HISTORY OF MATHEMATICS

MIDTERM 2

(1) (a) How did Archimedes express the formula $A = \pi r^2$ for the area of a circle?

He said that the area of a circle is equal to the area of a right angled triangle, where the sides forming the right angle have length r (radius) and C (circumference).

(b) Fermat and his contemporaries could compute the area bounded by the graph of $y = x^n$, the x-axis, and the line $x = a$ (in modern notation: $\int_0^a x^n dx$). Some of these mathematicians expressed this area in the same way Archimedes would have expressed it: how? (You probably do not know an answer, so you will have to make an educated guess).

We have $\int_0^a x^n dx = \frac{1}{n+1} a^{n+1}$. This is $\frac{1}{n+1}$ of the rectangle formed by the points $(0, 0)$, $(a, 0)$, (a, a^n) and $(0, a^n)$.

(2) The following is taken from Archimedes:

Given a series of magnitudes, each of which is equal to four times the next in order, all the magnitudes and one-third of the least added together will exceed the greatest one by one-third.

Translate this into a modern formula.

$$a + \frac{a}{4} + \frac{a}{16} + \ldots + \frac{a}{4^n} + \frac{a}{3 \cdot 4^n} = a + \frac{1}{3}a.$$

(3) This is Proposition 20 from Euclid's Book IX:

Prime numbers are more than any assigned multitude of prime numbers.

(a) How would we express this Proposition today?

There are infinitely many primes

(b) Why did Euclid express it in this strange way?

He wanted to avoid actual infinities. [Of course Euclid knew the concept of infinity: Aristotle, Zeno, Plato, Eudoxos etc. have discussed it at length. But since there were problems with actual infinities that the Greeks could not solve, they chose to use only potential infinity.]

(c) His proof starts as follows:

> Let A, B, and C be the assigned prime numbers. I say that there are more prime numbers than A, B, and C ...

Why did Euclid give the proof only for three primes?

Because he used a geometric language for multiplication; the product of two numbers was represented by an area, the product of three by a volume, and there was no geometric way of representing a product of four factors.

(4) List three classical unsolved problems of Greek mathematics, and explain (without going into details; keywords only) how and by whom they were finally solved.

- Squaring the circle: constructing a square with area equal to that of a given circle. Lindemann proved that this cannot be done using ruler and compass by showing that π is transcendental.
- Doubling the cube: constructing a cube with volume twice that of a given cube. This requires constructing $\sqrt[3]{2}$ with ruler and compass, which was shown to be impossible by Wantzel, who showed that $\sqrt[3]{2}$ cannot be written as 'nested square roots'.
- Trisecting the angle: dividing a given angle into three equal parts. Wantzel showed that e.g. and angle of 60^o cannot be trisected since $\cos 20^o$ satisfies an irreducible cubic polynomial, hence cannot be written as 'nested square roots'.

(5) Explain how to construct $\sqrt{\frac{5+\sqrt{5}}{2}}$ using the methods of Descartes.

No major problems here.

(6) Write a short essay on the discovery of the solution of cubic and quartic equations.

My impression was that at most 5 of you have read the article about Tartaglia's contest. Almost noone got the contributions of Bombelli right: there was a problem with cubics having three real solutions, since in this case Cardano's formulas involve complex numbers, as you proved in the next problem.

(7) Assume that $x = u + v$ is a real root of the cubic $x^3 + px + q$, where p, q are real numbers, and compute the coefficients * and ** in

$$x^3 + px + q = [x - (u + v)](x^2 + *x + **) \tag{1}$$

using long division.

We have $x^3+px+q = [x-(u+v)](x^2+(u+v)x+(p+(u+v)^2)$.

(a) Explain why the remainder is 0.

The remainder is $(u + v)^3 + p(u + v) + q$, which is 0 since $u + v$ is a root of $x^3 + px + q$.

(b) From the solution of the cubic we know that $p = -3uv$. Use this to compute the discriminant D of the quadratic factor in (1) in terms of u and v.

The quadratic has discriminant

$$b^2 - 4ac = (u + v)^2 - 4p - 4(u + v)^2 = -3(u - v)^2.$$

(c) Show that if u and v are real and distinct, then $D < 0$.

Obvious, since $(u - v)^2 \geq 0$.

(d) Show that if u and v are real and distinct, then the cubic has one real and two nonreal solutions.

Clear, since the quadratic has two nonreal roots in this case.

(e) Show that if the cubic has three distinct real roots, then u and v are nonreal complex numbers, and that $D > 0$ in this case.

If the cubic has three real roots, the quadratic factor must have two real roots, hence $D > 0$. But then u and v cannot be real by the above.

(8) (a) Which regular n-gons can be constructed using ruler and compass?

$n = 2^a P$, where P is a product of distinct Fermat primes $2^{2^m} + 1$. Thus, apart from factors of 2, we have $n = 3, 5, 15, 17, 51, 85, \ldots, 257, \ldots$.

(b) Which of these constructions were known to the Greeks?

Apart from factors of 2, they knew about $n = 3, 5, 15$.

(c) Who found the first construction of a regular n-gon not known to the Greeks, and when was that?

Gauss showed in 1801 how to construct a regular 17-gon with ruler and compass.

Note that Gauss and Wantzel showed that a regular 7-gon cannot be constructed using ruler and compass. Most of you claimed he constructed the 7-gon in this problem, and that he didn't in the next one.

(9) True or false:

	T	F
Pythagoras lived in Alexandria		x
28 is a perfect number	x	
Eudoxus invented the method of exhaustion	x	
For Euclid, a straight line was infinite		x
Eratosthenes computed the circumference of the earth	x	
Cardano studied medicine	x	
Complex numbers were invented so that the equation $x^2 + 1 = 0$ could be solved.		x
Bombelli published the book 'Ars Magna'		x
Gauss constructed the regular heptagon (7-gon) with ruler and compass.		x
The number π is transcendental	x	

Alexandria was founded by Alexander the great hundreds of years after Pythagoras died. 28 is the sum of its proper divisors, hence perfect. Euclid avoided actual infinites, straight lines for him were what we call line segments. Complex numbers came up in the solution of the cubic, not of the quadratic equation. Ars Magna was published by Cardano. Lindemann showed that π is transcendental, i.e., is not the root of a polynomial with rational coefficients.