Exhaustion: From Eudoxus to Archimedes

Franz Lemmermeyer

April 22, 2005

Abstract

Disclaimer: Eventually, I plan to polish this and use my own diagrams; so far, most of it is lifted from the web.

Exhaustion is a method invented by Eudoxus to prove results about lengths, areas and volumes of geometrical objects. In Euclid's elements (Book XII on the measurement of figures), the following propositions, among others, are proved by the method of exhaustion:

- XII.1. Similar polygons inscribed in circles are to one another as the squares on their diameters.
- XII.2. Circles are to one another as the squares on their diameters.
- XII.6. Pyramids of the same height with polygonal bases are to one another as their bases.
- XII.7. Corollary: Any pyramid is the third part of the prism with the same base and equal height.
- XII.10. Any cone is the third part of the cylinder with the same base and equal height.
- XII.18. Spheres are to one another in triplicate ratio of their respective diameters.

All proofs by exhaustion use the method of double contradiction (sort of an ε-δ-argument) where we today would apply limits.

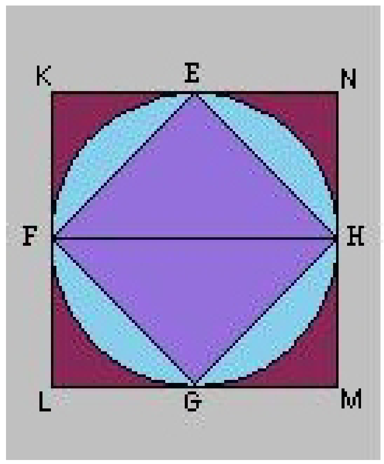
1 Euclid, Book XII, Proposition 2

Let us now give a slightly modernized version of Euclid's proof that circles are to one another as the squares on their diameters. Letting A_1 and A_2 denote the areas of circles with diameters d_1 and d_2, Euclid's claim translates into $A_1 : A_2 = d_1^2 : d_2^2$. This in turn means that the area of circles is proportional to the square of the diameter, that is, $A = c \cdot d^2$ for some constant c. In the next section we will sketch Archimedes' proof that this constant equals $c = \frac{1}{4}\pi$.

The idea of the proof is to inscribe regular 2^n-gons into circles. A look at the following figure immeadiately shows that

$$EFGH = \frac{1}{2}KLMN > \frac{1}{2}A,$$

where $EFGH$ denotes the area of the square with vertices E, F, G, H, and where A denotes the area of the circle.



We use this as the first step in the proof by induction of the formula

$$A_n > \left(1 - \frac{1}{2^{n-1}}\right)A, \qquad (1)$$

where A_n denotes the area of a regular 2^n-gon inscribed into the circle.

This is not in Euclid; what he proved is the inequality

$$A - A_n < \frac{1}{2}(A - A_{n-1}) \qquad (2)$$

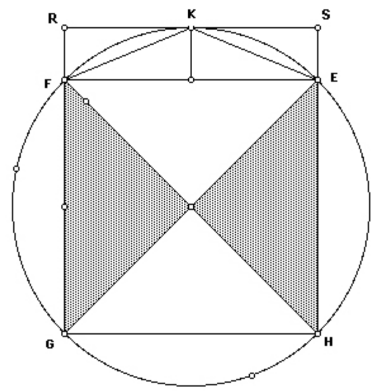
for $n \geq 3$. Applying this inequality repeatedly we find

$$\begin{aligned} A - A_n &< \frac{1}{2}(A - A_{n-1}) < \frac{1}{4}(A - A_{n-2}) < \ldots \\ &< 2^{2-n}(A - A_2) < 2^{1-n}A, \end{aligned}$$

where we have used that $A_2 > \frac{1}{2}A$ in the last line.

Exhaustion

Let us now prove Euclid's inequality (2). The following figure, which deals with inscribes squares ($n = 2$) and octagons ($n = 3$),



shows that

$$EKF = \frac{1}{2}EFRS > \frac{1}{2}\,\mathrm{seg}(EKF), \tag{3}$$

where $\mathrm{seg}(EKF)$ denotes the segment of the circle. Now the difference between the area A of the circle and the area A_n of the inscribed regular 2^n-gon is 2^n times the segment EKF:

$$A - A_n = 2^n \cdot \mathrm{seg}(EKF). \tag{4}$$

On the other hand, we have

$$A - A_{n+1} = 2^n(\mathrm{seg}(EKF) - EKF). \tag{5}$$

Plugging (4) into (5) we find

$$A - A_{n+1} = A - A_n - 2^n \cdot EKF. \tag{6}$$

Finally, combining this with (3), this yields

$$A - A_{n+1} < A - A_n - 2^{n-1}\,\mathrm{seg}(EKF),$$

and using (4) again we arrive at the desired inequality (2).

Double Contradiction

Euclid now finishes his (or, rather, Eudoxus') proof with a double contradiction. First observe that (1) says that the difference between the area of a circle and that of an inscribe 2^n-gon can be made as small as one wishes by choosing n large enough.

Now let A_1 and A_2 denote the areas of circles with diameters d_1 and d_2; we want to show that $A_1 : A_2 = d_1^2 : d_2^2$. Put $d_1^2 : d_2^2 = A_1 : C$; the claim is that $A_2 = C$. The proof by double contradiction works like this: assume that $A_2 > C$ or $A_2 < C$ and derive a contradiction in each case; then conclude that $A_2 = C$ (here Euclid uses another axiom, called the law of trichotomy, which is missing from his books: for any magnitudes A, B, we have exactly one of the cases $A < B$, $A = C$ or $A > C$).

Assume therefore that $A_2 > C$. Inscribe regular 2^n-gons with areas P_1 and P_2 into the circles with areas A_1 and A_2, and choose n so large that $A_2 > P_2 > C$. Then

$$\frac{d_1^2}{d_2^2} = \frac{P_1}{P_2}$$

by Proposition XII.1; but now $P_1 < A_1$ and $P_2 > C$ show that

$$\frac{d_1^2}{d_2^2} = \frac{P_1}{P_2} > \frac{A_1}{C},$$

contradicting the assumption that $d_1^2 : d_2^2 = A_1 : C$.

The case $A_2 < C$ can be proved similarly by circumscribing regular 2^n-gons about the circles.

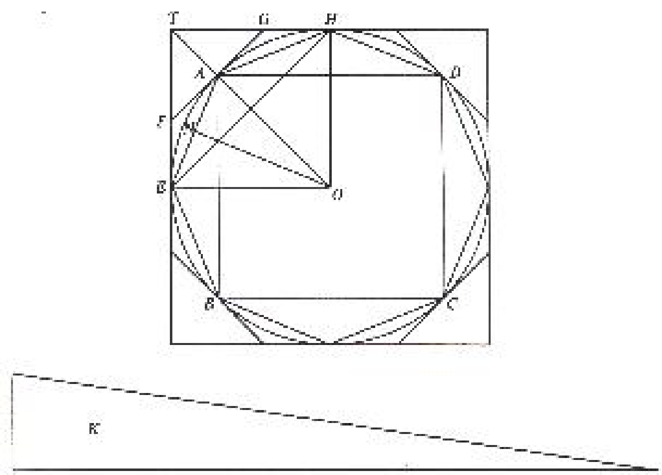
2 Archimedes and the Circle

This is Proposition 1 of Archimedes' *'On the measurement of the Circle*:

> The area of any circle is equal to a right-angled triangle in which one of the sides about the right angle is equal to the radius, and the other to the circumference, of the circle.

His proof went as follows.

Let ABCD be the given circle, K the triangle described.



Then, if the circle is not equal to K, it must be either greater or less.

I. If possible, let the circle be greater than K.

Inscribe a square ABCD, bisect the arcs AB, BC, CD, DA, then bisect (if necessary) the halves, and so on, until the sides of the inscribed polygon whose angular points are the points of division subtend segments whose sum is less than the excess of the area of the circle over K.

Thus the area of the polygon is greater than K.

Let AE be any side of it, and ON the perpendicular on AE from the centre 0.

Then ON is less than the radius of the circle and therefore less than one of the sides about the right angle in K. Also the perimeter of the polygon is less than the circumference of the circle, so less than the other side about the right angle in K.

Therefore the area of the polygon is less than K; which is inconsistent with the hypothesis.

Thus the area of the circle is not greater than K.

II. If possible, let the circle be less than K.

Circumscribe a square, and let two adjacent sides, touching the circle in E and H, meet in T. Bisect the arcs between adjacent points of contact and draw the tangents at the points of bisection. Let A be the middle point of the arc EH, and FAG the tangent at A.

Then the angle TAG is a right angle.

Therefore $TG > GA > GH$.

It follows that the triangle FTG is greater than half the area TEAH.

Similarly, if the arc AH be bisected and the tangent at the point of bisection be drawn, it will cut off from the area GAH more than one-half.

Thus, by continuing the process, we shall ultimately arrive at a circumscribed polygon such that the spaces intercepted between it and the circle are together less than the excess of K over the area of the circle.

Thus the area of the polygon will be less than K.

Now, since the perpendicular from 0 on any side of the polygon is equal to the radius of the circle, while the perimeter of the polygon is greater than the circumference of the circle, it follows that the area of the polygon is greater than the triangle K; which is impossible.

Therefore, the area of the circle is not less than K. Since then the area of the circle is neither greater nor less than K, it is equal to K.

3 Archimedes and the Spiral

For finding the area of his spiral, Archimedes needed a formula for the sum of the first n squares. Here is how he stated his result:

> If a series of any number of lines be given, which exceed one another by an equal amount, and the difference be equal to the least, and if other lines be given equal in number to these and in quantitiy to the greatest, the squares on the lines equal to the greatest, plus the square on the greatest and the rectangle contained by the least and the sum of all those exceeding one another by an equal amount will be the triplicate of all the squares on the lines exceeding one another by an equal amount.

What is this supposed to mean? First, in perfect Euclidean tradition, numbers are represented by lines. He starts with numbers that exceed one another by an equal amount, that is, with an arithmetic progression whose common difference equals the first term, say a, $2a$, $\ldots$, na. Then he considers a second sequence with an equal number of terms, but all equal to the greatest term: that is, na, na, $\ldots$, na (with n terms). Next "the squares on the lines equal to the greatest" is just $(na)^2+\ldots+(na)^2$, that is, $n\cdot(na)^2$. Then he adds the square of the greatest, getting $(n+1)\cdot(na)^2$. Now he adds "the rectangle contained by the least and the sum of all those exceeding one another by an equal amount", that is, the product of a (the least) and the sum of the first progression, that is $a + 2a + \ldots + na$, and says that this is equal to three times "all the squares on the lines exceeding one another by an equal amount", that is, $3[a^2 + (2a)^2 + \ldots + (na)^2]$.

In other words: the formula claimes by Archimedes, when translated into modern algebra, reads

$$(n+1)(na)^2 + a(a + 2a + \ldots + na) = 3[a^2 + (2a)^2 + \ldots + (na)^2].$$

Dividing through by a^2 shows that it is equivalent to

$$(n+1)n^2 + (1 + 2 + \ldots + n) = 3[1^2 + 2^2 + \ldots + n^2].$$

Using the fact that $1 + 2 + \ldots + n = \frac{n(n+1)}{2}$, we find

$$\begin{aligned} 1^2 + 2^2 + \ldots + n^2 &= \frac{1}{3}(n^3 + n^2 + \frac{n(n+1)}{2}) \\ &= \frac{2n^3 + 3n^2 + n}{6} = \frac{1}{6}n(n+1)(2n+1). \end{aligned}$$

Next we will give a translation of his proof. Define

$$\begin{aligned} S &= 1^2 + 2^2 + \ldots + n^2, \\ T &= 1 + 2 + \ldots + n. \end{aligned}$$

We know that $T = \frac{n(n+1)}{2}$. Now consider the following equations:

$$\begin{array}{rclcl} n^2 & = & (1 + (n-1))^2 & = & 1^2 + (n-1)^2 + 2 \cdot 1 \cdot (n-1) \\ & = & (2 + (n-2))^2 & = & 2^2 + (n-2)^2 + 2 \cdot 2 \cdot (n-2) \\ & = & (3 + (n-3))^2 & = & 3^2 + (n-3)^2 + 2 \cdot 3 \cdot (n-3) \\ & = & \ldots & & \\ & = & ((n-1) + 1)^2 & = & (n-1)^2 + 1^2 + 2 \cdot (n-1) \cdot 1 \end{array}$$

Adding these equations together and adding $2n^2 = n^2 + n^2$ gives

$$(n+1)n^2 = S + S + 2(n-1) + 4(n-2) + \ldots + (2n-2) \cdot 1 = 2S + Q,$$

where

$$Q = 2(n-1) + 4(n-2) + \ldots + (2n-2) \cdot 1.$$

Now

$$T + Q = n + 3(n-1) + 5(n-2) + \ldots + (2n-1) \cdot 1.$$

Next, Archimedes looks at the following equations

$$\begin{array}{rclcl} n^2 & = & n + (n-1)n & = & n + 2(1 + 2 + \ldots + n - 1) \\ (n-1)^2 & = & (n-1) + (n-2)(n-1) & = & n - 1 + 2(1 + 2 + \ldots + n - 2) \\ & \ldots & & & \\ 2^2 & = & 2 + 1 \cdot 2 & = & 2 + 2(1) \\ 1^2 & = & 1 & = & 1 + 1(0). \end{array}$$

Adding he gets $S = n + 3(n-1) + 5(n-2) + \ldots + (2n-1) \cdot 1 = T + Q$.

Subtracting this from $2S + Q = (n+1)n^2$ we get $2S - T = (n+1)n^2 - S$ or $3S = (n+1)n^2 + T$, from which

$$1^2 + 2^2 + \ldots + n^2 = \frac{1}{6}n(n+1)(2n+1)$$

follows with a little algebra.

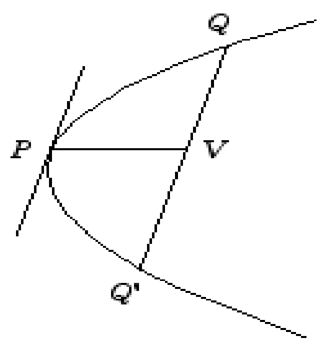
4 Archimedes and the Parabola

Sources:

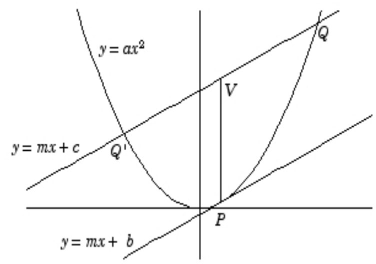
- http://jwilson.coe.uga.edu/EMT668/EMAT6680.F99/Erbas/emat6690/essay1/essay1.html
- http://cerebro.xu.edu/math/math147/02f/archimedes/archpartext.html

Archimedes starts out with several results on conics and parabolas; for proofs, Archimedes refers to the "elements of conics", apparently a standard reference, possible the lost *Conics* of Euclid. Since we do not have the time to prove these results geometrically, we will derive them using analytic geometry.

Proposition 1. *If from a point on a parabola a straight line be drawn which is either itself the axis or parallel to the axis, as PV, and if QQ' be a chord parallel to the tangent to the parabola at P and meeting PV in V, then $QV = VQ'$. Conversely, if $QV = VQ'$, the chord QQ' will be parallel to the tangent at P.*

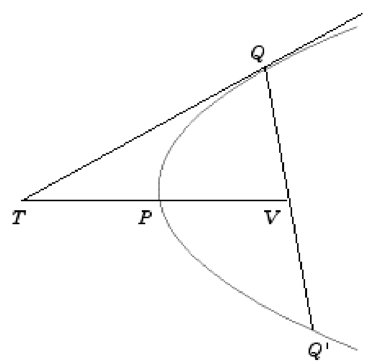


For our proof, we choose the coordinate system in such a way that the parabola has the y-axis as its axis of symmetry and $(0,0)$ as its vertex. Then it has the equation $y = ax^2$. Given $Q = (r, ar^2)$ and $Q' = (s, as^2)$, we get $V = (\frac{1}{2}(r+s), \frac{a}{2}(r^2+s^2))$. The slope of the line QQ' is $m = \frac{a(s^2-r^2)}{s-r} = a(r+s)$. This is the slope of the tangent at $x = \frac{1}{2}(r+s)$, and this proves the claim.



Proposition 2. *If in a parabola QQ' be a chord parallel to the tangent at P, and if a straight line be drawn through P which is either itself the axis or parallel to the axis, and which meets QQ' in V and the tangent at Q to the parabola in T, then $PV = PT$.*

Again, Archimedes refers to the Elements of Conics for a proof; this proposition is proved geometrically in the books of Appolonius.

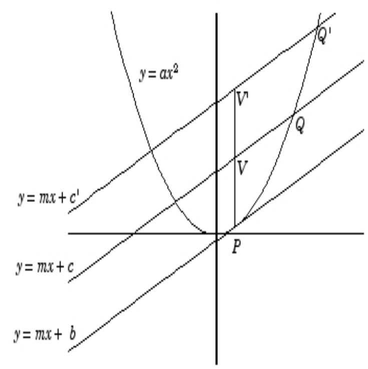


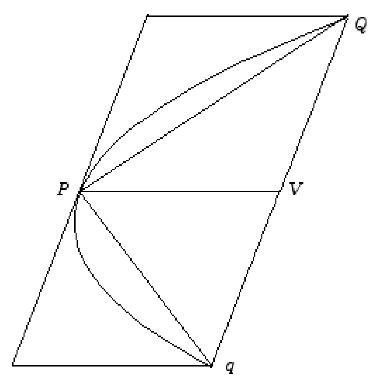
This time, let us use the parabola $y^2 = px$ for a change. Its tangent at $x = a$ has the equation $y = \frac{p}{2\sqrt{a}}(x - a) + p\sqrt{a}$. This line intersects the x-axis at $x = -a$.

Proposition 3. *If from a point on a parabola a straight line be drawn which is either itself the axis or parallel to the axis, as PV, and if from two other points Q, Q' on the parabola straight lines be drawn parallel to the tangent at P and meeting PV in V, V' respectively, then $PV : PV' = (QV)^2 : (Q'V')^2$.*

The proof is left as an exercise.

Proposition 20. *If Qq be the base, and P the vertex, of a parabolic segment, then the triangle PQq is greater than half the segment PQq.*





For the chord Qq is parallel to the tangent at P, and the triangle PQq is half the parallelogram formed by Qq, the tangent at P, and the diameters through Q, q. Therefore the triangle PQq is greater than half the segment.

Corollary. *It follows that it is possible to inscribe in the segment a polygon such that the segments leftover are together less than any assigned area.*

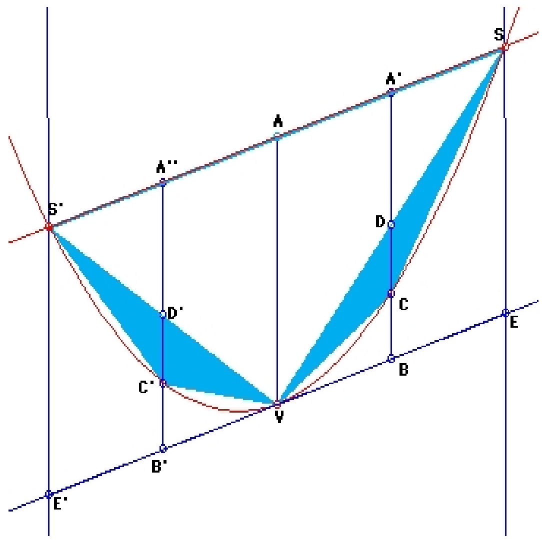
Consider the following figure:
We claim that

$$\text{area}(VCS) = \frac{1}{4}\text{area}(VAS). \tag{7}$$

Since BA' is parallel to the segment VA (and so to the segment ES) and B is the midpoint of VE, Proposition 3 implies

$$\frac{BC}{ES} = \frac{VB^2}{VE^2} = \frac{1}{4}.$$

The triangles VBD and VES are similar since $BD \parallel ES$ and $VE = 2VB$. Therefore, $BD : ES = VB : VE = 1/2$. Since $VESA$ is a paralleogram by construction, $BA' = ES$ and $BA' = 2BD$.



On the other hand, since $BC : 2BD = BC : BA' = 1/4$, we have $BD = 2BC$. In other words,

$$CD = \frac{1}{4}BA'.$$

Now the triangles VCA' and SCA' have the same altitude since $VESA$ is a parallelogram and B and A' are the midpoints of the sides VE and SA, respectively. On the other hand, since $DA' = 2CD$, we have

$$SA'D = 2SDC \quad \text{and} \quad VDA' = 2VCD.$$

Adding gives[1]

$$SA'V = SA'D + VDA' = 2(SDC + VCD).$$

Now $VAS = 2SA'V$ and $VCS = SDC + VDC$, hence

$$VCS = \frac{1}{4}VAS.$$

The Exhaustion

Given a segment of a parabola defined by the chord SS', construct the point V on the parabola in which the tangent is parallel to SS'. By Proposition 1, the line through V parallel to the axis of the parabola will intersect SS' in the midpoint of SS'. Now repeat this construction with the new segments defined by $S'V$ and VS; the areas of the new triangles are $\frac{1}{4}$ of the area A_1 of the original

[1] Thanks to Klaus Wulff for notifying me of a few missing apostrophes.

triangle $SS'V$. At the next step, the four new triangles added have area $\frac{1}{4}$ of the areas of the triangles constructed in the second step, that is, $\frac{1}{16}A_1$.

Now let the number of steps go to infinity (this limit argument is a proof by double contradiction in Archimedes); the difference between the area A of the segment and the areas of the triangles tends to 0 because of the corollary to Prop. 20, hence

$$\begin{aligned} A &= A_1 + \frac{1}{4}A_1 + \frac{1}{16}A_1 + \dots \\ &= A_1(1 + \frac{1}{4} + \frac{1}{16} + \dots) \\ &= \frac{4}{3}A_1. \end{aligned}$$