QUESTION

MATH 111

(1) Let $f(x)$ be a function satisfying the following conditions:
- $f'(x)$ is continuous for all x;
- $f(x)$ is constant on the interval $x < 0$;
- $f(1) = 1$, $f(2) = 2$.

Show that there is a point a with $f'(a) = 1$;

Showing that there exists some point such that the derivative has a certain value suggests using the mean value theorem. Since f is constant for $x < 0$, the derivative is 0 there; this part is just there to confuse you, it is not needed at all. The interesting interval is $[1, 2]$; applying the mean value theorem to f on this interval shows that there is some point $a \in (1, 2)$ such that $f'(a) = \frac{f(2)-f(1)}{2-1} = 1$.

(2) Consider the function $f(x) = \begin{cases} x^2|\cos\frac{\pi}{2x}| & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$

(a) Show that $f'(0)$ exists and find its value.
(b) Show that $f'(1/3)$ does not exist.
(c) For what values of x does $f'(x)$ fail to exist?

Showing that $f'(0)$ exists means proving that $\lim_{h\to 0} \frac{f(h)-f(0)}{h}$ exists. But

$$\lim_{h\to 0} \frac{f(h)-f(0)}{h} = \lim_{h\to 0} \frac{h^2|\cos\frac{\pi}{2h}|}{h} = \lim_{h\to 0} h|\cos\frac{\pi}{2h}| = 0$$

since $\cos z$ is bounded and since $h \to 0$: just apply the sandwich theorem to $0 \leq h|\cos\frac{\pi}{2h}| \leq h$. Thus $f'(0)$ exists and is equal to 0.

For $f'(1/3)$ we have to consider $\lim_{h\to 0} \frac{f(h+1/3)-f(1/3)}{h}$. We find

$$\frac{f(h+1/3)-f(1/3)}{h} = \frac{(h+1/3)^2|\cos\frac{\pi}{2(h+1/3)}| - 1/9|\cos\frac{3\pi}{2}|}{h} = \frac{(h+1/3)^2|\cos\frac{\pi}{2(h+1/3)}|}{h},$$

hence

$$\lim_{h\to 0} \frac{f(h+1/3)-f(1/3)}{h} = \lim_{h\to 0} \frac{(h+1/3)^2|\cos\frac{\pi}{2(h+1/3)}|}{h} = \frac{1}{9}\lim_{h\to 0} \frac{|\cos\frac{\pi}{2(h+1/3)}|}{h}.$$

Now $\cos\frac{\pi}{2(h+1/3)} > 0$ for small $h > 0$, and $\cos\frac{\pi}{2(h+1/3)} < 0$ for small $h < 0$. Thus

$$\lim_{h\to 0^+} \frac{f(h+1/3)-f(1/3)}{h} = \frac{1}{9}\lim_{h\to 0^+} \frac{\cos\frac{\pi}{2(h+1/3)}}{h},$$

and

$$\lim_{h\to 0^-} \frac{f(h+1/3)-f(1/3)}{h} = -\frac{1}{9}\lim_{h\to 0^-} \frac{\cos\frac{\pi}{2(h+1/3)}}{h}.$$

I'll leave it to you to show that these limits are nonzero, and that one is the negative of the other. Since they are not equal, the derivative of f at 1/3 does not exist.

The same thing happens for all $x \neq 0$ where $f(x) = 0$, namely for $x = \pm 1$, $\pm\frac{1}{3}$, $\pm\frac{1}{5}$, ... At points where $f(x) \neq 0$ the function is differentiable, since at points close to such an x the function is either positive or negative, hence we can omit the absolute values, and the function $x^2 \cos \frac{\pi}{2x}$ is clearly differentiable by the chain rule.