PROBLEMS

MATH 111

(1) $\int \frac{(t+1)^2-1}{t^4} dt$.

(2) $\int \sec t \tan t dt$.

(3) $\int \frac{\sec^2 x}{(1+7\tan x)^{2/3}}\ dx$.

(4) The graph of the implicit function $\sqrt{x}+\sqrt{y}=1$ cuts out an area from the first quadrant. Compute it.

(5) Find $\frac{d}{dx}\int_2^x \sqrt{2+\cos^3 t}dt$.

(6) Find $\frac{d}{dx}\int_2^{7x^2} \sqrt{2+\cos^3 t}dt$.

(7) Find $\frac{d}{dx}\int_x^1 \sqrt{2+\cos^3 t}dt$.

(8) Suppose that x and y are related by the equation

$$x = \int_0^y \frac{dt}{\sqrt{1+4t^2}}.$$

Show that d^2y/dx^2 is proportional to y, and find the constant of proportionality.

(9) The area of the region in the xy-plane enclosed by the x-axis, the curve $y=f(x)$, $f(x)\geq 0$, and the lines $x=1$ and $x=b$ is equal to $\sqrt{b^2+1}-\sqrt{2}$ for all $b>1$. Find $f(x)$.

(10) Prove that

$$\int_0^x \Big(\int_0^u f(t)dt\Big)du = \int_0^x f(u)(x-u)du.$$

(Hint: Express the integral on the right as a difference of two integrals and show that both sides have the same derivative.)

(11) Find $f(4)$ if $\int_0^{x^2} f(t)dt = x\cos\pi x$.

(1) $\int \frac{(t+1)^2-1}{t^4} dt$: Do not substitute; break up the fraction into pieces that you can handle individually.

(2) $\int \sec t \tan t dt$: substitute $u = \frac{1}{\cos t}$.

(3) $\int \frac{\sec^2 x}{(1+7\tan x)^{2/3}}\, dx$: substitute $u = 1 + 7\tan x$.

(4) The graph of the implicit function $\sqrt{x} + \sqrt{y} = 1$ cuts out an area from the first quadrant. Compute it.

Sketch the graph: $\sqrt{y} = 1 - \sqrt{x}$ gives $y = f(x) = (1-\sqrt{x}\,)^2 = 1 - 2\sqrt{x} + x$. Thus $f(0) = 1$ and $f(1) = 0$.

The area is given by $\int_0^1 f(x)dx$; integrating should not be a problem.

(5) Find $\frac{d}{dx}\int_2^x \sqrt{2+\cos^3 t}dt$.

Put $f(t) = \sqrt{2+\cos^3 t}$ and $F(x) = \int_2^x f(t)dt$. Then $F'(x) = f(x)$ by the fundamental theorem of calculus.

(6) Find $\frac{d}{dx}\int_2^{7x^2} \sqrt{2+\cos^3 t}dt$.

This time, substitute $u = 7x^2$ and use the chain rule.

(7) Find $\frac{d}{dx}\int_x^1 \sqrt{2+\cos^3 t}dt$.

Observe that $\int_x^1 \sqrt{2+\cos^3 t}dt = -\int_1^x \sqrt{2+\cos^3 t}dt$.

(8) Suppose that x and y are related by the equation

$$x = \int_0^y \frac{dt}{\sqrt{1+4t^2}}.$$

Show that d^2y/dx^2 is proportional to y, and find the constant of proportionality.

Tricky. First you compute dx/dy using the fundamental theorem of calculus. Invert to get dy/dx, then compute the second derivative (and don't forget that y is a function of x). In this formula, $y' = dy/dx$ appears, which you just have computed ... The constant should be 4.

(9) The area of the region in the xy-plane enclosed by the x-axis, the curve $y = f(x)$, $f(x) \geq 0$, and the lines $x = 1$ and $x = b$ is equal to $\sqrt{b^2+1} - \sqrt{2}$ for all $b > 1$. Find $f(x)$.

Expressing the area as an integral shows

$$\int_1^b f(x)dx = \sqrt{b^2+1} - \sqrt{2}.$$

By the fundamental theorem of calculus, this is $F(b) - F(1)$. Now differentiate with respect to b.

(10) Prove that

$$\int_0^x \Big(\int_0^u f(t)dt\Big)du = \int_0^x f(u)(x-u)du.$$

(Hint: Express the integral on the right as a difference of two integrals and show that both sides have the same derivative.)

This one is hard. The hint suggests writing the right hand side as a difference; the only way to do this is as $\int xf(u)du - \int uf(u)du$. But $\int xf(u)du = x\int f(u)du$ since x does not depend on u.

Now use the fundamental theorem to compute the derivatives. On the left hand side you should get $\int_0^x f(t)dt$ (write the inner integral as $g(u)$, then it should be clear). On the right hand side, use the Leibniz rule for differentiating $x\int_0^x f(u)du$. Your results should agree.

Since both sides attain the value 0 for $x = 0$ and have the same derivative, they must be equal.

(11) Find $f(4)$ if $\int_0^{x^2} f(t)dt = x\cos\pi x$.

Write $F(x) = \int_0^{x^2} f(t)dt$. Use the fundamental theorem to compute $F'(x)$. Compare it to the derivative of the right hand side and plug in $x =$? in such a way that the formula involves $f(4)$.