PROBLEMS

MATH 111

(1) 223-16: Does the function
$$f(x) = \begin{cases} \frac{1-\cos x}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$$
have a derivative at $x = 0$?

(2) 224-24: Suppose that a function f satisfies
  (a) $f(x + y) = f(x)f(y)$;
  (b) $f(x) = 1 + xg(x)$, with $\lim_{x\to 0} g(x) = 1$.

  Show that $f'(x)$ exists for every x and that $f'(x) = f(x)$.

(3) 310-12: For what values of k will $y = x^3 + kx^2 + 3x - 4$ have exactly one horizontal tangent?

(4) Inscribe a rectangle into an isosceles triangle in such a way that the base of the rectangle lies on the base of the triangle. Which of these rectangles has maximal area?

(5) 385-18: Compute $\int (\tan x)^{-3/2} \sec^2 x \, dx$.

(6) 392-19: Let $f(x) = \int_{1/x}^{x} \frac{dt}{t}$. Compute $f'(x)$.

(7) 392-20: Let $f(x) = \int_{\cos x}^{\sin x} \frac{dt}{1-t^2}$. Compute $f'(x)$.

(8) 392-22: For which value of x is $\int_x^{x+3} t(5 - t)dt$ maximal?

(1) 223-16: Does the function

$$f(x) = \begin{cases} \frac{1-\cos x}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$$

have a derivative at $x = 0$?

Let's first check whether f is continuous at $x = 0$ (if it's not, it cannot be differentiable, and we are done). To this end, check that $\lim_{x\to 0} f(x) = f(0)$, that is $\lim_{x\to 0} \frac{1-\cos x}{x} = 0$. Do this.

For deciding whether f is differentiable at $x = 0$ we have to check whether the limit $\lim_{h\to 0} \frac{f(h)-f(0)}{h} = \lim_{h\to 0} \frac{1-\cos h}{h^2}$ exists. Do this (one way is by multiplying through by $1 = \lim_{h\to 0} \frac{1+\cos h}{2}$). You should find that the limit exists and equals $\frac{1}{2}$. Thus $f'(0) = \frac{1}{2}$.

(2) 224-24: Suppose that a function f satisfies
(a) $f(x + y) = f(x)f(y)$;
(b) $f(x) = 1 + xg(x)$, with $\lim_{x\to 0} g(x) = 1$.
Show that $f'(x)$ exists for every x and that $f'(x) = f(x)$.

Showing that $f'(x)$ exists means showing that $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ exists. Use property (1) to transform the numerator, pull $f(x)$ in front of the limit (this is ok since x does not depend on h), then apply (2). Once you know what to do this is a very easy problem.

(3) 310-12: For what values of k will $y = x^3 + kx^2 + 3x - 4$ have exactly one horizontal tangent?

We have $y' = 3x^2 + 2kx + 3$. For which k does this have exactly one solution? Think of the discriminant. There are two solutions.

(4) Inscribe a rectangle into an isosceles triangle in such a way that the base of the rectangle lies on the base of the triangle. Which of these rectangles has maximal area?

Draw a picture. Denote the base of the rectangle by a and the height by b. Denote the base of the triangle by c and the height by h. You have to maximize the area $A = ab$. To find a relation between a and b, use similar triangles. The final answer should be $a = \frac{c}{2}$.

(5) 385-18: Compute $\int (\tan x)^{-3/2} \sec^2 x \, dx$.

Here you might run into trouble if you don't see that $\sec^2 x$ is the derivative of $\tan x$. If you see it, then of course you substitute $u = \tan x$.

(6) 392-19: Let $f(x) = \int_{1/x}^{x} \frac{dt}{t}$. Compute $f'(x)$.

Define $F(x) = \int_1^x \frac{dt}{t}$ (we cannot start at $t = 0$ since the function we are integrating is not defined there). Observe that $F'(x) = \frac{1}{x}$. Then

$$f(x) = \int_{1/x}^{x} \frac{dt}{t} = \int_1^x \frac{dt}{t} - \int_1^{1/x} \frac{dt}{t} = F(x) - F(\tfrac{1}{x})$$

Therefore $f'(x) = F'(x) - (-\frac{1}{x^2})F'(\frac{1}{x}) = \frac{1}{x} + \frac{1}{x^2}x = \frac{2}{x}$.

(7) 392-20: Let $f(x) = \int_{\cos x}^{\sin x} \frac{dt}{1-t^2}$. Compute $f'(x)$.

Same idea: put $F(X) = \int_0^x \frac{dt}{1-t^2}$; then $f(x) = F(\sin x) - F(\cos x)$. The result should be $f(x) = \frac{1}{\cos x} + \frac{1}{\sin x}$.

(8) 392-22: For which value of x is $f(x) = \int_x^{x+3} t(5-t)dt$ maximal?

Same idea: write $F(x) = \int_0^x t(5-t)dt$; then $f(x) = F(x+3) - F(x)$. The maximum is attained at a point where $f'(x) = 0$. A simple calculation shows $f'(x) = 6 - 6x$ and $x = 1$.

Why is this a maximum and not a minimum? Because f' changes sign from $+$ to $-$ at $x = 1$.