HISTORY OF MATHEMATICS

FINAL EXAM

Name: January 12, 2004

(1) Find the rational solutions of the equation $X^2 + 5Y^2 = 6$.

An obvious solution is $(1,1)$. The lines through this point with rational slope t are $Y = t(X-1)+1$. Intersecting these lines with the ellipse gives

$$0 = X^2 + 5[t(X-1)+1]^2 - 6 = X^2 + 5t^2(X-1)^2 + 10t(X-1) - 1$$
$$= (X-1)[X+1+5t^2(X-1)+10t]$$

(Remark: if you have to solve a quadratic equation and already know one solution, then you deserve to run into technical problems when you apply the quadratic formula.)

The equation $X-1=0$ gives the known point $(1,1)$, the second equation $X+1+5t^2(X-1)+10t = 0$ leads to

$$X = \frac{5t^2-10t-1}{5t^2+1}, \quad Y = t(X-1)+1 = \frac{1-2t-5t^2}{5t^2+1}.$$

(2) Describe Archimedes' computation of the area of a parabolic segment (statement of the result; his technique of proof).

The statement: Given a segment AB of a parabola, construct a tangent to the parabola at some point C parallel to AB. Then the area of the segment is $\frac{4}{3}$ that of the triangle.

The proof proceeds by exhaustion: the triangle defines two new segments, for which we construct two new small triangles as above. At each step, the sum of the areas of the newly constructed triangles is $\frac{1}{4}$ of the area of the triangles constructed in the step before. Taking limits, the area of the segment is the area of the first triangle multiplied by $1 + \frac{1}{4} + \frac{1}{4^2} + \ldots = \frac{4}{3}$.

Instead of using limits, Archimedes of course used the method of double contradiction.

(3) Explain Eratosthenes' idea for estimating the circumference of the earth.

He measured the angle of the shadow of a stick in some city A at a time when a vertical stick in city B cast no shadow. Assuming that the sun is very far away, the angle of the shadow and the angle AOB, where O is the center of the earth, are equal. Since Eratosthenes knew the distance AB (approximately), he could compute the circumference of the earth by simple geometry.

(4) List the names and authors of three important mathematical books published between 1200 and 1700.
- Fibonacci (Leonardo di Pisa), Liber abaci, 1202
- Cardano, Ars Magna, around 1545
- Descartes, Geometrie, 1637

(5) Shortly describe what (if anything) the following cultures and mathematicians contributed to the solution of polynomial equations:
- Babylonians: could solve quadratic equations with at least one positive root; they only knew how to find one solution, even if two existed.
- Chinese: numerical solution of cubic equations.
- Arabs: geometric solution of cubic equations.
- del Ferro: algebraic solution of certain cubics.
- Tartaglia: algebraic solution of cubics with one real root.
- Cardano: algebraic solution of cubics with one real root; provided proofs and published them in the Ars Magna.
- Ferrari: algebraic solution of quartic polynomials.
- Bombelli: algebraic solution of general cubics, even if they had three real roots (casus irreduciblis; this required using complex numbers). [I forgot him -(]
- Ruffini: proved that the general quintic cannot be solved using radicals. His proofs were not accepted (except by Cauchy).
- Abel: proved that the general quintic cannot be solved using radicals.
- Galois: proved that a polynomial f of degree n can be solved using radicals if and only if the Galois group of f is solvable.

(6) Why is non-Euclidean geometry important even if you're only interested in Euclidean geometry?

Because the existence of (consistent) geometries satisfying the first four, but not the fifth of Euclid's axioms, proves that the parallel axiom cannot be deduced from the other four axioms.

(7) Description of Presentation.

(8) The following is taken from a letter from Sophie Germain to the French mathematician Libri. Identify the student she is talking about:

> Finally, that student , who in spite of his impertinence displays a good disposition, has managed to be expelled from the Ecole Normale. He is without fortune and his mother has very little. He continues the injurious behavior of which he gave you a sample after your best lecture at the Academy. The poor woman has left her house, leaving him enough to live in a mediocre way, and has been forced to place herself as a *dame de compagnie*. One hears that he is becoming totally insane and I believe it.

The student apparently is French, got himself expelled from his school, and was on the border between genius and insanity: his name was Evariste Galois.

(9) The following is from a booklet on Diophantus by Bashmakova:

> "To divide a given number which is the sum of two squares into two other squares"
>
> Diophantus gives the number 13, which is equal to the sum $4+9$. Thus one solution, namely $(2,3)$, is already known. To find a second solution, Diophantus takes as the first number $x = t+2$ and as the second number $y = 2t-3$. In other words, he draws a straight line through the point $(2,-3)$ and notes, as before, that instead of the multiplier 2 one can choose any other number.
>
> It is worth noting that Diophantus takes as the known point not the point with positive coordinates, [...] but a point with a negative coordinate. [...] In general, in intermediate computations Diophantus gladly operates with negative numbers.

Explain why you do (or do not) agree with Bashamkova's analysis.

Coordinates were invented by Descartes and Fermat. While Diophantus used some substitution $y = 2(x-2)-3$, he did not interpret this geometrically as a line through $(2,-3)$. In particular, there are no negative numbers in Diophantus' text.

(10) Consider the factorization

$$\frac{\sin x}{x} = \Big(1 - \frac{x^2}{\pi^2}\Big)\Big(1 - \frac{x^2}{4\pi^2}\Big)\Big(1 - \frac{x^2}{9\pi^2}\Big)\cdots. \tag{1}$$

(a) How did Euler (at first) justify this equation?

He simply observed that both sides have the same zeros and the same value at $x = 0$, namely 1. For polynomials, this is sufficient to prove equality. For functions in general, it is not: the function $\frac{\sin x}{x}e^x$ has the same zeros as $\frac{\sin x}{x}$ and the same value at $x = 0$, but comparing the coefficients of x^2 would yield nonsense.

(b) The left hand side of (1) is known to be equal to

$$1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \ldots.$$

Compare the coefficients of x^2 on both sides and deduce that

$$1 + \frac{1}{4} + \frac{1}{9} + \ldots = \frac{\pi^2}{6}.$$

We know $(1+ax)(1+bx)(1+cx)\cdots = 1 + (a+b+c+\ldots)x + \ldots$, so the coefficient of x^2 of the power series on the right hand side of (1) is $-\frac{1}{\pi^2} - \frac{1}{4\pi^2} - \frac{1}{9\pi^2} - \ldots$, the coefficient of x^2 of the power series on the left hand side of (1) is $-\frac{1}{6}$. The claim follows.

(c) What do you get by comparing the coefficients of x^4 in (1) ?

We have $(1-ax)(1-bx)(1-cx)\cdots = 1-(a+b+c+\cdots)x+(ab+ac+bc+\ldots)x^2+\ldots$. Going on with this formula gives a correct, but not very beautiful formula. We'll get something nice if we observe that

$$2(ab+ac+bc+\ldots) = (a+b+c+\ldots)^2 - (a^2+b^2+c^2+\ldots).$$

In fact, comparing coefficients of x^4 in (1) now shows that

$$\frac{1}{120} = \frac{1}{2}\Big(\frac{1}{\pi^4}\Big[1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\ldots\Big]^2 - \Big[1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\ldots\Big]\Big),$$

which shows, after a little calculation, that

$$1+\frac{1}{2^4}+\frac{1}{3^4}+\ldots = \frac{\pi^4}{90}.$$

(11) The following is taken from a famous book:

Know then that in every equation there are as many distinct roots, that is, values of the unknown quantity, as is the number of dimensions of the unknown quantity.

Suppose, for example, x equal to 2, or $x-2$ equal to nothing, and again, x equal to 3, or $x-3$ equal to nothing. Multiplying together the two equations we have $xx-5x+6$ equal to nothing, or xx equal to $5x-6$. This is an equation in which x has the value 2 and at the same time has the value 3 ...

It often happens, however, that some roots are false, or less than nothing. Thus, if we suppose x to represent the defect of a quantity 5, we have $x+5$ equal to nothing which, multiplied by $x^3-9xx+26x-24$, gives $x^4-4x^3-19xx+106x-120$ equal to nothing, as an equation having four roots, namely three true roots, 2, 3, 4, and one false root, 5.

(a) Who wrote this?
(b) How is the theorem he is explaining called today?
(c) What is a false root?

(a) This is by Descartes. He was the one with false and true roots.
(b) The theorem he is explaining is the Fundamental Theorem of Algebra.
(c) A false root is a negative root.