PROBLEMS

MATH 111

The region bounded by $x = y^2$, $x = 3y^2 - 2$ and $y = 0$ is rotated about the x-axis. Compute the volume.

Note that the two parabolas intersect at $(1, 1)$.

Method 1: Cross sections. The solid is generated by rotating $x = 3y^2 - 2$ and then cutting away the part generated by $x = y^2$. Since $y^2 = \frac{x+2}{3}$ for the first and $y^2 = x$ for the second function, we get

$$\begin{aligned} V &= \pi \int_{-2}^{1} \frac{x+2}{3}\, dx - \pi \int_{0}^{1} x\, dx \\ &= \frac{\pi}{3}\Big(\frac{1}{2}x^2 + 2x\Big)\Big|_{-2}^{1} - \frac{1}{2}\pi x^2\Big|_{0}^{1} = \frac{3\pi}{2} - \frac{\pi}{2} = \pi. \end{aligned}$$

Method 2: Shells. Here the radius wanders in the y-direction, and the height is given by the difference of the x-values of the two functions. Thus

$$V = 2\pi \int_0^1 y(y^2 - (3y^2 - 2))\, dy = 4\pi \int_0^1 (y - y^3)\, dy = \pi$$

as before.

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Consider the region bounded by $y = x^2$, $y = -x^4$, and $x = 1$. Compute the volume of the solid generated by revolving this region about the y-axis.

Method 1: Shells. The radius is x, the height is $x^2 - (-x^4) = x^2 + x^4$, hence the volume is

$$V = 2\pi \int_0^1 x(x^2 + x^4)\, dx = \frac{5}{6}\pi.$$

Method 2: Cross sections. Here we have to compute the volume of the top and the bottom separately. For the top, the inner radius is $r = x = \sqrt{y}$, the outer radius is $R = 1$, hence

$$V_1 = \pi \int_0^1 (R^2 - r^2)\, dy = \pi\Big(y - \frac{1}{2}y^2\Big)\Big|_0^1 = \frac{1}{2}\pi.$$

For the bottom part, the inner radius is $r = x = y^{1/4}$ (we may replace $y = -x^4$ by $y = x^4$: this does not change the volume), hence

$$V_2 = \pi \int_0^1 (1 - y^{1/2})\, dy = \frac{1}{3}\pi.$$

Thus the overall volume is $V = V_1 + V_2 = \frac{5}{6}\pi$.