Some Problems of Diophantus

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It is believed that Diophantus worked around 250 AD; apart from this we think that he lived for 84 years, since a puzzle given by by Metrodorus around 500 AD says

> his boyhood lasted 1/6th of his life;
> he married after 1/7th more;
> his beard grew after 1/12th more,
> and his son was born 5 years later;
> the son lived to half his father's age,
> and the father died 4 years after the son.

This gives rise to a linear equation in Diophantus' age x (much simpler than anything Diophantus has done) with $x = 84$ as the solution.

Diophantus' main claim to fame rests on his book "Arithmetika", which consists of 13 parts. Six of them were known since Fermat's times, another four have been discovered in Arabic translation. In these books, Diophantus solves "indeterminate equations": a determinate equation is an equation that determines the solutions, like $x^2 - 2x + 3 = 0$; an example of an indeterminate equation is $x^2 + y^2 = 1$, which has many rational solutions.

Diophantus invented algebraic notation: he had a symbol for one unknown x, and other symbols for x^2, …, x^6. This bold step was unsurpassed until Viète (1540–1603) improved upon it by using vowels for unknown and consonants for known quantitites; Descartes later introduced the modern variant where unknowns are denoted by x, y, z and known quantities by a, b, c ….

The Problems

One of the most famous problems that Diophantus treated was writing a square as the sum of two squares (book II, problem 8):

> To divide a given square into a sum of two squares.

This problem became important when Fermat, in his copy of Diophantus' Arithmetica edited by Bachet, noted that he had this wonderful proof that cubes can't be written as a sum of two cubes, fourth powers not as a sum of two fourth powers, and so on, but that the margin of this book was too small to contain it. (This observation was later published by his son Samuel; Fermat publicly only claimed to have proofs for exponents 3 and 4).

Diophantus' Solution

Diophantus' solution proceeds as follows: since his notation allows only one unknown, he cannot treat the general case, so he starts by assuming that the given square is 16; letting x^2 denote one of the squares whose sum is 16, the other square must be $16 - x^2$. He writes this square in the form $(2x-4)^2$, and then has to solve $16 - x^2 = (2x-4)^2 = 4x^2 - 16x + 16$, which gives $5x^2 = 16x$, hence $5x = 16$ (Diophantus has no problems with canceling x since he doesn't know 0). Now $x = \frac{16}{5}$, so $2x - 4 = \frac{12}{5}$, and in fact $\frac{256}{25} + \frac{144}{25} = \frac{400}{25} = 16$.

Diophantus is aware of the fact that his method produces many more solutions: he writes

> I form the square from any number of x minus as many units as there are in the side of 16.

The side of 16 is the square root of 16 (think of 16 as the area of square), so in modern terms his statement means: instead of $2x - 4$, the substitution $mx - 4$ will produce a rational solution (of course m has to be chosen positive and rational).

Solution with modern algebraic notation

Now let us solve the problem $x^2 + y^2 = a^2$, where a is a given positive rational number, using modern notation. Diophantus' idea is to write $y = mx - a$ (for us, $mx + a$ would work equally well), which gives $a^2 = x^2 + m^2x^2 - 2amx + a^2$, that is, $x^2(1+m^2) = 2amx$. Thus either $x = 0$ or $x = \frac{2am}{1+m^2}$. Note that the substitution $y = mx + a$ would have given $x = -\frac{2am}{1+m^2}$, which, for a and m positive rational numbers, would produce negative values. Thus Diophantus had good reasons for picking $y = mx - a$ instead: he does not know negative numbers.

Modern Geometric Interpretation

It has been discovered at the beginning of the 20th century (the earliest instance I could find was a book of Klein; but there is a much more general theorem on 'curves of genus 0' containing all this and more by Hilbert and Hurwitz from the 1890's) that there is a geometric interpretation of the substitutions that Diophantus (and his 'modern students', like Fermat, Euler, Lagrange, Cauchy, Lucas, Sylvester etc.) was using.

Here's how: consider the points $(x, y) \in \mathbb{R} \times \mathbb{R}$ satisfying $x^2 + y^2 = a^2$; this circle has an obvious rational point, namely $P = (0, -a)$. Now consider the lines through this point P; their equations are $y = mx - a$ (with the exception of the tangent at P, which has 'slope ∞' and is described by $x = a$). Intersecting this line with the circle will give two points of intersection, one of them being – of course – P. Let's compute the second point: plugging the equation for the line into $x^2 + y^2 = a^2$ and solving for x gives $x^2(m^2 + 1) = 2amx$; the solution $x = 0$ corresponds to the original point $P = (0, -a)$, the x-coordinate of the second point of intersection is $x = a\frac{2m}{1+m^2}$. Plugging this into our equation $y = mx - a$ gives $y = a\frac{m^2-1}{m^2+1}$. Since a is rational, this means that whenever we plug in a rational number for m, these formulas gives us a rational point satisfying $x^2 + y^2 = a^2$. In fact, it is quite easy to see that conversely, every rational point $\neq P$ corresponds to some rational value of m.

This is a very neat 20th-century solution of Diophantus' problem, and techniques such as this one (actually, the techniques have become much mure sophisticated during the last 100 years) are today subsumed under the title 'arithmetic geometry'.

Another Problem

Consider the following problem:

> To add the same number to two given numbers so as to make each of them a square.

Again, lack of algebraic notation forces Diophantus to explain the solution using an example, and he picks the numbers 2 and 3 for this. Then he has to find a number x such that $2 + x$ and $3 + x$ are squares. Using modern notation, let us assume that $2+x = u^2$ and $3+x = v^2$; then $1 = 3-2 = v^2-u^2 = (v-u)(v+u)$. Now put $v + u = 4$; then $v - u = \frac{1}{4}$, and we find $x = \frac{97}{64}$, as well as $2 + x = \frac{225}{64} = (\frac{15}{8})^2$ and $3 + x = \frac{289}{64} = (\frac{17}{8})^2$.

Observe that the equation $1 = v^2 - u^2$ describes a hyperbola; modern mathematicians could use the technique of sweeping lines to determine all solutions of this (and related) diophantine equations.