MATH 111

PRACTICE EXAM 1

(1) Find the equation of the line through $(3,0)$ that is normal to the parabola $y = x^2$.

Slope of tangent at $P(a,b)$ is $2a$, hence the slope of the normal is $-\frac{1}{2a}$, and the equation of the normal is $y - b = -\frac{1}{2a}(x-a)$, where $b = a^2$.

This normal must go through $(3,0)$, hence $0 - a^2 = -\frac{1}{2a}(3-a)$, that is, $2a^3 + a - 3 = 0$. Then $a = 1$ is a solution, and since $2a^3 + a - 3 = (a-1)(2a^2 + 2a + 3)$ and the quadratic factor has negative discriminant, this is the only solution.

Answer: $y = -\frac{1}{2}(x-1) + 1$ is the equation of the normal.

(2) Find the local minima, local maxima, and inflection points of the function

$$f(x) = \begin{cases} x^2 & \text{if } x \le 1 \\ (2-x)^3 & \text{if } x > 1 \end{cases}$$

Check that the function is continuous at $x = 1$. The derivative is

$$f(x) = \begin{cases} 2x & \text{if } x < 1 \\ -3(2-x)^2 & \text{if } x > 1 \end{cases}$$

What happens at $x = 1$? The limits from the left and the right do not agree, therefore $f'(1)$ is not defined.

In order to find local extrema, check all x with $f'(x) = 0$ or $f'(x)$ undefined. We find that $f'(x) = 0$ for $x = 0$ and $x = 2$; the point $(0,0)$ is a local minimum, the point $(2,0)$ is not a local extremum since f' does not change sign there.

Finally we have to consider $x = 1$. For $x = 1 - h$ and small $h > 0$, we find that $f'(x) > 0$, so f is strictly increasing there. For $x = 1 + h$ and small $h > 0$, we find that $f'(x) < 0$, hence f is strictly increasing there. This shows that f has a local maximum at $x = 1$.

It is easily checked that $x = 2$ is an inflection point.

(3) A wire with length 3 m is used for making a circle and an equilateral triangle. How should this be done if the sum of the areas of circle and triangle is maximal?

Let r denote the radius of the circle and s the side length of the equilateral triangle. The height of the triangle is $\frac{\sqrt{3}}{2}s$ (Pythagoras), hence the sum of the areas is $A = \pi r^2 + \frac{\sqrt{3}}{4}s^2$.

Now we use $2\pi r + 3s = 3$; solving for s and plugging this into the formula for A gives

$$A(r) = \pi r^2 + \frac{\sqrt{3}}{4}\Big(1 - \frac{2\pi}{3} r\Big)^2.$$

Now compute $A'(r)$, set it equal to 0, and solve the linear equation for r. Done.

Actually, no: We have to check that this value of r gives a maximum. But it is easily checked that $A'(0) < 0$ and $A'(2\pi/3) > 0$, hence the derivative changes sign from $-$ to $+$, and this implies that what we have found is a *minimum*! Thus the maximum occurs at one of the endpoints: we have to compute $A(0)$ ($r = 0$, $s = 1$) and $A(2\pi/3)$ ($r = 2\pi/3$ and $s = 0$) and compare.

(4) Consider the functions $f(x) = \sqrt{x+a}$ and $g(x) = |x|$.
  (a) Which condition on a must be satisfied so that that graphs of f and g enclose a nonzero area?
  (b) Assuming that this condition is satisfied, compute the area enclosed by f and g.

Intersecting the functions gives $x + a = |x|^2 = x^2$, that is, $x^2 - x - a = 0$. This equation has two solutions if and only if the discriminant $1 + 4a > 0$. Thus the two functions enclose an area if and only of $a > -\frac{1}{4}$.

In this case, the two intersection points are $x_1 = \frac{1}{2}(1 - \sqrt{1+4a}\,)$ and $x_2 = \frac{1}{2}(1 + \sqrt{1+4a}\,)$. Note that $x_2 > 0$, and that $x_1 < 0$ if and only if $a > 0$. Thus there are two cases:
  (a) $-\frac{1}{4} < a \le 0$: then $0 < x_1 < x_2$, and the area we have to compute is $\int_{x_1}^{x_2} (\sqrt{x+a} - |x|)dx = \int_{x_1}^{x_2} (\sqrt{x+a} - x)dx$ since $|x| > 0$ here.
  (b) $a > 0$: then $x_1 < 0$, and the area is

$$\begin{aligned}\int_{x_1}^{x_2} (\sqrt{x+a} - |x|)dx &= \int_{x_1}^{0} (\sqrt{x+a} - |x|)dx + \int_{0}^{x_2} (\sqrt{x+a} - |x|)dx \\ &= \int_{x_1}^{0} (\sqrt{x+a} + x)dx + \int_{0}^{x_2} (\sqrt{x+a} - x)dx.\end{aligned}$$