AREA AND CIRCUMFERENCE OF CIRCLES

Having seen how to compute the area of triangles, parallelograms, and polygons, we are now going to consider curved regions. Areas of polygons can be found by first dissecting them into squares or triangles and then adding up the areas of each of the individual pieces. But not every region can be so dissected; for example, a ***disk*** (i.e., a bounded region in the plane whose circumference is a circle) cannot be divided into (a finite number of) polygons. The basic idea, now, is to find the exact area of a disk by approximating it by the areas of inscribed polygons. We will give more details below, but the reader should now realize why the discussion of the area of a disk will have a different flavor than that of polygonal areas.

Let us begin by seeing how one might have discovered the formula for the area of a disk. Let D be a disk with radius r and circumference c. If $a(D)$ denotes the area of D, then our goal is to find the usual area formula

$$a(D) = \frac{1}{2}cr$$

(if $c = 2\pi r$, then $\frac{1}{2}cr = \pi r^2$).

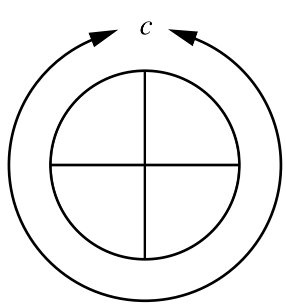

Figure 1:

Divide D into 4 equal sectors and rearrange them in a row:

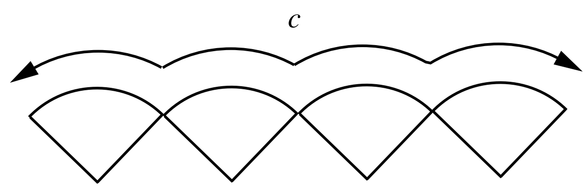

Figure 2:

Of course, the area remains unchanged, and the total length of the top 4 arcs is still c. Now double the area by adding 4 shaded sectors.

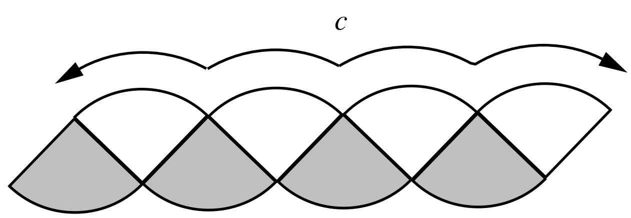

Figure 3:

If we denote $a(D)$ by A, then the area of this new figure is $2A$. It looks a bit like a rectangle: each of the scalloped top and bottom edges has length c, and each of the two side edges has length r (for they are radii of D). Now divide D into more equal sectors, and rearrange them in the same way.

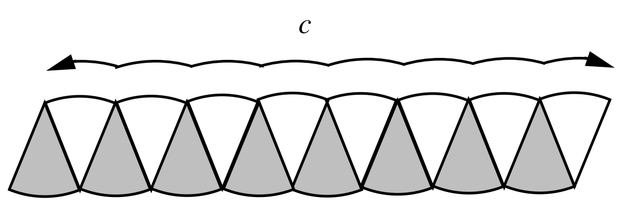

Figure 4:

The rearrangement, which now looks more like a rectangle, still has area $2A$, top and bottom of length c, and sides of length r. If we divide D into a larger number of equal sectors, then the rearranged figure is “almost” a rectangle with sides of lengths c and r. There are two ways to compute the area of this rectangle. On the one hand, it is $2A$; on the other hand, its area is cr. Thus, $2A = cr$, and so $A = \frac{1}{2}cr$. This formula is reminiscent of the formula for the area of a triangle, and it was so viewed in ancient times.

There are two flaws in this preliminary discussion. The circumference c must be computed, and the formula $c = 2\pi r$ is not so easy to prove. But the obvious defect in our preamble is the passage from an "almost" rectangle to an honest rectangle. The modern way to deal with approximations, using limits, was introduced by Newton (1642–1727) and Leibniz (1646–1716), independently, in the late seventeenth century; it is the fundamental new idea in calculus. The basic idea of limit is to use approximations to a number A by a sequence of "simpler" numbers to get exact information about A. For example, we will approximate the area of a circle by areas of inscribed polygons.

Before we continue, we recall how to manipulate inequalities.

Theorem 0.1. *Assume that $b < B$ are real numbers.*

(i) *If m is positive, then $mb < mB$, whereas if m is negative, then $mb > mB$.*

(ii) *For any number N, positive, negative, or zero, we have*

$$N + b < N + B \qquad \textit{and} \qquad N - b > N - B.$$

(iii) *Let c and d be positive numbers. If $d < c$, then $1/d > 1/c$, and, conversely, if $1/c < 1/d$, then $c > d$.*

Perhaps the first successful study of limits was done by Eudoxus (ca. 400–347 BC), who enunciated a principle, the *method of exhaustion*, that can be found in Euclid's *Elements*. The basic, quite reasonable, assumption made by Eudoxus is, for any positive number A, that the sequence $\frac{1}{2}A, \frac{1}{4}A, \ldots, (\frac{1}{2})^n A, \ldots$ becomes arbitrarily small.

Here is a modern paraphrase version of this classical Greek notion.

Definition. Given a positive number A and an increasing sequence of positive numbers $k_1 < k_2 < \cdots < k_n < \cdots < A$, then we say that k_n ***converges to*** A, denoted by $k_n \to A$, if $A - k_n < (\frac{1}{2})^n A$ for all $n \geq 1$.

We call the next statement the ***Method of Exhaustion***.

Definition. If $k_n \to A$ and $B < A$, then there is k_ℓ with $B < k_\ell < A$.

The next criterion for approximation may be easier to use than the definition.

Lemma 0.2. *Let A be a positive number, and let $k_1 < k_2 < k_3 < \cdots < A$ be an increasing sequence. If $A - k_1 < \frac{1}{2}A$ and, for every $n \geq 1$,*

$$A - k_{n+1} < \tfrac{1}{2}(A - k_n),$$

then $k_n \to A$.

Proof. We prove by induction on $n \geq 1$ that $A - k_n < (\frac{1}{2})^n A$.

The base step is the given inequality $A - k_1 < \frac{1}{2}A$. Let us prove the inductive step. We are assuming that $A - k_{n+1} < \frac{1}{2}(A - k_n)$; the inductive hypothesis $A - k_n < (\frac{1}{2})^n A$ now gives

$$A - k_{n+1} < \tfrac{1}{2}(A - k_n) < \tfrac{1}{2}\left[(\tfrac{1}{2})^n A\right] = (\tfrac{1}{2})^{n+1} A. \quad \bullet$$

Let $B = .99999\cdots$ be the number whose decimal expansion is all 9s.

Proposition 0.3. *If* $B = .99999\cdots$, *then* $B = 1$.

Proof. It is plain that $B \le 1$, so that either $B < 1$ or $B = 1$. Our strategy is to eliminate the first possibility.

We show that the increasing sequence $k_1 = 0.9 < k_2 = 0.99 < k_3 = 0.999 < \ldots < 1$ converges to 1. Note that $k_n = 1 - (\frac{1}{10})^n$ for all $n \ge 1$. Hence,

$$1 - k_n = 1 - [1 - (\tfrac{1}{10})^n] = (\tfrac{1}{10})^n < (\tfrac{1}{2})^n.$$

Therefore, $k_n \to 1$.

If $B < 1$, then the method of exhaustion says that there is some k_ℓ with $B < k_\ell$; that is, there is some $k_\ell = 0.9\cdots 90$ (there are ℓ 9s) with $B < k_\ell$. But $B - k_\ell = .0\cdots 09\cdots > 0$, so that $B > k_\ell$, a contradiction . We have eliminated the possibility $B < 1$, and the only remaining option is $B = 1$. $\bullet$

This indirect proof is called ***reductio ad absurdum***.

We are now going to use the method of exhaustion to prove that if D and D' are disks with radius r and r', respectively, then $a(D')/a(D) = r'^2/r^2$; it will then follow easily that $a(D) = \pi r^2$. The proof we give is the proof given in Book XII of Euclid's *Elements*.

The proof of the area formula for a disk will use the fact that areas of inscribed polygons approximate the area of the disk from below (to prove the circumference formula, we will also need the fact that areas of circumscribed polygons approximate the area of the disk from above).

Let D be a disk of radius r, and let P_1 be a square inscribed in D. Bisect all the sides of P_1 to get an inscribed regular octagon P_2, as in Figure 5.

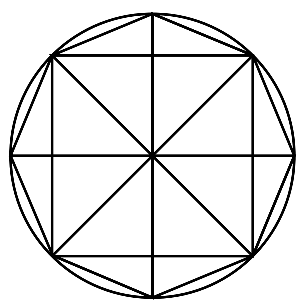

Figure 5:

Continue this process, so that there is a growing sequence of inscribed regular polygons $P_1, P_2, P_3, \ldots$ inside of D. Note that P_1 has 4 sides, P_2 has 8 sides, and more generally, P_n is a regular polygon having 2^{n+1} sides.

Theorem 0.4. *$a(P_n) \to a(D)$, where $a(P_n)$ and $a(D)$ denote the area of P_n and D, respectively.*

Proof. We use Lemma 0.4.

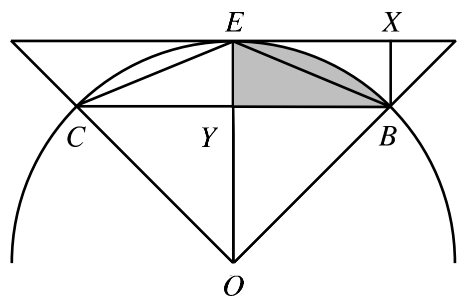

Figure 6:

It is clear that $a(P_n) < a(D)$ for all n, for each polygon P_n is inscribed in D and, hence, has smaller area than D. Moreover, the sequence of areas is an increasing sequence, for $a(P_n) < a(P_{n+1})$ because P_n is inside of P_{n+1}.

If Q_1 is a circumscribed square, then it is easy to see that $2a(P_1) = a(Q_1) > a(D)$, so that $a(P_1) = \frac{1}{2}a(Q_1) > \frac{1}{2}a(D)$. Hence, $a(D) - a(P_1) < a(D) - \frac{1}{2}(D) = \frac{1}{2}a(D)$.

It remains to check the inequalities $A - a(P_{n+1}) < \frac{1}{2}[A - a(P_n)]$ for all $n \geq 1$. We first describe this inequality geometrically, using Figure 6. Let CB be a side of the 2^{n+1}-gon P_n, and let CE and EB be sides of the 2^{n+2}-gon P_{n+1}. On each of the 2^{n+1} edges of P_n, consider the replica of the region bounded by CB and the arc from C to B through E. We focus on Z_n, the shaded half of this region, which is bounded by YB, YE, and the arc from E to B. Thus, if $k_n = a(P_n)$, then there being two copies of Z_n on each of the 2^{n+1} edges of P_n gives

$$A - k_n = 2^{n+1}[2 \times a(Z_n)] = 2^{n+2}a(Z_n).$$

Similarly, if $k_{n+1} = a(P_{n+1})$, then

$$A - k_{n+1} = 2^{n+2}a(Y_{n+1}),$$

where Y_{n+1} is bounded by the line EB and the arc from E to B. Therefore, to see that $A - k_{n+1} < \frac{1}{2}(A - k_n)$, it suffices to show that $a(Y_{n+1}) < \frac{1}{2}a(Z_n)$. If this inequality holds, then

$$A - k_{n+1} = 2^{n+2}a(Y_{n+1}) < \tfrac{1}{2}2^{n+2}a(Z_n) = \tfrac{1}{2}(A - k_n).$$

Now that the needed inequality has been described geometrically, we prove that it holds.

$$a(Z_n) < a(\square BYEX) = 2a(\Delta BYE),$$

so that $\frac{1}{2}a(Z_n) < a(\Delta BYE)$. Hence, $a(Z_n) = a(Y_{n+1}) + a(\Delta BYE) > a(Y_{n+1}) + \frac{1}{2}a(Z_n)$.
Subtracting $\frac{1}{2}a(Z_n)$ from both sides gives $\frac{1}{2}a(Z_n) > a(Y_{n+1})$. We have proved that $a(P_n) \to a(D)$. •

We now prepare for an application of the method of exhaustion.

Lemma 0.5. *Let D and D' be disks with radius r and r', respectively. If P is a regular n-gon inscribed in D, and if P' is a similar polygon inscribed in D', then*

$$\frac{a(P')}{a(P)} = \frac{r'^2}{r^2}.$$

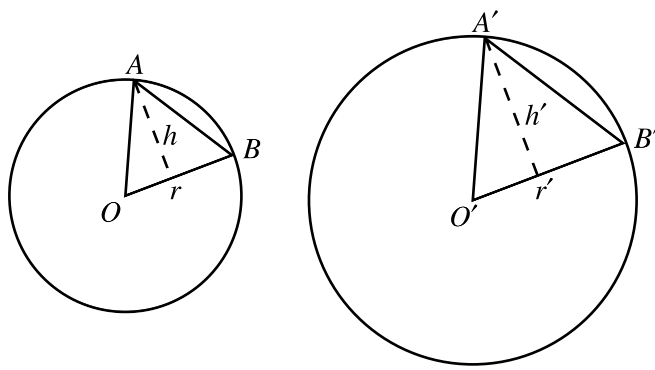

Figure 7:

Proof. Let us first consider a triangle T having one vertex at the center O of the disk D and the other two vertices on the circle, and let T' be a similar triangle in D'. Let the height of T from A to the radius OB be denoted by h, and let the height of T' from A' to the radius $O'B'$ be denoted by h'. Because the triangles T and T' are similar, $h/r = h'/r'$, so that $h' = hr'/r$. Therefore,

$$\begin{aligned}\frac{a(T')}{a(T)} &= \frac{\frac{1}{2}h'r'}{\frac{1}{2}hr}\\ &= \frac{\frac{1}{2}(hr'/r)r'}{\frac{1}{2}hr}\\ &= \frac{r'^2}{r^2}.\end{aligned}$$

We have shown that $a(T') = (r'^2/r^2)a(T)$.

Now dissect P into n triangles congruent to T, one for each side of P, and do a similar dissection of P' into triangles T'. Since all the triangles dissecting P are congruent to T, we have $a(P) = na(T)$; similarly, $a(P') = na(T')$. Therefore,

$$\frac{a(P')}{a(P)} = \frac{na(T')}{na(T)} = \frac{r'^2}{r^2}. \quad \bullet$$

We give two proofs of the next theorem, which is the heart of the area formula. The first, more geometric, proof is essentially the ingenious one given by Eudoxus; the second, more algebraic, proof is more in the modern spirit.

Theorem 0.6. *If D and D' are disks with radius r and r', respectively, then*

$$\frac{a(D')}{a(D)} = \frac{r'^2}{r^2}.$$

Proof. Let us denote $a(D)$ by A and $a(D')$ by A'. If, on the contrary, $A'/A \neq r'^2/r^2$, then either $A'/A < r'^2/r^2$ or $A'/A > r'^2/r^2$. In the first case, there is thus some number M with $A'/M = r'^2/r^2$ (M has no obvious geometric interpretation), and $A'/A < A'/M$. Multiplying both sides by $1/A'$ now gives $1/A < 1/M$, and so $M < A$.

We have seen that $a(P_n) \to a(D)$, where P_n is the inscribed regular 2^{n+1}-gon we constructed. Because $M < A$, the method of exhaustion says that there is some inscribed polygon P_ℓ with $M < a(P_\ell)$. Let P'_ℓ be the corresponding polygon in D'. Now

$$\frac{a(P'_\ell)}{a(P_\ell)} = \frac{r'^2}{r^2}.$$

But $A'/M = r'^2/r^2$, so that $a(P'_\ell)/a(P_\ell) = A'/M$. Hence

$$\frac{a(P'_\ell)}{A'} = \frac{a(P_\ell)}{M}.$$

The left side is smaller than 1 [because P'_ℓ is inside of the disk D', hence $a(P'_\ell) < a(D') = A'$], whereas the right side is greater than 1 [for P_ℓ was chosen so that $M < a(P_\ell)$], and this is a contradiction.

The other possibility $A'/A > r'^2/r^2$ also leads to a contradiction (one merely switches the roles of D and D'). There is some number M' with $M'/A = r'^2/r^2$, so that $A'/A > M'/A$. Multiplying both sides by A gives $A' > M'$, and one can now repeat the argument above beginning with polygons in D' instead of in D. $\bullet$

The technique of proving equality of numbers a and b by showing that each assumption $a < b$ and $a > b$ leads to a contradiction is called ***double reduction ad absurdum***.

Notice that the key idea was to replace the limiting disks D and D' by the approximating polygons P_ℓ and P'_ℓ. This avoidance of the limiting figures if often called the ***horror of the infinite***.

Here is a more modern proof.

Theorem 0.7. *If D and D' are disks with radius r and r', respectively, then*

$$\frac{a(D')}{a(D)} = \frac{r'^2}{r^2}.$$

Proof. We know that $a(P_n) \to a(D)$, and that $a(P_n')/a(P_n) = r'^2/r^2$ for all $n \geq 1$. Hence, $(r'^2/r^2)a(P_n) \to (r'^2/r^2)a(D)$. But $r'^2/r^2 a(P_n) = a(P_n')$, so that

$$\lim r'^2/r^2 a(P_n) = \lim r'^2/r^2 a(P_n') = a(D').$$

Therefore, $a(D') = r'^2/r^2 a(D)$, for a convergent sequence has exactly one limit. •

The familiar definition of π as circumference/diameter assumes something about π that is not at all obvious, namely, that this ratio is the same for all disks: If D and D' are disks with circumferences c and c' and diameters d and d', why is $c'/d' = c/d$? The definition of π given below avoids this question; the area of the unit disk is one specific number.

Definition. The number π is the area of the unit disk; that is, π is the area of a disk with radius 1.

Corollary 0.8. *If D is a disk with radius r, then*

$$a(D) = \pi r^2.$$

Proof. If D' is the unit disk, then

$$\tfrac{1}{2}ca(D')a(D) = \tfrac{1}{2}cr'^2 r^2 = \tfrac{1}{2}c1r^2.$$

Since $a(D') = \pi$, we have $a(D) = \pi r^2$, as desired. •

We feel superior to the Greeks because, nowadays, the area formula can be derived routinely using calculus. But let us see whether we have a right to be so smug. The area of a disk of radius r is given by the definite integral

$$A = 2\int_{-r}^{r} \sqrt{r^2 - x^2}\, dx.$$

Using the substitution: $x = r\sin\theta$, so that $dx = r\cos\theta d\theta$, we see that the indefinite integral is

$$2\int \sqrt{r^2 - r^2\sin^2\theta}\; r\cos\theta d\,\theta = 2\int r^2\cos^2\theta d\,\theta;$$

one now uses the double angle formula, $\cos^2\theta = \frac{1}{2}(1 - \cos 2\theta)$, to obtain

$$\int r^2(1 - \cos 2\theta)\, d\theta = r^2(\theta - \tfrac{1}{2}\sin 2\theta).$$

To evaluate the original definite integral, we must find the new limits of integration: as x varies over the interval $[-r, r]$, θ varies over the interval $[0, \pi]$. But why is that? For us, π means the area of the unit disk; we have not yet established any connection between the area of a circle and its circumference. It follows that it is not yet legitimate to use radian measure, and so it is premature to use limits of integration for the new definite integral that mention π. Perhaps we should not feel so superior to the Greeks!

We now set the stage for establishing the formula $c = 2\pi r$ for the circumference c of a circle of radius r. The proof of this formula was first given by Archimedes (287–212 BC), one of the greatest scientists of antiquity, in the century after Euclid. Like the area formula just proven, it, too, is a formula often quoted but rarely proved in high schools.

Let Q_1 denote a circumscribed square, so that its sides are tangent to D. Define Q_2 to be the circumscribed octagon which is constructed by "cutting off corners" of Q_1. In more detail, referring to Figure 8, draw the line ℓ joining the center O of the disk to a vertex F of Q_1; ℓ cuts the circle in the point M, and the line HK, tangent to the circle at M, is defined to be one of the sides of Q_2; throw away ΔFHK from Q_1. If one repeats this procedure at the other vertices of Q_1, one has constructed Q_2 by throwing away four triangles (corners) from Q_1, one triangle from each vertex of Q_1. There are 8 sides: HK; RS; TU; VW, and the 4 remnants of the original sides of Q_1, namely, KR; ST; UV; WH.

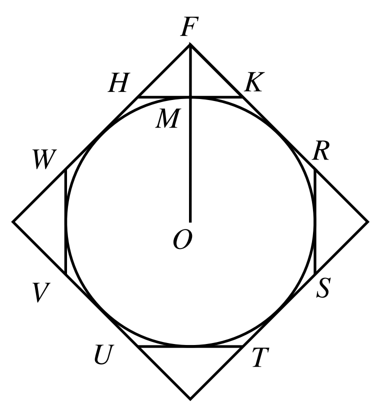

Figure 8:

This construction can be repeated. Given a circumscribed polygon Q_n, which has 2^{n+1} sides of equal length, construct Q_{n+1} by connecting O to each of the 2^{n+1} vertices of Q_n, and then throwing away corners of Q_n in the same way as in the construction of Q_2 from Q_1.

Figure 9 shows a portion of Q_n and the smaller Q_{n+1};

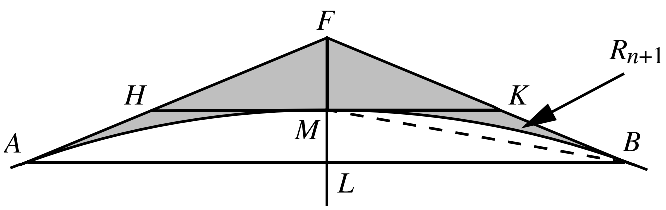

Figure 9:

AF and FB are remnants of sides of Q_n, tangent to D at points A and B, respectively. The point M is the midpoint of the arc joining A and B, and the tangent HK to the circle at M is one of the sides of Q_{n+1}.

We now generalize the Eudoxus definition of convergence so that it applies to decreasing sequences.

Definition. Given a positive number A and an decreasing sequence of positive numbers $K_1 > K_2 > \cdots < K_n < \cdots > A$, then we say that K_n ***converges to*** A, denoted by $K_n \to A$, if $K_n - A < (\frac{1}{2})^n K_1$ for all $n \geq 1$.

Here is the decreasing version of the ***Method of Exhaustion***.

Definition. If $K_n \to A$ and $B > A$, then there is K_m with $B > K_m > A$.

Lemma 0.9. *Let A be a positive number, and let $K_1 > K_2 > K_3 > \cdots > A$ be a decreasing sequence. If $K_1 - A > \frac{1}{2}K_1$ and, for every $n \geq 1$,*

$$K_{n+1} - A > \tfrac{1}{2}(K_n - A),$$

then $K_n \to A$.

Proof. Similar to the proof of Lemma 0.2. •

Theorem 0.10. *The decreasing sequence $a(Q_1), a(Q_2), a(Q_3), \ldots$ converges to $a(D)$.*

Proof. First, it is clear that $a(Q_n) > a(D)$ for all $n \geq 1$ because D is inside of each Q_n. Moreover, the sequence of areas is decreasing, $a(Q_n) > a(Q_{n+1})$ for all $n \geq 1$ because Q_n contains Q_{n+1}. Second, an easy argument using inscribed and circumscribed squares gives $a(D) > a(P_1) = \frac{1}{2}a(Q_1)$, so that $2a(D) > a(Q_1)$. It now follows, from Theorem 0.1, that

$$a(Q_1) - a(D) < a(Q_1) - \tfrac{1}{2}a(Q_1) = \tfrac{1}{2}a(Q_1).$$

Before we check the last criterion in Lemma 0.9, let us describe $a(Q_n) - a(D)$ geometrically. In Figure 9, let R_n denote the area of the shaded region bounded by AF, FB, and the arc joining AB. There are 2^{n+1} regions congruent to R_n, one for each of the vertices of Q_n, and

$$a(Q_n) - a(D) = 2^{n+1} a(R_n).$$

Of course,

$$a(Q_{n+1}) - a(D) = 2^{n+2} a(R_{n+1}),$$

where R_{n+1} is the smaller area bounded by MK, KB, and the arc joining M and B.

Bisect everything with the line LF, the perpendicular-bisector of AB (which joins F to the center of D), and let Z_n be the right side of R_n. Notice that R_n contains two copies of R_{n+1}, so that one copy of R_{n+1} lies inside of Z_n. Thus,

$$a(Q_n) - a(D) = 2^{n+1} a(R_n) = 2^{n+2} a(Z_n),$$

for all $n \geq 1$. It suffices to show that $a(R_{n+1}) < \frac{1}{2} a(Z_n)$, for then

$$\begin{aligned} a(Q_{n+1}) - a(D) &= 2^{n+2} a(R_{n+1}) \\ &< \tfrac{1}{2} 2^{n+2} a(Z_n) \\ &= \tfrac{1}{2}[a(Q_n) - a(D)]. \end{aligned}$$

Let us return to the proof. We begin by showing that

$$a(\Delta BKM) < a(\Delta FKM).$$

Because LM is the altitude of ΔBKM to the base KM,

$$\begin{aligned} \tfrac{1}{2} c a(\Delta BKM) a(\Delta FKM) &= \tfrac{1}{2} c \tfrac{1}{2} |LM||KM| \tfrac{1}{2} |KM||FM| = \tfrac{1}{2} c |LM||FM| \\ &= \tfrac{1}{2} c |FL| - |FM||FM| = \tfrac{1}{2} c |FL||FM| - 1. \end{aligned}$$

By similarity of the triangles ΔBFL and ΔFKM,

$$\tfrac{1}{2} c |FL||FM| - 1 = \tfrac{1}{2} c |BF||FK| - 1 = \tfrac{1}{2} c |BF| - |FK||FK = \tfrac{1}{2} c |BK||FK|.$$

But the construction of Q_{n+1} from Q_n gives $|BK| = |KM|$. Therefore,

$$\tfrac{1}{2} c |BK||FK| = \tfrac{1}{2} c |KM||FK| < 1,$$

because the hypotenuse FK of the right triangle ΔFKM is longer than the leg KM. Now

$$a(R_{n+1}) < a(\Delta BKM) < a(\Delta FKM),$$

the last inequality having just been proved. Therefore,

$$2a(R_{n+1}) < a(R_{n+1}) + a(\Delta FKM) = a(Z_n),$$

and so

$$a(R_{n+1}) < \tfrac{1}{2} a(Z_n). \quad \bullet$$

Let c denote the circumference of D, let p_n denote the perimeter of P_n, and let q_n denote the perimeter of Q_n. In Figure 9, it is obvious that

$$p_n < c; \tag{1}$$

after all, p_n is made up of the sides of P_n of the form AB, and if α is the arc from A to B through M, then $|AB| < \text{length}(\alpha)$, because a straight line is the shortest path between two points. It is not so obvious that $q_n > c$. Why is $|AF| + |FB| > \text{length}(\alpha)$? Archimedes focused on a special property of curves like arcs of circles, namely, the chord joining any two points on a circle lies inside the disk.

Definition. Let α be a curve joining a pair of points A and B. We say that α is ***concave*** with respect to AB if every chord UV, where U and V are points on α, lies inside the region bounded by α and AB.

Figure 10 shows a concave curve, whereas Figure 11 gives an example of a curve that is not concave.

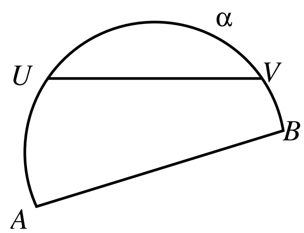

Figure 10:

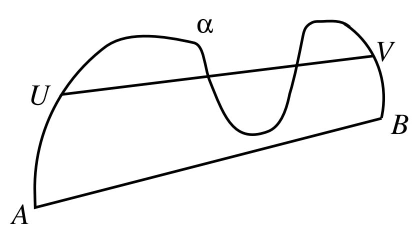

Figure 11:

Concavity Principle (Archimedes). Let α be a curve joining points A and B that is concave with respect to AB. If β is another concave curve joining A and B that lies inside the region bounded by α and AB, then

$$\text{length}(\alpha) > \text{length}(\beta).$$

We are going to use this principle in the special case when α is a path consisting of 2 edges and β is an arc of a circle, as in Figure 12.

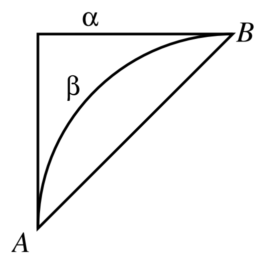

Figure 12:

As with the method of exhaustion enunciated by Eudoxus, the Concavity Principle of Archimedes was not proved in classical times (it requires an analysis of what one means by the length of a curve, defined nowadays using the arclength formula of calculus). It is remarkable how these ancient thinkers were able to state exactly what was needed to give a coherent proof.

Before proving the perimeter formula, let us show that the Concavity Principle is plausible. Suppose that α is a (necessarily concave) 2-edged path and that the inside path β also consists of 2 edges, as in Figure 13. We claim that $|AC| + |CB| > |AD| + |DB|$.

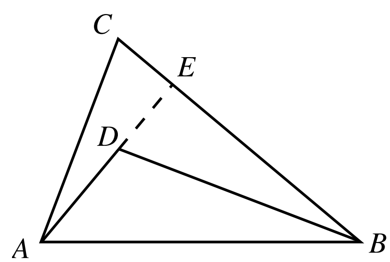

Figure 13:

This inequality holds, for

$$|AC| + |CB| = |AC| + |CE| + |EB| > |AE| + |EB|$$

(the shortest path from A to E is the straight line AE). But

$$|AE| + |EB| = |AD| + |DE| + |EB| > |AD| + |DB|$$

(the shortest path from D to B is the straight line DB). We have verified the inequality $|AC| + |CB| > |AD| + |DB|$.

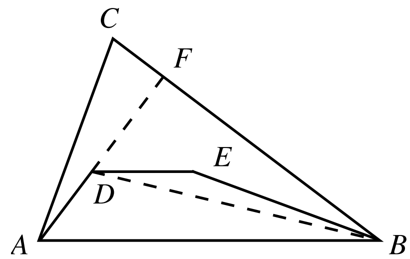

Figure 14:

Now let β be a 3-edged concave path, as in Figure 14.

Extend AD, and let it meet CB in the point F. Because the 3-edged path β is concave, it is entirely inside the triangle ΔAFB. Let us compute.

$$\begin{aligned} |AC| + |CB| &= |AC| + |CF| + |FB| \\ &> |AF| + |FB| \\ &= |AD| + |DF| + |FB|. \end{aligned}$$

But ΔFDB, with the 2-edged path DE and EB inside it, is just a replica of Figure 13, and so $|DF| + |FB| > |DE| + |EB|$. Therefore,

$$|AC| + |CB| > |AD| + |DF| + |FB| > |AD| + |DE| + |EB|.$$

This argument can be extended to all intermediate n-edged concave paths β from A to B, where $n \geq 2$. It should also be clear, in Figure 14, that there are such multi-edged paths with lengths very close to the length of the arc of a circle joining A and B; this is the key point in a (modern) proof of the Concavity Principle.

Lemma 0.11. *If q_n denotes the perimeter of the circumscribed polygon Q_n, then $q_n > c$.*

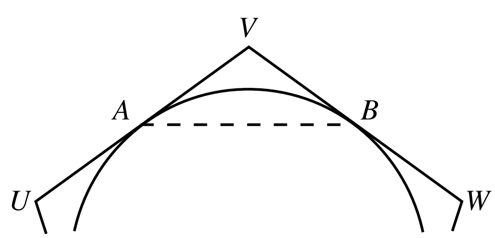

Figure 15:

Proof. As in Figure 15, the perimeter of Q_n consists of pairs of edge-pieces (AV is a piece of the edge UV, VB is a piece of the edge VW), each tangent to the circle from a vertex of Q_n. The Concavity Principle says that the length of such paths $AV + VB$ is longer than the arc of the circle joining A and B. •

We are going to give two proofs of the circumference formula; the first is in the classical spirit, the second is in the more modern spirit.

Theorem 0.12. *Let D be a disk of radius r and circumference c. If Δ is a right triangle with legs of lengths r and c, then*

$$a(D) = a(\Delta).$$

Proof. If $a(\Delta) = \frac{1}{2}rc \neq a(D)$, then either $\frac{1}{2}rc > a(D)$ or $\frac{1}{2}rc > a(D)$. We shall reach a contradiction in each of these two cases; equality will then be the only possibility.

Assume first that $\frac{1}{2}rc < a(D)$. Now $a(P_n) \to a(D)$, by Theorem 0.4, so that the method of exhaustion gives a polygon P_ℓ with

$$\tfrac{1}{2}rc < a(P_\ell).$$

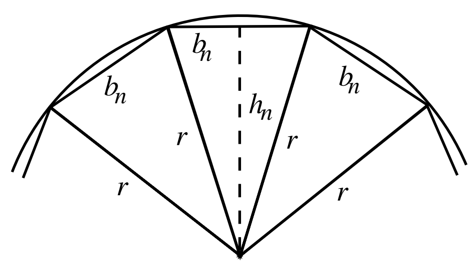

Figure 16:

Now P_ℓ is divided into $2^{\ell+1}$ congruent isosceles triangles, each having height h_ℓ and base b_ℓ. Thus, $a(P_\ell) = 2^{\ell+1}(\frac{1}{2}h_\ell b_\ell)$. The perimeter p_ℓ of P_ℓ is equal to $2^{\ell+1}b_\ell$, so that

$$a(P_\ell) = \tfrac{1}{2}h_\ell p_\ell.$$

Because $h_\ell < r$ (r is the hypotenuse of a right triangle having h_ℓ as a leg) and $p_\ell < c$, by Inequality (1), we have

$$a(P_\ell) = \tfrac{1}{2}h_\ell p_\ell < \tfrac{1}{2}rc, \tag{2}$$

and this contradicts the choice of P_ℓ.

Assume now that $\frac{1}{2}rc > a(D)$. Now $a(Q_n) \to a(D)$, by Theorem 0.10, so that the method of exhaustion gives a polygon Q_m with

$$\tfrac{1}{2}rc > a(Q_m).$$

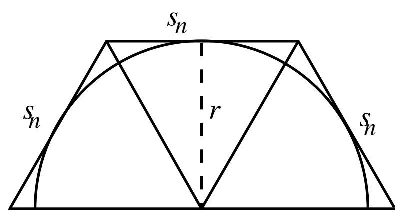

Figure 17:

Now Q_m is divided into 2^{m+1} congruent isosceles triangles, each having height r and base s_m, say. Thus, $a(Q_m) = 2^{m+1}(\frac{1}{2}rs_m)$. The perimeter q_m of Q_m is equal to $2^{m+1}s_m$, so that

$$a(Q_m) = \tfrac{1}{2}rq_m.$$

By Lemma 0.11, $q_m > c$. It follows that

$$a(Q_m) = \tfrac{1}{2}rq_m > \tfrac{1}{2}rc, \tag{3}$$

and this contradicts the choice of Q_m. It follows that $\frac{1}{2}rc = a(D)$, as desired. •

Thus, the circumscribed polygons Q_n are needed to eliminate the possibility $\frac{1}{2}rc > a(D)$. Again, we have seen a double reductio ad absurdum, as well as a horror of the infinite.

Here is a more modern proof of the circumference formula.

Theorem 0.13. *If D is a disk of radius r and circumference c, then*

$$c = 2\pi r.$$

Proof. Inequality (2) in the preceding proof gives $a(P_n) < \frac{1}{2}rc$, and inequality (3) gives $\frac{1}{2}rc < a(Q_n)$. Hence, for all $n \geq 1$, we have

$$a(P_n) < \tfrac{1}{2}rc < a(Q_n).$$

Now $a(P_n) \to a(D)$, by Theorem 0.4, and $a(Q_n) \to a(D)$, by Theorem 0.10. It follows that $\frac{1}{2}rc = a(D) = \pi r^2$, and so $c = 2\pi r$. •

Corollary 0.14. *If c is the circumference of a circle having diameter d, then $\pi = c/d$.*

Proof. Since $c = 2\pi r$ and $d = 2r$, we have $c/d = \pi$. •

Only now, having determined that the circumference of the unit disk is 2π, can one use radian measure.

The perimeter of the unit disk being 2π is the reason π is so denoted, for it is the first letter of the Greek word meaning *p*erimeter. The symbol π was introduced by William Jones in 1706; some earlier notations were π/δ (Oughtred, 1652) and c/r (for 2π) (De Moivre, 1698).

Let us compare the classical determination of the circumference of a circle of radius r with the standard calculation nowadays done in calculus. First of all, one develops the arclength formula (the idea behind which is plainly visible in the work of Archimedes, for arclength is defined as a limit of lengths of polygonal paths). If a curve γ has a parametrization:

$$x = f(t),\ y = g(t), \text{ where } a \le t \le b,$$

then the length L of γ is given as

$$L = \int_a^b \sqrt{f'(t)^2 + g'(t)^2}\, dt.$$

Parametrize the circle by $x = f(t) = r\cos t$ and $y = g(t) = r\sin t$, where $0 \le t \le 2\pi$; now $f'(t)^2 + g'(t)^2 = r^2(\sin^2 t + \cos^2 t) = r^2$, and $L = 2\pi r$. However, this is more subtle than it first appears, for one must ask how the number π, defined as the area of the unit circle, enters into the parametrization. The answer, of course, is via radian measure, which presupposes the perimeter formula (it is a bad pun to call this circular reasoning).