Let $f(x)$ be a function satisfying the following conditions:

- $f''(x)$ is continuous for all $x$;
- $f(x)$ is constant for $x < 0$;
- $f(1) = 1$, $f(2) = 2$.

(1) Show that there is a point $a$ with $f'(a) = 1$
(2) Show that for all $k \in (0, \frac{1}{2}]$ there is a $c$ such that $f''(c) = k$.

The first part is easy. Now let us look at the second claim. Here's what we know about

- $f$: $f(1) = 1$, $f(2) = 2$;
- $f'$: first, $f'(x) = 0$ for all $x < 0$ since $f$ is constant there. Since $f''$ exists, $f'$ must be continuous, hence $f'(0) = \lim_{x \to 0^-} f'(x) = 0$. Moreover, we know $f'(a) = 1$ for some $a \in (1, 2)$.
- $f''$: we know that $f''(x) = 0$ for all $x < 0$, and since $f''$ is continuous, we also have $f''(0) = 0$.

In order to prove something about $f''$, let us apply the mean value theorem to $g(x) = f'(x)$. We have $g(0) = 0$ and $g(a) = 1$, hence there is some $b \in (0, a)$ such that $g'(b) = \frac{g(a)-g(0)}{a-0} = \frac{1}{a}$. But we know $1 < a < 2$, hence $\frac{1}{a} > \frac{1}{2}$. Thus $g' = f''$ attains all the values between $f''(0) = 0$ and $f''(b) = \frac{1}{a} > \frac{1}{2}$, in particular all $k \in (0, \frac{1}{2}]$.