PROBLEMS

MATH 111

(1) Find the length of $y = x^{1/2} - \frac{1}{3}x^{3/2}$ for $1 \leq x \leq 4$.
(2) A solid is generated by revolving about the x-axis the region bounded by the graph of a continuous function $y = f(x)$, the x-axis, $x = 0$ and $x = a$. Its volume for all $a > 0$ is $a^2 + a$. Find f.
(3) The base of a solid is the region in the first quadrant between the line $y = x$ and the parabola $y = 2\sqrt{x}$. The cross sections of the solid perpendicular to the x-axis are equilateral triangles whose bases stretch from the line to the curve. Find the volume.
(4) Compute the volumes of the solids generated by rotating the region in the first quadrant bounded by $x = y - y^3$, $x = 1$ and $y = 1$ about
   (a) the x-axis;
   (b) the y-axis;
   (c) the line $x = 1$;
   (d) the line $y = 1$.
(5) Find the length of $x = t^2$, $y = 2t$ for $0 \leq t \leq 1$.

(1) Find the length of $y = x^{1/2} - \frac{1}{3}x^{3/2}$ for $1 \le x \le 4$.

$y' = \frac{1}{2}(\frac{1}{\sqrt{x}} - \sqrt{x})$, hence $1 + (y')^2 = \frac{1}{4}(\frac{1}{\sqrt{x}} + \sqrt{x})^2$. Since the expression inside the brackets is positive for $1 \le x \le 4$, we get

$$\begin{aligned} L &= \int_1^4 \sqrt{1 + (y')^2}\, dx = \frac{1}{2}\int_1^4 \Big(\frac{1}{\sqrt{x}} + \sqrt{x}\Big)\, dx \\ &= \Big(\frac{1}{3}x^{3/2} + x^{1/2}\Big)\Big|_1^4 = \Big(\frac{8}{3} + 2\Big) - \Big(\frac{1}{3} + 1\Big) = \frac{10}{3}. \end{aligned}$$

(2) A solid is generated by revolving about the x-axis the region bounded by the graph of a continuous function $y = f(x)$, the x-axis, $x = 0$ and $x = a$. Its volume for all $a > 0$ is $a^2 + a$. Find f.

The volume is $V(a) = \pi \int_0^a f(x)^2 dx$. Since $V(a) = a$, differentiating using the first fundamental theorem of calculus gives $\pi f(a)^2 = 2a + 1$, from which we easily find $f(x) = \sqrt{\frac{2x+1}{\pi}}$.

(3) The base of a solid is the region in the first quadrant between the line $y = x$ and the parabola $y = 2\sqrt{x}$. The cross sections of the solid perpendicular to the x-axis are equilateral triangles whose bases stretch from the line to the curve. Find the volume.

The area of an equilateral triangle with base s is $\frac{\sqrt{3}}{4}s^2$. Moreover, the base of our triangles is $s = 2\sqrt{x} - x$. Thus the volume of the solid is

$$\begin{aligned} A &= \frac{\sqrt{3}}{4}\int_0^4 (2\sqrt{x} - x)^2\, dx = \frac{\sqrt{3}}{4}\int_0^4 (4x - 4x^{3/2} + x^2)\, dx \\ &= \frac{\sqrt{3}}{4}\Big(2x^2 - \frac{8}{5}x^{5/2} + \frac{1}{3}x^3\Big)\Big|_0^4 = \frac{\sqrt{3}}{4}\Big(32 - \frac{8}{5}\cdot 32 + \frac{64}{3}\Big) = \frac{32}{15}\sqrt{3}. \end{aligned}$$

(4) Compute the volumes of the solids generated by rotating the region in the first quadrant bounded by $x = y - y^3$, $x = 1$ and $y = 1$ about a) the x-axis; b) the y-axis; c) the line $x = 1$; and d) the line $y = 1$.

(a) the x-axis: Using cross sections we get $V = \pi \int_0^1 (1^2 - (y - y^3)^2)\, dy$. (The radius wanders from $y = 0$ to $y = 1$, so we integrate over y).

(b) the y-axis: here we will use shells; the radius wanders on the y-axis, and we find $r = y$ and $h = 1 - x = 1 - (y - y^3)$, hence the volume of the solid is $V = 2\pi \int_0^1 y(1 - y + y^3)\, dy$.

(c) the line $x = 1$: using cross sections perpendicular to the x-axis we find that the radius of the disc is given by $1 - (y - y^3)$, hence the volume of the solid is $V = \pi \int_0^1 (1 - y + y^3)^2\, dy$.

(d) the line $y = 1$. We use shells again; the radius wanders along the y-axis and is given by $1 - y$; the height is x as before, hence the volume of the solid is $V = 2\pi \int_0^1 (1 - y)(1 - y + y^3)\, dy$.

(5) Find the length of $x = t^2$, $y = 2t$ for $0 \le t \le 1$.

$L = 2\int_0^1 \sqrt{1 + t^2}\, dt$.