

# Essential cohomology for elementary abelian $p$ -groups

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## Abstract

For an odd prime  $p$  the cohomology ring of an elementary abelian  $p$ -group is polynomial tensor exterior. We show that the ideal of essential classes is the Steenrod closure of the class generating the top exterior power. As a module over the polynomial algebra, the essential ideal is free on the set of Mùì invariants.

*Key words:* elementary abelian  $p$ -group, essential ideal, Mùì invariants

*2000 MSC:* 20J06 (primary), 13A50, 55S10

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## 1. Introduction

Let  $G$  be a finite group and  $k$  a field whose characteristic  $p$  divides the order of  $G$ . A cohomology class  $x \in H^n(G, k)$  is called *essential* if its restriction  $\text{Res}_H(x)$  is zero for every proper subgroup  $H$  of  $G$ . The essential classes form an ideal, called the essential ideal and denoted by  $\text{Ess}(G)$ . It is standard that restriction to a Sylow  $p$ -subgroup of  $G$  is a split injection (see for example Theorem XII,10.1 of [4]), and so the essential ideal can only be nonzero if  $G$  is a  $p$ -group. Many  $p$ -groups have nonzero essential ideal, for instance the quaternion group of order eight. The essential ideal plays an important role and has therefore been the subject of many studies: two such being Carlson's work

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<sup>1</sup>Supported by a Ph.D. research scholarship from the Scientific and Technical Research Council of Turkey (TÜBİTAK-BAYG)

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on the depth of a cohomology ring [2], and the cohomological characterization due to Adem and Karagueuzian of those  $p$ -groups whose order  $p$  elements are all central [1].

The nature of the essential ideal depends crucially on whether or not the  $p$ -group  $G$  is elementary abelian. If  $G$  is not elementary abelian, then a celebrated result of Quillen (Theorem 7.1 of [10]) implies that  $\text{Ess}(G)$  is a nilpotent ideal. By contrast, the essential ideal of an elementary abelian  $p$ -group contains non-nilpotent classes. Work to date on the essential ideal has concentrated on the non-elementary abelian case. In this paper we give a complete treatment of the outstanding elementary abelian case. As we shall recall in the next section, the case  $p = 2$  is straightforward and well known. So we shall concentrate on the case of an odd prime  $p$ .

So let  $p$  be an odd prime and  $V$  a rank  $n$  elementary abelian  $p$ -group. We may equally well view  $V$  as an  $n$ -dimensional  $\mathbb{F}_p$ -vector space. Recall that the cohomology ring has the form

$$H^*(V, \mathbb{F}_p) \cong S(V^*) \otimes_{\mathbb{F}_p} \Lambda(V^*), \quad (1)$$

where the exterior copy of the dual space  $V^*$  is  $H^1(V, \mathbb{F}_p)$ , and the polynomial copy lies in  $H^2(V, \mathbb{F}_p)$ : specifically, the polynomial copy is the image of the exterior copy under the Bockstein boundary map  $\beta$ . Our first result is as follows:

**Theorem 1.1.** *Let  $p$  be an odd prime and  $V$  a rank  $n$  elementary abelian  $p$ -group. Then the essential ideal  $\text{Ess}(V)$  is the Steenrod closure of  $\Lambda^n(V^*)$ . That is,  $\text{Ess}(V)$  is the smallest ideal in  $H^*(V, \mathbb{F}_p)$  which contains the one-dimensional space  $\Lambda^n(V^*) \subseteq H^n(V, \mathbb{F}_p)$  and is closed under the action of the Steenrod algebra.*

Our second result concerns the structure of  $\text{Ess}(V)$  as a module over the polynomial subalgebra  $S(V^*)$  of  $H^*(V, \mathbb{F}_p)$ . It was conjectured by Carlson (Question 5.4 in [3]) – and earlier in a less precise form by Mui [8] – that the essential ideal of an arbitrary  $p$ -group is free and finitely generated as a module over a certain polynomial subalgebra of the cohomology ring. In [6], the second author demonstrated finite generation, and for most  $p$ -groups of a given order was able to prove freeness as well: specifically the method works provided the group is not a direct product in which one factor is elementary abelian of rank at least two. Our second result states that Carlson’s conjecture holds for elementary abelian  $p$ -groups too, and gives explicit free generators.

**Theorem 1.2.** *Let  $p$  be an odd prime and  $V$  a rank  $n$  elementary abelian  $p$ -group. Then as a module over the polynomial part  $S(V^*)$  of the cohomology ring  $H^*(V, \mathbb{F}_p)$ , the essential ideal  $\text{Ess}(V)$  is free on the set of Mùì invariants, as defined in Definition 3.3.*

*Structure of the paper.* In §2 we briefly cover the well-known case  $p = 2$ . We introduce the Mùì invariants in §3. After proving Theorem 1.2 in §4 we consider the action of the Steenrod algebra on the Mùì invariants in order to prove Theorem 1.1 in §5.

## 2. Elementary abelian $p$ -groups and the case $p = 2$

The cohomology group  $H^1(G, \mathbb{F}_p)$  may be identified with the set of group homomorphisms  $\text{Hom}(G, \mathbb{F}_p)$ . This set is an  $\mathbb{F}_p$ -vector space, and – assuming that  $G$  is a  $p$ -group – the maximal subgroups of  $G$  are in bijective correspondence with the one-dimensional subspaces: the maximal subgroup corresponding to  $\alpha: G \rightarrow \mathbb{F}_p$  being  $\ker(\alpha)$ . Of course, the cohomology class  $\alpha \in H^1(G, \mathbb{F}_p)$  has zero restriction to the maximal subgroup  $\ker(\alpha)$ . Note that in order to determine  $\text{Ess}(G)$  it suffices to consider restrictions to maximal subgroups.

*Definition.* Denote by  $L_n$  the polynomial

$$L_n(X_1, \dots, X_n) = \det \begin{vmatrix} X_1 & X_2 & \cdots & X_n \\ X_1^p & X_2^p & \cdots & X_n^p \\ \vdots & \vdots & \ddots & \vdots \\ X_1^{p^{n-1}} & X_2^{p^{n-1}} & \cdots & X_n^{p^{n-1}} \end{vmatrix} \in \mathbb{F}_p[X_1, \dots, X_n].$$

There is a well-known alternative description of  $L_n$ .

**Lemma 2.1.**  *$L_n$  is the product of all monic linear forms in  $X_1, \dots, X_n$ . So for an  $n$ -dimensional  $\mathbb{F}_p$ -vector space  $V$  we may define  $L_n(V) \in S(V^*)$  up to a nonzero scalar multiple by*

$$L_n(V) = \prod_{[x] \in \mathbb{P}V^*} x. \quad (2)$$

*Proof.* First part: Here we call a linear form monic if the first nonzero coefficient is one. The right hand side divides the left. Both sides have the same total degree. And the coefficient of  $X_1 X_2^p X_3^{p^2} \cdots X_n^{p^{n-1}}$  is +1 in both cases. The second part follows.  $\square$

Let  $V$  be an elementary abelian 2-group. Then  $H^*(V, \mathbb{F}_2) \cong S(V^*)$ , where the dual space  $V^*$  is identified with  $H^1(V, \mathbb{F}_2)$ . Pick  $x_1, \dots, x_n$  to be a basis for  $H^1(V, \mathbb{F}_2)$ . The following is well-known:

**Lemma 2.2.** *For an elementary abelian 2-group  $V$ , the essential ideal is the principal ideal in  $H^*(V, \mathbb{F}_2)$  generated by  $L_n(x_1, \dots, x_n)$ .*

*Moreover,  $\text{Ess}(V)$  is the free  $S(V^*)$ -module on  $L_n(V)$ , and the Steenrod closure of this one generator.*

*Proof.*  $L_n(V)$  is essential, because every nonzero linear form is a factor and every maximal subgroup is the kernel of a nonzero linear form. Now suppose that  $y$  is essential, and let  $x \in V^*$  be a nonzero linear form. Let  $U \subseteq V^*$  be a complement of the subspace spanned by  $x$ . So  $y = y'x + y''$  with  $y' \in S(V^*)$  and  $y'' \in S(U)$ . Hence  $\text{Res}_H(y'') = 0$  for  $H = \ker(x)$ , as  $y$  is essential and  $\text{Res}_H(x) = 0$ . But the map  $\text{Res}_H: V^* \rightarrow H^*$  satisfies  $\ker(\text{Res}_H) \cap U = 0$ , and so  $\text{Res}_H$  is injective on  $S(U)$ . Hence  $y'' = 0$ , and  $x$  divides  $y$ . By unique factorization in  $S(V^*)$  it follows that  $L_n(V)$  divides  $y$ . So  $\text{Ess}(V)$  is the principal ideal generated by  $L_n(V)$ , and the free module on this one generator. Finally, the definition of the essential ideal means that it is closed under the action of the Steenrod algebra.  $\square$

We finish off this section by recalling the action of the Steenrod algebra on the cohomology of an elementary abelian  $p$ -group in the case of an odd prime. So let  $p$  be an odd prime and  $V$  an elementary abelian  $p$ -group. Recall that the mod- $p$ -cohomology ring is the free graded commutative algebra

$$H^*(V, \mathbb{F}_p) \cong \mathbb{F}_p[x_1, \dots, x_n] \otimes_{\mathbb{F}_p} \Lambda(a_1, \dots, a_n),$$

where  $a_i \in H^1(V, \mathbb{F}_p)$ ,  $x_i \in H^2(V, \mathbb{F}_p)$ , and  $n$  is the rank of  $V$ . That is,  $a_1, \dots, a_n$  is a basis of the exterior copy of  $V^*$ , and  $x_1, \dots, x_n$  is a basis of the polynomial copy. The product  $a_1 a_2 \cdots a_n \in H^n(V, \mathbb{F}_p)$  is a basis of the top exterior power  $\Lambda^n(V^*)$ . The Steenrod algebra  $\mathcal{A}$  acts on the cohomology ring, making it an unstable  $\mathcal{A}$ -algebra with  $\beta(a_i) = x_i$  and  $\mathcal{P}^1(x_i) = x_i^p$ . Observe that  $L_n(x_1, \dots, x_n)$  is essential, for the same reason as in the case  $p = 2$ .

### 3. The Mùì invariants

Let  $k$  be a finite field and  $V$  a finite dimensional  $k$ -vector space. Consider the natural action of  $GL(V)$  on  $V^*$ . The Dickson invariants generate the invariants for the induced action of  $GL(V)$  on the polynomial algebra  $S(V^*)$ . But there is also an induced action on the polynomial tensor exterior algebra  $S(V^*) \otimes_k \Lambda(V^*)$ , and the Mùì invariants are  $SL(V)$ -invariants of this action: see Mùì's original paper [7] as well as Crabb's modern treatment [5].

We shall need several properties of the Mùì invariants. For the convenience of the reader, we rederive these from scratch: but see Mùì's papers [7, 9] and Sum's work [11].

*Notation.* Often we shall work with the direct sum decomposition

$$H^*(V, \mathbb{F}_p) = \bigoplus_{r=0}^n N_r(V),$$

where  $n$  is the rank of  $V$  and we set

$$N_r(V) = S(V^*) \otimes_{\mathbb{F}_p} \Lambda^r(V^*).$$

Observe that restriction to each subgroup respects this decomposition. This means that the essential ideal is well-behaved with respect to this decomposition:

$$\text{Ess}(V) = \bigoplus_{r=0}^n N_r(V) \cap \text{Ess}(V). \quad (3)$$

*Definition.* Recall that  $L_n(x_1, \dots, x_n)$  is the determinant of the  $n \times n$ -matrix

$$C = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{p^{n-1}} & x_2^{p^{n-1}} & \cdots & x_n^{p^{n-1}} \end{pmatrix},$$

where  $C_{s,i} = x_i^{p^{s-1}}$  for  $1 \leq s \leq n$ . For each such  $s$ , define  $E(s)$  to be the matrix obtained from  $C$  by deleting row  $s$  and then prefixing  $(a_1 \ a_2 \ \cdots \ a_n)$  as new first row: so

$$\det E(s) = \sum_{i=1}^n (-1)^{i+1} \gamma_{s,i} a_i,$$

where  $\gamma_{s,i}$  is the determinant of the minor of  $C$  obtained by removing row  $s$  and column  $i$ .

Now define the Mù invariant  $M_{n,s} \in H^*(V, \mathbb{F}_p)$  by  $M_{n,s} = \det E(s)$ . Note that our indexing differs from Mù's: our  $M_{n,s}$  is his  $M_{n,s-1}$ .

$$\text{Example. So } M_{4,3} = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ x_1 & x_2 & x_3 & x_4 \\ x_1^p & x_2^p & x_3^p & x_4^p \\ x_1^{p^3} & x_2^{p^3} & x_3^{p^3} & x_4^{p^3} \end{vmatrix} \text{ and } \gamma_{2,3} = \begin{vmatrix} x_1 & x_2 & x_4 \\ x_1^{p^2} & x_2^{p^2} & x_4^{p^2} \\ x_1^{p^3} & x_2^{p^3} & x_4^{p^3} \end{vmatrix}.$$

**Lemma 3.1.**  $M_{n,s} \in N_1(V) \cap \text{Ess}(V)$ .

*Proof.* By construction  $M_{n,s} \in N_1(V)$ . Restricting to a maximal subgroup of  $V$  involves killing a nonzero linear form on  $V^*$ : that is, one imposes a linear dependence on the  $a_i$  and consequently the same linear dependency on the  $x_i$ . So one obtains a linear dependency between the columns of  $E(s)$ , meaning that restriction kills  $M_{n,s} = \det E(s)$ .  $\square$

**Lemma 3.2.**  $\text{Ess}(V)^2 = L_n(V) \cdot \text{Ess}(V)$ .

*Proof.* As  $L_n(V)$  is essential, the left hand side contains the right. Now let  $H$  be a maximal subgroup of  $V$ . Then  $H = \ker(a)$  for some nonzero  $a \in H^1(V, \mathbb{F}_p)$ . Let  $x = \beta(a) \in H^2$ . Observe that the kernel of restriction to  $H$  is generated by  $a, x$ . Suppose that  $f, g$  both lie in this kernel: then we may write  $f = f'a + f''x$ ,  $g = g'a + g''x$ , and so  $fg = (f''g' \pm f'g'')ax + f''g''x^2$ , that is  $fg = xh$  for  $h = (f''g' \pm f'g'')a + f''g''x \in \ker \text{Res}_H$ .

Since  $H^*(V, \mathbb{F}_p)$  is a free module over the unique factorization ring  $S(V^*)$ , this means that  $fg = L_n(V) \cdot y$  for some  $y \in H^*(V, \mathbb{F}_p)$ . So  $h = \frac{L_n(V)}{x} \cdot y$ . As  $\text{Res}_H(h) = 0$  and  $\text{Res}_H\left(\frac{L_n(V)}{x}\right)$  is a non-zero divisor, we deduce that  $\text{Res}_H(y) = 0$ . So  $y \in \text{Ess}(V)$ .  $\square$

*Definition 3.3.* Let  $S = \{s_1, \dots, s_r\} \subseteq \{1, \dots, n\}$  be a subset with  $s_1 < s_2 < \dots < s_r$ . In view of Lemmas 3.1 and 3.2 we may define the Mù invariant  $M_{n,S} \in N_r(V) \cap \text{Ess}(V)$  by

$$M_{n,S} = \frac{1}{L_n(V)^{r-1}} M_{n,s_1} M_{n,s_2} \cdots M_{n,s_r}.$$

Note in particular that  $M_{n,\emptyset} = L_n(V)$ .

*Remark.* Observe that

$$M_{n,S} M_{n,T} = \begin{cases} \pm L_n(V) M_{n,S \cup T} & \text{if } S \cap T = \emptyset; \\ 0 & \text{otherwise.} \end{cases} \quad (4)$$

#### 4. Joint annihilators

In this section we study the joint annihilators of the  $M_{n,S}$  with  $|S| = r$  as a means to prove Theorem 1.2.

**Lemma 4.1.** *The joint annihilator of  $M_{n,1}, \dots, M_{n,n}$  is  $N_n(V)$ .*

*Proof.* The element  $a_1 \dots a_n$  is a basis for  $\Lambda^n(V)$  and is clearly annihilated by each  $M_{n,s}$ . Conversely, suppose that  $y \neq 0$  is annihilated by every  $M_{n,s}$ . As  $M_{n,s}N_r(V) \subseteq N_{r+1}(V)$  we may assume without loss of generality that  $y \in N_r(V)$  for some  $r$ . Multiplying once or more by suitably chosen elements  $a_i$ , we reduce to the case  $y \in N_{n-1}(V)$ .

Denote by  $K$  the field of fractions of  $S(V^*)$ , and let  $W = K \otimes_k \Lambda^{n-1}(V^*)$ . Each  $M_{n,s}$  induces a linear form  $\phi_s: W \rightarrow K$  given by  $\phi_s(w)a_1 \dots a_n = M_{n,s}w$ . By assumption,  $y \neq 0$  lies in the kernel of every  $\phi_s$ . A basis for  $W$  consists of the elements  $a_1 \dots \widehat{a_r} \dots a_n$  for  $1 \leq r \leq n$ , where the hat denotes omission. Now,

$$M_{n,s} \cdot a_1 \dots \widehat{a_r} \dots a_n = (-1)^{r+1} \gamma_{s,r} a_r \cdot a_1 \dots \widehat{a_r} \dots a_n,$$

and so

$$\phi_s(a_1 \dots \widehat{a_r} \dots a_n) = \gamma_{s,r}.$$

Now consider the matrix  $\Gamma \in M_n(K)$  given by  $\Gamma_{s,r} = \gamma_{s,r}$ . If one transposes and then multiplies the  $i$ th row by  $(-1)^i$  and the  $j$ th column by  $(-1)^j$ , then one obtains the adjugate matrix of  $C$ . As the determinant of  $C$  is  $L_n(V)$  and in particular nonzero, it follows that  $\det \Gamma \neq 0$ .

So by construction of  $\Gamma$ , the  $\phi_s$  form a basis of  $W^*$ . So their common kernel is zero, contradicting our assumption on  $y$ .  $\square$

**Corollary 4.2.** *The joint annihilator of  $\{M_{n,S} : |S| = r\}$  is  $\bigoplus_{s \geq n-r+1} N_s(V)$ .*

*Proof.* By induction on  $r$ , Lemma 4.1 being the case  $r = 1$ . As  $M_{n,S} \in N_{|S|}(V)$  and  $N_r(V)N_s(V) \subseteq N_{r+s}(V)$ , the annihilator is at least as large as claimed. Now suppose that  $y \in H^*(V, \mathbb{F}_p)$  does not lie in  $\bigoplus_{s \geq n-r+1} N_s(V)$ . We may therefore write

$$y = \sum_{s=0}^n y_s$$

with  $y_s \in N_s(V)$ , and we know that  $s_0 \leq n - r$  for  $s_0 = \min\{s \mid y_s \neq 0\}$ . As  $y_{s_0} \neq 0$  and  $y_{s_0} \notin N_n(V)$ , Lemma 4.1 tells us that  $y_{s_0} M_{n,t} \neq 0$  for some  $1 \leq t \leq n$ . As  $y_{s_0} M_{n,t} \in N_{s_0+1}(V)$ , we conclude that  $y M_{n,t}$  lies outside  $\bigoplus_{s \geq n-r+2} N_s(V)$ . So the inductive hypothesis means that there is some  $T$  with  $|T| = r - 1$  and  $y M_{n,t} M_{n,T} \neq 0$ . So  $y M_{n,S} \neq 0$  for  $S = T \cup \{t\}$  and  $|S| = r$ : note that  $t \in T$  is impossible.  $\square$

**Corollary 4.3.** *Every  $M_{n,S}$  is nonzero. For  $S = \underline{n} = \{1, \dots, n\}$  we have*

$$M_{n,\underline{n}} \text{ is a nonzero scalar multiple of } a_1 a_2 \cdots a_n.$$

*Proof.* Observe that  $M_{n,\underline{n}}$  is a scalar multiple of  $a_1 \cdots a_n$  for degree reasons. The case  $r = n$  of Corollary 4.2 says that  $1 \in N_0(V)$  does not annihilate  $M_{n,\underline{n}}$  and therefore  $M_{n,\underline{n}} \neq 0$ . But from Eqn (4) we see that every  $M_{n,S}$  divides  $L_n(V) M_{n,\underline{n}} \neq 0$ .  $\square$

*Proof of Theorem 1.2.* In view of Eqn (3) it suffices to show that for each  $r$  the Müi invariants  $M_{n,S}$  with  $|S| = r$  are a basis of the  $S(V^*)$ -module  $N_r(V) \cap \text{Ess}(V)$ . We observed in Definition 3.3 that these  $M_{n,S}$  lie in this module.

So suppose that  $y \in N_r(V) \cap \text{Ess}(V)$ . We should like there to be  $f_S \in S(V^*)$  such that

$$y = \sum_{|S|=r} f_S M_{n,S}. \quad (5)$$

Note that for  $T = \underline{n} - S$  we have  $M_{n,S} M_{n,T} = \pm L_n(V) M_{n,\underline{n}}$  by Eqn (4). Define  $\varepsilon_S \in \{+1, -1\}$  by  $M_{n,S} M_{n,T} = \varepsilon_S L_n(V) M_{n,\underline{n}}$ . So Eqn. (5) implies that we should define  $f_S$  by

$$f_S M_{n,\underline{n}} = \frac{1}{L_n(V)} \varepsilon_S y M_{n,T},$$

since  $T \cap S' \neq \emptyset$  and therefore  $M_{n,S'} M_{n,T} = 0$  for all  $S' \neq S$  with  $|S'| = r$ . Note that this definition of  $f_S$  makes sense, as  $y M_{n,T}$  lies in both  $N_r(V) N_{n-r}(V) = N_n(V)$  and  $L_n(V) \text{Ess}(V)$ , the latter inclusion coming from Lemma 3.2.

With this definition of  $f_S$  we have

$$\left( y - \sum_{|S|=r} f_S M_{n,S} \right) M_{n,T} = 0$$

for every  $|T| = n - r$ . As  $y - \sum_{|S|=r} f_S M_{n,S}$  lies in  $N_r(V)$ , this means that  $y = \sum_{|S|=r} f_S M_{n,S}$  by Corollary 4.2.

Finally we show linear independence. Suppose that  $g_S \in S(V^*)$  are such that  $\sum_{|S|=r} g_S M_{n,S} = 0$ . Pick one  $S$  and set  $T = \underline{n} - S$ . Multiplying by  $M_{n,T}$ , we deduce that  $g_S = 0$ .  $\square$

## 5. The action of the Steenrod algebra

To prepare for the proof of Theorem 1.1 we shall study the operation of the Steenrod algebra on the Mù invariants.

**Lemma 5.1.**

$$\beta(M_{n,s}) = \begin{cases} L_n(V) & s = 1 \\ 0 & \text{otherwise} \end{cases} \quad \beta(L_n(V)) = 0. \quad (6)$$

For  $0 \leq s \leq n - 2$  we have:

$$\mathcal{P}^{p^s}(M_{n,r}) = \begin{cases} M_{n,r-1} & r = s + 2 \\ 0 & \text{otherwise} \end{cases} \quad \mathcal{P}^{p^s}(L_n(V)) = 0. \quad (7)$$

*Proof.* One sees Eqn (6) by inspecting the determinants in the definition of  $M_{n,s}$  and  $L_n(V)$ . The proof of Eqn (7) is also based on an inspection of these determinants. Recall that  $\mathcal{P}^m(a_i) = 0$  for every  $m > 0$ , and that  $\mathcal{P}^m(x_i^{p^s})$  is zero too except for  $\mathcal{P}^{p^s}(x_i^{p^s}) = x_i^{p^{s+1}}$ . We may use the Cartan formula

$$\mathcal{P}^m(xy) = \sum_{a+b=m} \mathcal{P}^a(x)\mathcal{P}^b(y)$$

to distribute  $\mathcal{P}^{p^s}$  over the rows of the determinant. As  $p^s$  cannot be expressed as a sum of distinct smaller powers of  $p$ , we only have to consider summands where all of  $\mathcal{P}^{p^s}$  is applied to one row and the other rows are unchanged. This will result in two rows being equal unless it is the row consisting of the  $x_i^{p^{s+1}}$  that is missing.  $\square$

**Lemma 5.2.** Let  $S = \{s_1, \dots, s_r\}$  with  $1 \leq s_1 < s_2 < \dots < s_r \leq n$ .

1. Suppose that  $1 \notin S$ . Then  $M_{n,S} = \beta(M_{n,S \cup \{1\}})$ .
2.  $L_n(V)^{r-1} \mathcal{P}^m(M_{n,S}) = \mathcal{P}^m(M_{n,s_1} \cdots M_{n,s_r})$  for each  $m < p^{n-1}$ .

3. For  $2 \leq u \leq n$  set  $X = \{s \in S \mid s \leq u\}$  and  $Y = \{s \in S \mid s > u\}$ . Then

$$L_n(V)\mathcal{P}^{p^{u-2}}(M_{n,S}) = \mathcal{P}^{p^{u-2}}(M_{n,X}) \cdot M_{n,Y}.$$

4. For  $1 \leq r \leq n$  and  $0 < m < p^{n-1}$  one has  $\mathcal{P}^m(M_{n,\{1,\dots,r\}}) = 0$ .

5. For  $2 \leq u \leq n$  one has  $\mathcal{P}^{p^{u-2}}(M_{n,\{1,\dots,u-2,u\}}) = M_{n,\{1,\dots,u-1\}}$ .

*Proof.* Recall that

$$L_n(V)^r M_{n,S} = L_n(V)M_{n,s_1} \cdots M_{n,s_r}. \quad (8)$$

The first two parts follow by applying Equations (6) and (7).

Recall that by the Adem relations each  $\mathcal{P}^m$  may be expressed in terms of the  $\mathcal{P}^{p^s}$  with  $p^s \leq m$ . So the third part follows from the second, since we deduce from Eqn. (7) that  $\mathcal{P}^m(M_{n,s}) = 0$  if  $0 < m \leq p^{u-2}$  and  $s > u$ .

Fourth part: By induction on  $r$ . Follows for  $r = 1$  from the Adem relations and Eqn (7). Inductive step: Enough to consider  $\mathcal{P}^{p^s}$  for  $0 \leq s \leq n - 2$ . By the inductive hypothesis and a similar argument to the third part, deduce that

$$L_n(V)\mathcal{P}^{p^s}(M_{n,\{1,\dots,r\}}) = M_{n,\{1,\dots,r-1\}}\mathcal{P}^{p^s}(M_{n,r}).$$

But this is zero by Eqn (7), since  $M_{n,\{1,\dots,r-1\}}M_{n,r-1} = 0$ .

Fifth part: Using the fourth part and an argument similar to the third part, deduce that

$$L_n(V)\mathcal{P}^{p^{u-2}}(M_{n,\{1,\dots,u-2,u\}}) = M_{n,\{1,\dots,u-2\}}\mathcal{P}^{p^{u-2}}(M_{n,u}) = M_{n,\{1,\dots,u-2\}}M_{n,u-1} :$$

but this is  $L_n(V)M_{n,\{1,\dots,u-1\}}$ . □

*Proof of Theorem 1.1.* We shall show that for every  $M_{n,S}$  there is an element  $\theta$  of the Steenrod algebra with  $M_{n,S} = \theta(M_{n,\underline{n}})$ . We do this by decreasing induction on  $r = |S|$ . It is trivially true for  $r = n$ , so assume now that  $r < n$ . Amongst the  $S$  with  $|S| = r$  we shall proceed by induction over  $u$ , the smallest element of  $\underline{n} - S$ . So

$$S = \{1, \dots, u-1\} \cup Y \quad \text{with } s > u \text{ for every } s \in Y.$$

Part 1 of Lemma 5.2 covers the case  $u = 1$ , so assume that  $u \geq 2$ . Set  $T = \{1, \dots, u-2, u\}$ . We complete the induction by showing that  $M_{n,S} = \mathcal{P}^{p^{u-2}}(M_{n,T \cup Y})$ . Part 3 of

Lemma 5.2 gives us

$$L_n(V)P^{p^{\mu-2}}(M_{n,T \cup Y}) = P^{p^{\mu-2}}(M_{n,T})M_{n,Y}.$$

But  $P^{p^{\mu-2}}(M_{n,T}) = M_{n,\{1, \dots, \mu-1\}}$ , by Part 5 of that lemma. So  $P^{p^{\mu-2}}(M_{n,T \cup Y}) = M_{n,S}$ , as claimed.  $\square$

*Remark.* Theorem 1.2 shows that the  $S(V^*)$ -module generated by the Mù invariants  $M_{n,S}$  is the essential ideal and therefore closed under the action of the Steenrod algebra. One may however see more directly that this  $S(V^*)$ -module is Steenrod closed. This is observed for example in [11]. In view of Lemma 5.2 and Equations (6) and (7) it only remains to show that  $\mathcal{P}^{p^{n-1}}(M_{n,s})$  lies in our  $S(V^*)$ -module. Now  $P^{p^{n-1}}(M_{n,n}) = 0$  by the unstable condition, so suppose  $s < n$ . Recall that  $M_{n,s}$  is a determinant, the last row of the matrix having entries  $x_i^{p^{n-1}}$ . So applying  $P^{p^{n-1}}$  replaces these entries by  $x_i^{p^n}$ . But it is well known that  $x_i^{p^n}$  is an  $S(V^*)$ -linear combination of the  $x_i^{p^r}$  for  $r \leq n-1$ , and that the coefficients are independent of  $i$ : this is the “fundamental equation” in the sense of [12], and the coefficients are the Dickson invariants  $c_{n,r}$  in  $S(V^*)$ . Applying  $S(V^*)$ -linearity of the determinant in the bottom row of the matrix, one deduces that  $P^{p^{n-1}}(M_{n,s})$  is an  $S(V^*)$ -linear combination of the  $M_{n,r}$ .

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