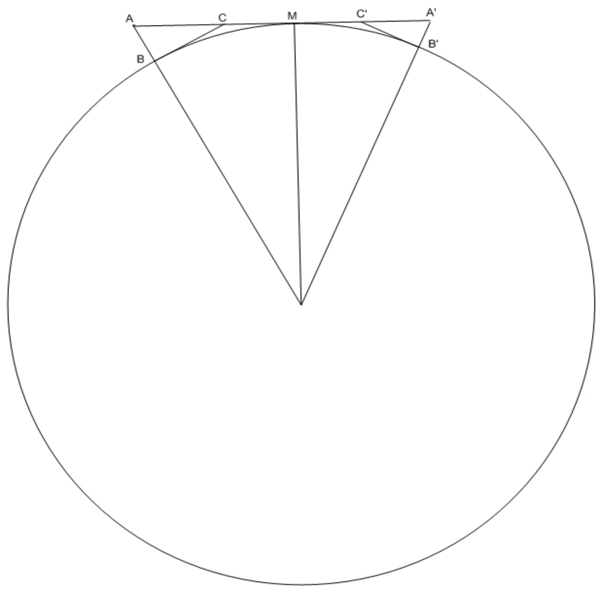
This is about Euclid's proof that circles are to each other as the squares of their diameters (Book XII, Proposition 2).

(1) Prove (similarly to what we did in class) that the area A_n of a 2^n-gon circumscribed about a circle with area A satisfies
$$A_n < (1 + \frac{1}{2^{n-2}})A.$$
(2) Complete the proof that $A_1 : A_2 = d_1^2 : d_2^2$ using the formula above.
(3) Study Euclid's original proof. Which parts of our proof occurs in Euclid?
(4) Did Euclid use circumscribed polygons to finish his proof? How exactly did he finish his proof?

Solutions

1. We will prove the inequality using induction; the first step is therefore showing that $A_2 < 2A$, which was shown in class. Note that you are not allowed to use the formula $A = \pi r^2$, since this was not known to Euclid.

Now consider the following figure.



Here AA' is a side of a 2^n-gon circumscribed about the circle. CC' is a side of a 2^{n+1}-gon, BC and $B'C'$ are half of such sides. Since the triangle BCM has a right angle at B, the theorem of Pythagoras implies that $AC > BC$, and since BC and CM are half sides of a regular 2^{n+1}-gon, we conclude that $BC = CM$ and hence $AC > CM$. The triangles ABC and BCM have the same height above AM, hence $\text{area}(ABC) > \text{area}(BCM)$.

Thus

$$\begin{aligned}\text{segm}(ABM) &= \text{area}(ABM) - \text{segm}(BM) = \text{area}(ABC) + \text{area}(BCM) - \text{segm}(BM) \\ &= \text{area}(ABC) + \text{segm}(BCM) > \text{area}(BCM) + \text{segm}(BCM) \\ &> 2\,\text{segm}(BCM),\end{aligned}$$

i.e. $\text{segm}(BCM) < \frac{1}{2}\,\text{segm}(ABM)$.

Now we claim that

$$A_n < (1 + \frac{1}{2^{n-2}})A. \tag{1}$$

Assume this holds for some $n \geq 2$; then $A_n - A < 2^{2-n}A$ and

$$A_n - A = 2^{n+1}\,\text{segm}(ABM),$$

$$A_{n+1} - A = 2^{n+1}\,\text{segm}(BCM) < 2^n\,\text{segm}(ABM) = \frac{1}{2}(A_n - A) < 2^{1-n}A$$

and this implies that (1) holds for $n + 1$.

2. Let A_1 (A_2) be the area of a circle with diameter d_1 (d_2). We claim that $A_1 : A_2 = d_1^2 : d_2^2$. Define C by $d_1^2 : d_2^2 = A_1 : C$; then the claim is that $C = A_2$. If $C \neq A_2$, then there are two cases:

- $C < A_2$. This was done in class: we can inscribe regular 2^n-gons with areas P_1 and P_2 into the circles, and we can choose n so large that $C < P_2 < A_2$. By Euclid's Proposition 1, we have $d_1^2 : d_2^2 = P_1 : P_2$. But clearly $P_1 < A$, and since $P_2 > C$ we find $d_1^2 : d_2^2 = P_1 : P_2 < A : C$ contradicting the definition of C.
- $C > A_2$. We can circumscribe polygons with areas P_1 and P_2 about the circles, and choose n so large that $A_2 < P_2 < C$. Then, using Proposition 1 again, we get $d_1^2 : d_2^2 = P_1 : P_2 > A_1 : C$ since $P_1 > A_1$ and $P_2 < C$.

3. Let us now study Euclid's proof. He starts with

> Inscribe the square EFGH in the circle EFGH. Then the inscribed square is greater than the half of the circle EFGH.

This corresponds to our inequality $A_2 > \frac{1}{2}A$.

> Bisect the circumferences EF, FG, GH, and HE at the points K, L, M, and N.

Here he compares an inscribed square to an inscribed octagon. This corresponds to our comparison of a regular 2^n-gon with a regular 2^{n+1}-gon. Note that Euclid cannot deal with variables, so he has to give a proof by examples; here he takes $n = 2$.

> Each of the triangles EKF, FLG, GMH, and HNE is greater than the half of the segment of the circle about it.

This inequality corresponds to the special case $n = 2$ of our

$$A - A_{n+1} < \frac{1}{2}(A - A_n).$$

Note that Euclid's proof would work just as well for arbitrary n; he only lacks the language to express himself.

> It was proved in the first theorem of the tenth book that if two unequal magnitudes are set out, and if from the greater there is subtracted a magnitude greater than the half, and from that which is left a greater than the half, and if this is done repeatedly, then there will be left some magnitude which is less than the lesser magnitude set out.

This is Eudoxus' lemma: If $0 < \varepsilon < m$, and if you form $m - m_1 - m_2 - m_3 - \ldots - m_n$, where $m_1 > \frac{1}{2}m$, $m_2 > \frac{1}{2}(m - m_1)$, $m_3 > \frac{1}{2}(m - m_1 - m_2)$ etc., then for sufficiently large n we will have $0 < m - m_1 - m_2 - m_3 - \ldots - m_n < \varepsilon$. It is used to prove that the difference between the area A of the circle and the area P_n of the inscribed 2^n-gon can be made as small as one wishes.

Euclid does this, and then proves the case $C < A_2$ as we did.

4. For the proof of the case $C > A_2$, Euclid does not use circumscribed polygons as we did: he reduces this case to the case $C < A_2$ with a little trick.

In fact, if $d_1^2 : d_2^2 = A_1 : C$ with $C > A_2$, then $A_1 : C = D : A_2$ with $D < A_1$. Switching the roles of the circles A_1 and A_2 shows that this is a contradiction.