PROBLEMS

MATH 111

Q. If $y = \int_0^{\sin x} f(t)dt$, then why is $\frac{dy}{dx}$ not equal to $f(\sin t)$?

A. Because of the chain rule: put $u = sinx$; then $dy/du = f(u)$, hence $dy/dx = dy/du \cdot du/dx = f(\sin x) \cdot \cos x$.

In general, $\frac{d}{dx} \int_0^{g(x)} f(t)\ dt = g'(x)f(g(x))$.

While we're at it: replacing the integral above by $\int_1^{\sin x} f(t)dt$, or, more generally, by $\int_a^{\sin x} f(t)dt$ for any constant a, does not change the derivative: if F is an antiderivative for f, then $\int_0^{g(x)} f(t)\ dt = F(g(x)) - F(0)$ (this also explains why you have to use the chain rule when you compute the derivative of F), and since $\int_a^{g(x)} f(t)\ dt = F(g(x)) - F(a)$, both integrals have the same derivative as long as a is constant.

Q. 411-14. Let $g(x) = \begin{cases} (\tan x)^2/x & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$.

a) Show that $xg(x) = (\tan x)^2$ for $0 \leq x \leq \pi/4$;
b) Find the volume of the solid generated by revolving the region below the graph and between $x = 0$ and $x = \pi/4$ about the y-axis.

A. a) This is trivial if $x > 0$, and also trivial if $x = 0$ since in this case both sides are 0.

b) We use shells and find $V = 2\pi \int_0^{\pi/4} xg(x)\ dx = 2\pi \int_0^{\pi/4} (\tan x)^2\ dx$.

Now we have to compute the integral. Substitution seems to make things worse, so we have to think of something else. There aren't many things to think of, except addition formulas (which don't seem to help here) and the standard relation $\sin^2 x + \cos^2 x = 1$. In fact, $\tan^2 x = \frac{\sin^2 x}{\cos^2 x} = \frac{1-\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} - 1$. Now $V = 2\pi[\tan x - x]_0^{\pi/4} = 2\pi(1 - \frac{\pi}{4})$.

Q. 452-15. A round hole of radius $\sqrt{3}$ is bored through the center of a solid sphere of a radius 2. Find the volume of material removed from the sphere.

A. The biggest part of the material is just a cylinder. Pythagoras shows that it has height $2 \cdot 1$, hence its volume is $V_C = \pi \cdot \sqrt{3}^2 \cdot 2 = 6\pi$.

It remains to compute the volume of the two 'caps'. You get the upper cap by revolving a piece of a circle around the y-axis, namely the piece between $y = 1$ (half the height) and $y = 2$. The equation of the circle is $x^2 + y^2 = 2^2$, or $x = \sqrt{4 - y^2}$. Thus $V_2 = \pi \int_1^2 \sqrt{4 - y^2}^2\ dy = \pi(4y - \frac{1}{3}y^3)|_1^2 = \frac{5}{3}\pi$.

Thus the volume of the removed material is $6\pi + 2 \cdot \frac{5}{3}\pi = \frac{28}{3}\pi$.

The volume V_2 can also be computed using shells. The radius is x, the height is $\sqrt{4-x^2}-1$, hence $V_2 = 2\pi \int_0^{\sqrt{3}} x(\sqrt{4-x^2}-1)\ dx$. The integration should not be a problem, and you will get the same answer as before.

Q. 454-4. Show that for $0 < \alpha \leq \pi/2$ we have

$$\int_0^\alpha \sqrt{1+\cos^2\theta}\ d\theta > \sqrt{\alpha^2+\sin^2\alpha}.$$

A. The idea is to interpret both sides. The integral on the left hand side computes the length of something, namely the length of some curve with $dy/d\theta = \cos\theta$. This is the derivative of the sine function, so the left hand side is the length of the sine curve $y = \sin\theta$ from $\theta = 0$ to $\theta = \alpha$. This length is definitely greater than the distance between the endpoints $(0,0)$ and $(\alpha, \sin\theta)$ (draw a diagram!), which is exactly what we have on the right hand side.

This can of course be generalized: since the line is the shortest path between any two points, we always have

$$\int_0^x \sqrt{1+(f'(x))^2}\ dx \geq \sqrt{x^2+f(x)^2}$$

for all $x \geq 0$ in the domain of f, where $y = f(x)$ is any differentiable function with $f(0) = 0$.