A THEOREM OF JON F. CARLSON ON FILTRATIONS OF MODULES

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Abstract. We give an alternative proof to a theorem of Jon F. Carlson [5] which states that if G is a finite group and k is a field of characteristic p, then any kG-module is a direct summand of a module which has a filtration whose sections are induced from elementary abelian p-subgroups of G. We also prove two new theorems which are closely related to Carlson's theorem.

1. Introduction

Let G be a finite group and let k be a field of characteristic $p > 0$. Let $\mathcal{H}$ be a collection of subgroups of G. We say that a kG-module M is filtered by modules induced from $\mathcal{H}$ if there is a filtration

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{n-1} \subseteq M_n = M$$

such that for each $i = 1, 2, \ldots, n$, there is a subgroup $H_i \in \mathcal{H}$ and a kH_i-module W_i such that $M_i/M_{i-1} \cong W_i \uparrow_{H_i}^G$. We consider the following theorem of Jon F. Carlson.

Theorem 1.1 (Carlson [5])**.** *Any kG-module M is a direct summand of a module that is filtered by modules induced from elementary abelian p-subgroups.*

This theorem provides another way to see the role of elementary abelian p-groups in modular representation theory, and has many applications, including the Chouinard's theorem for finitely generated modules (see Theorem 8.2.12 in [7]). Carlson proves Theorem 1.1 by first reducing it to p-groups and then by showing that it follows from the following statement by induction.

Theorem 1.2 (Carlson [5])**.** *Suppose that G is a p-group which is not elementary abelian. Then there is a sequence $H_1, \ldots, H_n$ of maximal subgroups of G such that $k \oplus \Omega^{1-n}(k) \oplus (proj)$ has a filtration*

$$0 = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_n = k \oplus \Omega^{1-n}(k) \oplus (proj)$$

where $L_i/L_{i-1} \cong \Omega^{1-i}(k) \uparrow_{H_i}^G$ for $i = 1, \ldots, n$.

The maximal subgroups $H_1, \ldots, H_n$ in the above theorem are not necessarily distinct. In fact, for $p > 2$ we take $n = 2k$ and $H_{2i} = H_{2i-1}$ for all $i = 1, \ldots, k$. Carlson proves the existence of such a filtration using Serre's theorem on vanishing products in group cohomology (see Theorem 3.1 for the statement of Serre's theorem) and a hypercohomology calculation with coefficients in a chain complex obtained from Serre's theorem. Theorem 1.2 plays an important role in [6] where the authors use this filtration to find an upper bound for the dimensions of critical endo-trivial modules.

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In this paper we give an alternative proof to Theorem 1.2 using L_ζ-modules. The L_ζ-modules are defined as follows: Associated to a cohomology class $\zeta \in H^n(G,k)$, there is a module L_ζ defined as the kernel of the representing homomorphism $\hat{\zeta} : \Omega^n(k) \to k$. Modules of this form are called L_ζ-modules. They are commonly used to relate cohomology theory with modular representation theory; for example, they appear in many results about varieties of modules (see [1], [2], [3], [7], [8]).

In our proof for Theorem 1.2, we still use Serre's theorem, but we avoid the hypercohomology calculation. Given $\zeta_1, \dots, \zeta_n \in H^1(G, \mathbb{F}_p)$ satisfying the conclusion of Serre's theorem, i.e., $\beta(\zeta_1)\cdots\beta(\zeta_n) = 0$ where β is the Bockstein map, we observe that $L_{\beta(\zeta_1)\cdots\beta(\zeta_n)} = \Omega(k) \oplus \Omega^{2n}(k)$ has a filtration whose sections are isomorphic to Heller shifts of $L_{\beta(\zeta_i)}$'s. This is an easy consequence of a known exact sequence for the L_ζ when ζ is a product of two cohomology classes (see Proposition 2.3). Next, we show that for every $\zeta \in H^1(G, \mathbb{F}_p)$ with kernel $H \leq G$, there is a 2-step filtration for $L_{\beta(\zeta)} \oplus (proj)$ such that the sections are induced from H (see Proposition 2.4 and 2.5). In fact, for $p = 2$, the argument is much simpler since, in this case, Serre's theorem is true without Bocksteins, and we have $L_\zeta \cong k \uparrow_H^G$ for every $\zeta \in H^1(G, \mathbb{F}_2)$ with kernel H. We present our alternative proof in Section 3.

In the rest of the paper, we prove two theorems which are variations of Carlson's theorems. The first one is a generalization of Proposition 2.4 and 2.5, and it is strong enough to imply Theorem 1.2 when it is applied to a suitable extension.

Theorem 1.3. *Let ζ be the cohomology class in $H^n(G,k)$ which is represented by the extension*

$$E : 0 \to k \to M_{n-1} \to \cdots \to M_0 \to k \to 0.$$

Then, $L_\zeta \oplus (proj)$ has a filtration

$$0 = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_n = L_\zeta \oplus (proj)$$

with $L_i/L_{i-1} \cong \Omega^{n-i+1}(M_{i-1})$ for $i = 1, \dots, n$.

Here the notation $(proj)$ means that the statement is true after adding a suitable projective summand. We will be using this notation throughout the paper.

In section 5, we introduce the varieties of modules, and prove

Theorem 1.4. *Let $\mathcal{H}$ be a collection of subgroups of G. Then, for a finitely generated kG-module M, the following are equivalent:*

(i) $V_G(M) = \bigcup_{H\in\mathcal{H}} \operatorname{res}^*_{G,H}(V_H(M \downarrow_H^G))$.

(ii) There exists a finitely generated kG-module V such that $M \oplus V$ is filtered by modules induced from $\mathcal{H}$.

We conclude the paper with the following application:

Corollary 1.5. *Let G be an elementary abelian 2-group. If $\zeta \in H^n(G,k)$ is represented by the extension*

$$E : 0 \to k \to M_{n-1} \to \cdots \to M_0 \to k \to 0$$

where M_i's are direct sums of modules induced from proper subgroups, then ζ is a scalar multiple of a product of one dimensional classes in $H^1(G, \mathbb{F}_2)$. In particular, E is equivalent to an extension coming from a topological group action on a sphere.

There is a similar result for $p > 2$ under stronger conditions. This result is also proved in Section 5.

Throughout this paper, G always denotes a finite group, k is a field of characteristic $p > 0$. We assume that all kG-modules are finitely generated, and all tensor products are over k unless otherwise is stated clearly.

2. Preliminaries

Given a kG-module M, the Heller shift of M is defined as the kernel of the surjection $P(M) \to M$ where $P(M)$ denotes the projective cover of M. We denote the Heller shift of M by $\Omega(M)$. The n-th Heller shift of M is defined inductively by $\Omega^n(M) = \Omega(\Omega^{n-1}(M))$ for positive n. Similarly, the negative shift is defined inductively by $\Omega^{-n}(M) = \Omega^{-1}(\Omega^{-n+1}(M))$ where minus one Heller shift $\Omega^{-1}(M)$ of a kG-module M is defined as the cokernel of the injection $M \to I(M)$, where $I(M)$ is the injective hull of M. Uniqueness of projective cover and injective hull gives the uniqueness of the modules $\Omega^n(M)$ up to isomorphism. The details about Heller shifts can be found in many books on modular representation theory. We will use the standard properties of Heller shifts without listing them here. We refer the reader to Proposition 4.4 of [4] for a complete list of these properties.

Given a projective resolution

$$\cdots \to P_{n+1} \xrightarrow{\partial_{n+1}} P_n \xrightarrow{\partial_n} P_{n-1} \xrightarrow{\partial_{n-1}} \cdots \to P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\varepsilon} k \to 0$$

of k as a kG-module, we say that it is minimal if P_0 is the projective cover of k, P_1 is the projective cover of $\ker \varepsilon$, and P_n is the projective cover of $\ker \partial_{n-1}$ for $n \geq 2$. So, by the above description of Heller shifts, we have $\Omega(k) = \ker \varepsilon$, and $\Omega^n(k) = \ker \partial_{n-1}$ for all $n \geq 2$. Note that the cohomology group $H^n(G,k)$ is the n-th cohomology of the cochain complex $\mathrm{Hom}_{kG}(P_*, k)$. Let $f \in \mathrm{Hom}_{kG}(P_n, k)$ be a cocycle representing $\zeta \in H^n(G,k)$, then the cocycle condition gives that f restricted to the image of ∂_{n+1} is zero. Thus f gives a map $\hat{\zeta} : \Omega^n(k) \to k$ called the representing homomorphism for $\zeta \in H^n(G,k)$. Two homomorphisms $\hat{\zeta}$ and $\hat{\zeta}'$ represent the same cohomology class if they differ by a homomorphism which factors through a projective module. The only homomorphism $\Omega^n(k) \to k$ that factors through a projective module is the zero homomorphism, so the representing homomorphism is unique (see page 140 in [1] or page 16-17 of [4] for details). The L_ζ-modules are defined as follows:

Definition 2.1. Let ζ be a cohomology class in $H^n(G,k) - \{0\}$ for $n \geq 1$ and let $\hat{\zeta} : \Omega^n(k) \to k$ be the homomorphism representing ζ. We define L_ζ as the kernel of the homomorphism $\hat{\zeta}$. When $\zeta = 0$, we set $L_\zeta = \Omega(k) \oplus \Omega^n(k)$.

Since the representing homomorphism $\hat{\zeta}$ is uniquely defined, L_ζ is well defined up to isomorphism. As a consequence of the definition we have the following diagram:

$$\begin{array}{ccccccccccccc}
 & & L_\zeta & = & L_\zeta & & & & & & & & \\
 & & \downarrow & & \downarrow & & & & & & & & \\
0 & \longrightarrow & \Omega^n(k) & \longrightarrow & P_{n-1} & \longrightarrow & P_{n-2} & \longrightarrow & . & \longrightarrow & k & \longrightarrow & 0 \\
 & & \hat{\zeta}\downarrow & & \downarrow & & \| & & \| & & \| & & \\
0 & \longrightarrow & k & \longrightarrow & P_{n-1}/L_\zeta & \longrightarrow & P_{n-2} & \longrightarrow & ... & \longrightarrow & k & \longrightarrow & 0
\end{array}$$

In particular, we have

Lemma 2.2. *For every $\zeta \in H^n(G,k) - \{0\}$, there is an exact sequence*

$$0 \to k \to \Omega^{-1}(L_\zeta) \oplus (proj) \to \Omega^{n-1}(k) \to 0$$

with an extension class corresponding to ζ under the isomorphisms

$$H^n(G,k) \cong \mathrm{Ext}^1_{kG}(\Omega^{n-1}(k), k).$$

When $L_\zeta \neq 0$, the above sequence is exact without a $(proj)$ summand.

Proof. The above diagram gives the short exact sequence

$$0 \to k \to P_{n-1}/L_\zeta \to \Omega^{n-1}(k) \to 0.$$

which gives the desired sequence after applying the isomorphism $P_{n-1}/L_\zeta \cong \Omega^{-1}(L_\zeta) \oplus (proj)$.

When $L_\zeta \neq 0$, the module P_{n-1} is the injective hull of L_ζ, hence $P_{n-1}/L_\zeta \cong \Omega^{-1}(L_\zeta)$. So, when ζ is not a periodicity generator (for example when G is not a periodic group), then the above sequence is exact without a $(proj)$ summand in the middle. □

In our alternative proof for Theorem 1.2, the main ingredient is the following exact sequence:

Proposition 2.3. *If $\zeta_1 \in H^r(G,k)$ and $\zeta_2 \in H^s(G,k)$, then there is an exact sequence*

$$0 \to \Omega^r(L_{\zeta_2}) \to L_{\zeta_1 \cdot \zeta_2} \oplus (proj) \to L_{\zeta_1} \to 0.$$

Proof. See Lemma 5.9.3 on page 191 of [2]. □

For the rest of the section, we assume G is a (finite) p-group. Recall that

$$H^1(G, \mathbb{F}_p) \cong \mathrm{Hom}(G, \mathbb{Z}/p).$$

Given a one dimensional class $\zeta \in H^1(G, \mathbb{F}_p)$, the kernel of the corresponding homomorphism is usually referred as the kernel of ζ.

Lemma 2.4. *Let G be a 2-group, and ζ be a cohomology class in $H^1(G, \mathbb{F}_2)$ considered as a class in $H^1(G,k)$. Then $L_\zeta \cong \Omega(k) \uparrow_H^G$ where H is the kernel of ζ.*

Proof. We have the following commutative diagram

$$\begin{array}{ccccccccc}
 & & L_\zeta & = & L_\zeta & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \Omega(k) & \longrightarrow & P_0 & \longrightarrow & k & \longrightarrow & 0 \\
 & & \hat{\zeta}\downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & k & \longrightarrow & P_0/L_\zeta & \longrightarrow & k & \longrightarrow & 0.
\end{array}$$

from which we obtain that the extension class of the sequence $0 \to k \to P_0/L_\zeta \to k \to 0$ corresponds to ζ under the isomorphism $H^1(G,k) \cong \mathrm{Ext}^1_{kG}(k,k)$. There is an extension of the form

$$0 \to k \to k \uparrow_H^G \to k \to 0$$

with extension class equal to ζ. From the equivalence of the exact sequences we get $k \uparrow_H^G \cong P_0/L_\zeta \cong \Omega^{-1}(L_\zeta) \oplus (proj)$. Taking the first Heller shift of this isomorphism, we obtain $L_\zeta \cong \Omega(k \uparrow_H^G)$. Note that for p-groups, $\Omega(k \uparrow_H^G) \cong \Omega(k) \uparrow_H^G$. We conclude that $L_\zeta \cong \Omega(k) \uparrow_H^G$. □

In the case where p is an odd prime we have a similar result.

Lemma 2.5. *Let G be a finite p-group, and let $\beta(\zeta)$ be the Bockstein of a one dimensional class $\zeta \in H^1(G, \mathbb{F}_p)$. Consider $\beta(\zeta)$ as a class in $H^2(G,k)$. Then, $L_{\beta(\zeta)} \oplus (proj)$ has a filtration $0 = M_0 \subseteq M_1 \subseteq M_2 = L_{\beta(\zeta)} \oplus (proj)$ with the property $M_2/M_1 \cong \Omega(k) \uparrow_H^G$ and $M_1 \cong \Omega^2(k) \uparrow_H^G$ where H is the kernel of ζ.*

Proof. By Proposition 5.7.6 in [7], there is an extension of the form

$$0 \to k \to k\uparrow_H^G \to k\uparrow_H^G \to k \to 0$$

with extension class equal to $\beta(\zeta)$. Thus, we have a diagram

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \Omega^2(k) & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & k & \longrightarrow & 0 \\
 & & \widehat{\beta(\zeta)}\downarrow & & \downarrow f_1 & & \downarrow f_0 & & \| & & \\
0 & \longrightarrow & k & \longrightarrow & k\uparrow_H^G & \longrightarrow & k\uparrow_H^G & \longrightarrow & k & \longrightarrow & 0
\end{array}$$

where the left most homomorphism is the representing homomorphism for $\beta(\zeta)$.

Thus, we have the diagram

$$\begin{array}{ccccccccccc}
 & & & & \ker\tilde{f}_1 & \xrightarrow{\cong} & \ker f_0 & & & & \\
 & & & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & k & \longrightarrow & P_1/L_{\beta(\zeta)} & \longrightarrow & P_0 & \longrightarrow & k & \longrightarrow & 0 \\
 & & \| & & \downarrow \tilde{f}_1 & & \downarrow f_0 & & \| & & \\
0 & \longrightarrow & k & \longrightarrow & k\uparrow_H^G & \longrightarrow & k\uparrow_H^G & \longrightarrow & k & \longrightarrow & 0
\end{array}$$

where $\ker f_0 = \Omega(k\uparrow_H^G) \oplus (proj)$. The first vertical short exact sequence in the above diagram gives us the sequence

$$0 \to \Omega(k\uparrow_H^G) \oplus (proj) \to \Omega^{-1}(L_{\beta(\zeta)}) \oplus (proj) \to k\uparrow_H^G \to 0.$$

If we tensor this exact sequence with $\Omega(k)$ over k, we get

$$0 \to \Omega^2(k\uparrow_H^G) \oplus (proj) \to L_{\beta(\zeta)} \oplus (proj) \to \Omega(k\uparrow_H^G) \oplus (proj) \to 0.$$

Note that we can cancel projective modules at both ends of the sequence since projective kG-modules are also injective. Thus, we get an exact sequence

$$0 \to \Omega^2(k\uparrow_H^G) \to L_{\beta(\zeta)} \oplus (proj) \to \Omega(k\uparrow_H^G) \to 0$$

which gives the desired filtration. □

Throughout the paper, we will come across the situations, where we will need to cancel projective modules from both ends of exact sequences as we did in the above proof. More generally, we will need to cancel projective modules from sections of filtrations of modules. We quote the following lemma from [7]. The proof easily follows from the fact that projective kG-modules are injective.

Lemma 2.6. *Suppose that M is a kG-module which has a filtration*

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$$

with $M_i/M_{i-1} \cong X_i \oplus P_i$ for some projective modules P_i. Then, $M \cong M' \oplus P$ for some projective module P such that M' has a filtration

$$0 = M'_0 \subseteq M'_1 \subseteq \cdots \subseteq M'_n = M'$$

with $M'_i/M'_{i-1} \cong X_i$ for all $i = 1, \ldots, n$.

3. The Alternative Proof

The aim of this section is to give a proof for Theorem 1.2 using L_ζ-modules. First, we recall Serre's theorem on vanishing products in group cohomology.

Theorem 3.1 (Serre [9]). *Suppose that G is a p-group which is not elementary abelian. Then there is a sequence $\zeta_1, \dots, \zeta_n \in H^1(G, \mathbb{F}_p)$ of nonzero elements such that*

$$\zeta_1\zeta_2\cdots\zeta_n = 0 \quad \text{if} \quad p = 2,$$
$$\beta(\zeta_1)\beta(\zeta_2)\cdots\beta(\zeta_n) = 0 \quad \text{if} \quad p > 2.$$

Now, we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. First lets assume $p = 2$. Let $\zeta_1, \dots, \zeta_n$ be classes in $H^1(G, \mathbb{F}_2)$ satisfying the conclusion of Serre's theorem. Then, we have

$$L_{\zeta_1\cdots\zeta_n} \cong \Omega(k) \oplus \Omega^n(k).$$

By Proposition 2.3, for each $i = 1, \dots, n-1$, there is an exact sequence of the form

$$0 \to \Omega^{n-i}(L_{\zeta_i}) \to L_{\zeta_i\cdots\zeta_n} \oplus P_i \to L_{\zeta_{i+1}\cdots\zeta_n} \to 0$$

where $P_1, \dots, P_{n-1}$ are projective modules. By adding projective summands to the last two terms of the exact sequences above, we find exact sequences of the form

$$0 \to \Omega^{n-i}(L_{\zeta_i}) \to L_{\zeta_i\cdots\zeta_n} \oplus Q_i \to L_{\zeta_{i+1}\cdots\zeta_n} \oplus Q_{i+1} \to 0$$

for $i = 1, \dots, n-1$ where

$$Q_i = \oplus_{k=i}^{n-1} P_k.$$

Using these exact sequences, we obtain a filtration for

$$L_{\zeta_1\cdots\zeta_n} \oplus Q_1 \cong \Omega(k) \oplus \Omega^n(k) \oplus Q_1$$

as follows: Let $M_n = L_{\zeta_1\cdots\zeta_n} \oplus Q_1$ and $M_1 = \Omega^{n-1}(L_{\zeta_1})$. Then, $M_n/M_1 \cong L_{\zeta_2\cdots\zeta_n} \oplus Q_2$. Choose M_2 such that $M_2/M_1 \cong \Omega^{n-2}(L_{\zeta_2})$. Then the exact sequence

$$0 \to \Omega^{n-2}(L_{\zeta_2}) \to L_{\zeta_2\cdots\zeta_n} \oplus Q_2 \to L_{\zeta_3\cdots\zeta_n} \oplus Q_3 \to 0$$

gives that $M_n/M_2 \cong L_{\zeta_3\cdots\zeta_n} \oplus Q_3$ which will be the middle term of the next exact sequence. Continuing this way we obtain a filtration

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = \Omega(k) \oplus \Omega^n(k) \oplus Q_1$$

with $M_n/M_i \cong L_{\zeta_{i+1}\cdots\zeta_n} \oplus Q_{i+1}$ and $M_i/M_{i-1} \cong \Omega^{n-i}(L_{\zeta_i})$ for $i = 1, \dots, n$. Tensoring the entire filtration by $\Omega^{-n}(k)$, and cancelling the projective summands from sections as in Lemma 2.6, we obtain a filtration

$$0 = L_0 \subseteq \cdots \subseteq L_n = \Omega^{1-n}(k) \oplus k \oplus (proj)$$

with $L_i/L_{i-1} \cong \Omega^{-i}(L_{\zeta_i})$. Note that by Proposition 2.4, we have $L_{\zeta_i} \cong \Omega^1(k\uparrow_{H_i}^G)$. Thus,

$$L_i/L_{i-1} \cong \Omega^{-i}(\Omega(k\uparrow_{H_i}^G)) \cong \Omega^{1-i}(k\uparrow_{H_i}^G) \cong \Omega^{1-i}(k)\uparrow_{H_i}^G$$

where the last isomorphism is true because G is a p-group. So, the proof for $p = 2$ is complete.

Now, assume $p > 2$. As above, we can obtain a filtration

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = \Omega(k) \oplus \Omega^{2n}(k) \oplus (proj)$$

such that $M_i/M_{i-1} \cong \Omega^{2n-2i}(L_{\beta(\zeta_i)})$ for $i = 1, \dots, n$. By Proposition 2.5, $L_{\beta(\zeta_i)} \oplus (proj)$ has a filtration with sections isomorphic to $\Omega^2(k)\uparrow_H^G$ and $\Omega^1(k)\uparrow_H^G$. After adding projective modules to each M_i, we can assume M_i/M_{i-1} has a filtration with sections isomorphic to

$\Omega^{2n-2i+2}(k)\uparrow_H^G$ and $\Omega^{2n-2i+1}(k)\uparrow_H^G$. For each $i=1,\dots,n$, let N_i be the kG-module satisfying $M_{i-1}\subseteq N_i\subseteq M_i$ with $N_i/M_{i-1}\cong\Omega^{2n-2i+2}(k)\uparrow_{H_i}^G$ and $M_i/N_i\cong\Omega^{2n-2i+1}(k)\uparrow_{H_i}^G$. By taking $L_{2i}=M_i$ and $L_{2i-1}=N_i$, tensoring everything with $\Omega^{-2n}(k)$, and cancelling the projective summands on the sections, we obtain a filtration

$$0=L_0\subseteq\cdots\subseteq L_{2n}=k\oplus\Omega^{1-2n}(k)\oplus(proj)$$

where

$$L_j/L_{j-1}\cong\Omega^{1-j}(k)\uparrow_{H_i}^G$$

when $j=2i$ or $j=2i-1$. This completes the proof. □

Now, we explain briefly how Theorem 1.1 follows from Theorem 1.2. The details of this argument can be found on page 166 of [7]. First note that it is enough to prove Theorem 1.1 for $M=k$. The general case follows by tensoring everything with M. Also note that if P is a Sylow p-subgroup of G, then k is a summand of $k\uparrow_P^G$. So, it is enough to prove Theorem 1.1 for p-groups. To see this, suppose that there is a kP-module V such that $k\oplus V$ has a filtration whose sections are induced from elementary abelian p-subgroups. Inducing the entire filtration to G, we get a filtration for $k\uparrow_P^G\oplus V\uparrow_P^G$, and hence conclude that k is a direct summand of a finitely generated module which has a filtration with the desired properties.

When G is a p-group, and M is the trivial module k, Theorem 1.1 follows from Theorem 1.2 by an induction. Note that by Theorem 1.2, there is a filtration

$$0=L_0\subseteq L_1\subseteq\cdots\subseteq L_n\cong k\oplus\Omega^{1-n}(k)\oplus(proj)$$

where $L_i/L_{i-1}\cong\Omega^{1-i}(k)\uparrow_{H_i}^G$ for $i=1,\dots,n$. If any of the subgroups H_i is not an elementary abelian p-group, then we can apply Theorem 1.2 to H_i and refine the above sequence further until we reach the stage that all the subgroups involved are elementary abelian p-subgroups.

We note that the following version of Theorem 1.1 is also true.

Theorem 3.2 (Carlson [5])**.** *There exists a finitely generated kG-module V such that $k\oplus V$ has a filtration*

$$0=L_0\subseteq L_1\subseteq\cdots\subseteq L_n=k\oplus V$$

where for every $i=1,\dots,n$, the sections L_i/L_{i-1} are isomorphic to $\Omega^{n_i}(k)\uparrow_{E_i}^G$ for some integer n_i and some maximal elementary abelian p-subgroup E_i of G.

We will use this version later in Section 5.

4. A Filtration Theorem for L_ζ-Modules

The main purpose of this section is to prove Theorem 1.3 stated in the introduction. We also prove an important corollary which will be useful later in Section 5.

Definition 4.1. Let E be an n-fold extension of kG-modules with extension class α in $\mathrm{Ext}_{kG}^n(A,B)$. Suppose that $\tilde{E}:0\to\Omega^{-n+1}(B)\to M\to A\to 0$ is an extension whose class is associated to α under the isomorphism

$$\mathrm{Ext}_{kG}^n(A,B)\cong\mathrm{Ext}_{kG}^1(A,\Omega^{-n+1}(B)).$$

Then, we say $\tilde{E}$ is a contraction of E.

Lemma 4.2. *Let E be an n-fold extension of kG-modules*

$$E : 0 \to B \to M_{n-1} \to M_{n-2} \to \cdots \to M_0 \to A \to 0$$

and let

$$\tilde{E} : 0 \to \Omega^{-n+1}(B) \to M \to A \to 0$$

be a contraction of E. Then, $M \oplus (proj)$ has a filtration

$$0 = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_n = M \oplus (proj)$$

with $L_i/L_{i-1} \cong \Omega^{1-i}(M_{i-1})$ for $i = 1, \ldots, n$.

Proof. Consider the following commutative diagram

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & B & \longrightarrow & M_{n-1} & \longrightarrow & M_{n-2} & \longrightarrow & M_{n-3} & \longrightarrow & \ldots \\
 & & \| & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & B & \longrightarrow & I(M_{n-1}) & \longrightarrow & K_{n-2} & \longrightarrow & M_{n-3} & \longrightarrow & \ldots \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & \Omega^{-1}(M_{n-1}) & = & \Omega^{-1}(M_{n-1}) & & & &
\end{array}$$

where K_{n-2} is the push out and $I(M_{n-1})$ is the injective hull of M_{n-1}. From the second horizontal exact sequence, we obtain an extension

$$E' : 0 \to \Omega^{-1}(B) \oplus (proj) \to K_{n-2} \to M_{n-3} \to \cdots \to M_0 \to A \to 0$$

whose extension class corresponds to α under the isomorphism

$$\mathrm{Ext}^{n-1}_{kG}(A, \Omega^{-1}(B)) \cong \mathrm{Ext}^{n}_{kG}(A, B).$$

Let

$$\tilde{E} : 0 \to \Omega^{-n+1}(B) \to M \to A \to 0$$

be a contraction of E. Note that $\tilde{E}$ is also a contraction of E'. So, by induction, $M \oplus (proj)$ has a filtration

$$0 = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_{n-1} = M \oplus (proj)$$

with $L_i/L_{i-1} \cong \Omega^{1-i}(M_{i-1})$ for $i = 1, \ldots, n-2$ and $L_{n-1}/L_{n-2} \cong \Omega^{2-n}(K_{n-2})$.

To finish the proof, we need to refine the above filtration at the L_{n-1}/L_{n-2} section. For this, consider the exact sequence $0 \to M_{n-2} \to K_{n-2} \to \Omega^{-1}(M_{n-1}) \to 0$. After tensoring this exact sequence with $\Omega^{2-n}(k)$, and cancelling projective summands from both ends of the sequence, we get a filtration for $\Omega^{2-n}(K_{n-2}) \oplus (proj)$ with sections isomorphic to $\Omega^{2-n}(M_{n-2})$ and $\Omega^{1-n}(M_{n-1})$. By adding projective summands to L_n if necessary, we can assume L_{n-1}/L_{n-2} also have a similar filtration. So, there exists a kG-module $\tilde{L}_{n-1}$ such that

$$0 = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_{n-2} \subseteq \tilde{L}_{n-1} \subseteq L_{n-1} = M \oplus (proj)$$

is a filtration having the desired properties. □

Now, we are ready to prove Theorem 1.3.

Proof of Theorem 1.3. Let ζ be the cohomology class in $H^n(G, k)$ which is represented by the extension

$$E : 0 \to k \to M_{n-1} \to M_{n-2} \to \cdots \to M_0 \to k \to 0.$$

Tensoring the exact sequence in Lemma 2.2 with $\Omega^{1-n}(k)$, and cancelling the projective summands from both ends, we obtain a short exact sequence

$$\tilde{E} : 0 \to \Omega^{1-n}(k) \to \Omega^{-n}(L_\zeta) \oplus (proj) \to k \to 0$$

with extension class corresponding to ζ under the isomorphism

$$\mathrm{Ext}^n_{kG}(k,k) \cong \mathrm{Ext}^1_{kG}(k, \Omega^{1-n}(k)).$$

So, $\tilde{E}$ is a contraction of E. Applying Lemma 4.2, we obtain a filtration

$$0 = T_0 \subseteq T_1 \subseteq \cdots \subseteq T_n = \Omega^{-n}(L_\zeta) \oplus (proj)$$

with $T_i/T_{i-1} = \Omega^{1-i}(M_{i-1})$ for $i = 1, \ldots, n$. Tensoring the entire system with $\Omega^n(k)$, and eliminating the projective summands if necessary, gives the desired filtration. □

Note that Theorem 1.2 follows from Theorem 1.3 as a corollary. To see this, first observe that for a p-group which is not elementary abelian, there is a sequence of maximal subgroups $H_1, \ldots, H_n$ and an exact sequence

$$E : 0 \to k \to C_{2n-1} \to \cdots \to C_1 \to C_0 \to k \to 0$$

such that $C_{2i-2} \cong C_{2i-1} \cong k \uparrow^G_{H_i}$ for $i = 1, \ldots, n$ and the class of E in $\mathrm{Ext}^{2n}_{kG}(k,k)$ is zero. This is just a consequence of Serre's theorem (see Corollary 3.4 in Carlson [5]). We now apply Theorem 1.3 to this sequence and conclude Theorem 1.2.

In the rest of the section, we study some consequences of Theorem 1.3. We first introduce some more terminology: Given a kG-module M, let $J_G(M)$ denote the kernel of the homomorphism

$$M \otimes_k - : \mathrm{Ext}^*_{kG}(k,k) \to \mathrm{Ext}^*_{kG}(M,M)$$

defined by tensoring an extension with M (over k). Note that $J_G(M)$ can also be considered as the annihilating ideal of $\mathrm{Ext}^*_{kG}(M,M)$ as a $\mathrm{Ext}^*_{kG}(k,k)$-module. We now recall the following well known theorem.

Theorem 4.3. *Let $\zeta \in H^n(G,k)$ and let M be a kG-module. Then, $\zeta \in J_G(M)$ if and only if*

$$L_\zeta \otimes M \cong \Omega^n(M) \oplus \Omega(M) \oplus (proj).$$

Proof. See Proposition 9.7.5 in [7]. □

Combining this theorem with Theorem 1.3, we obtain

Corollary 4.4. *Let M be a finitely generated kG-module such that $J_G(M)$ includes a product of Bocksteins of one dimensional classes. Then, for each integer s, there exists a finitely generated kG-module V such that $\Omega^s(M) \oplus V$ has a filtration*

$$0 = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_m = \Omega^s(M) \oplus V$$

where for every $i = 1, \ldots, m$, we have $L_i/L_{i-1} \cong \Omega^{s+i-1}(M \downarrow^G_{H_i}) \uparrow^G_{H_i}$ for some maximal subgroup H_i of G.

Proof. Suppose that $\zeta = \beta(u_1) \cdots \beta(u_n) \in J_G(M)$ where u_i's are one dimensional classes in $H^1(G, \mathbb{F}_p)$. Note that ζ is represented by an extension which is the Yoneda splice of extensions of the form

$$0 \to k \to k \uparrow^G_{H_i} \to k \uparrow^G_{H_i} \to k \to 0$$

where H_i is the kernel of u_i. So, by Theorem 1.3, there is a filtration for $L_\zeta \oplus (proj)$

$$0 = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_{2n} = L_\zeta \oplus (proj)$$

where the jth section L_j/L_{j-1} is isomorphic to $\Omega^{n+1-j}(k)\uparrow_{H_i}^G$ when $j=2i$ or $j=2i-1$. Tensoring L_i's with M, we obtain a filtration

$$0=(L_0\otimes M)\subseteq(L_1\otimes M)\subseteq\cdots\subseteq(L_{2n}\otimes M)=(L_\zeta\otimes M)\oplus(proj)$$

such that

$$(L_j\otimes M)/(L_{j-1}\otimes M)\cong(\Omega^{n+1-j}(k)\uparrow_{H_i}^G)\otimes M\cong\Omega^{n+1-j}(M\downarrow_{H_i}^G)\uparrow_{H_i}^G\oplus(proj)$$

where $j=2i$ or $j=2i-1$. By Theorem 4.3, we have $L_\zeta\otimes M\cong\Omega^n(M)\oplus\Omega(M)\oplus(proj)$. So, tensoring the entire system with $\Omega^{s-n}(k)$, and eliminating the projective summands, we obtain the desired filtration. □

Note that the filtration length m in Corollary 4.4 depends on the number of one dimensional classes whose product of Bocksteins is in $J_G(M)$. This suggests the following definition:

Definition 4.5. The cohomology length of a kG-module M, denoted by $chl_G(M)$, is defined as the smallest positive integer n such that there exist non-zero elements $u_1,u_2,\ldots,u_n\in H^1(G,\mathbb{F}_p)$ such that

$$\begin{aligned} u_1u_2\cdots u_n\in J_G(M) \quad &\text{if} \quad p=2,\\ \beta(u_1)\beta(u_2)\cdots\beta(u_n)\in J_G(M) \quad &\text{if} \quad p>2.\end{aligned}$$

If no such integer exists, then we set $chl_G(M)=\infty$.

It is easy to see that if $chl_G(M)=n$, then there is a filtration as in Corollary 4.4 of length $2n$. Note that for $p=2$, we have a filtration of length n. So, the cohomology length of a module is an interesting invariant to consider if one is interested in finding filtrations like in Corollary 4.4 of shortest length.

Note that there is a notion of cohomology length for p-groups (which are not elementary abelian) as a consequence of Serre's theorem. The cohomology length of a p-group G, denoted by $chl(G)$, is defined as the minimal m such that the product in Serre's theorem vanish. We can declare $chl(G)=\infty$ for groups where the Serre's theorem does not hold. Then, it is clear that $chl(G)=chl_G(k)$. Also note that for any kG-module M, we have $chl_G(M)\leq chl(G)$. In general, the cohomology length of a kG-module M can be much smaller then the cohomology length of G. For example, if G is an elementary abelian p-group and H a maximal subgroup of G, then $chl(G)=\infty$ whereas $chl_G(k\uparrow_H^G)=1$.

5. Varieties of Modules

In this section we introduce the varieties of modules, and prove Theorem 1.4 and Corollary 1.5 stated in the introduction. As in the previous sections, k denotes a field of characteristic $p>0$. We do not assume that k is algebraically closed and denote the algebraic closure of k by K. Let $V_G(k)$ denote the maximal ideal spectrum of $H^\bullet(G,k)$ where $H^\bullet(G,k)=H^*(G,k)$ for $p=2$, and $H^\bullet(G,k)=H^{ev}(G,k)$, the ring of even dimensional classes, for $p>2$. Since $H^\bullet(G,k)$ is a finitely generated commutative k-algebra, $V_G(k)$ is a finite dimensional homogeneous affine variety, where a point in $V_G(k)$ can be viewed as a k-linear ring homomorphism $H^\bullet(G,k)\to K$. Two such ring homomorphisms correspond to the same point in $V_G(k)$ if and only if they are in the same orbit under the Galois action of $\mathrm{Aut}_k(K)$.

¿From the above description, it is easy to see that any ring homomorphism $f:H^\bullet(G,k)\to H^\bullet(H,k)$ induces a continuous map $f^*:V_H(k)\to V_G(k)$ of corresponding varieties. In particular, for every $H\leq G$, the restriction homomorphism $\mathrm{res}_{G,H}:H^\bullet(G,k)\to H^\bullet(H,k)$ induces a map $\mathrm{res}^*_{G,H}:V_H(k)\to V_G(k)$ on varieties.

For a finitely generated kG-module M, the support variety of M, denote by $V_G(M)$, is defined as the variety of the ideal $J_G(M)$ where $J_G(M)$ is the annihilator of $\mathrm{Ext}^*_{kG}(M,M)$ in $H^\bullet(G,k)$. Observe that we have $\mathrm{res}^*_{G,H}(V_H(M)) \subseteq V_G(M)$ as a consequence of the obvious inclusion for ideals $\mathrm{res}_{G,H}\big(J_G(M)\big) \subseteq J_H(M)$. So, $\mathrm{res}^*_{G,H}$ induces a map $\mathrm{res}^*_{G,H} : V_H(M) \to V_G(M)$ on the varieties of the module M.

The following is a list of properties of varieties which we will need in this section.

Lemma 5.1. *Suppose that k is a field of characteristic $p > 0$, G is a finite group, and H is a subgroup of G. Let M, M_1, M_2, M_3 be kG-modules, and N a kH-module.*

(i) $V_G(M) = 0$ if and only if M is projective.

(ii) $V_G(M) = V_G(\Omega^n(M))$ for all integers n.

(iii) If $0 \to M_1 \to M_2 \to M_3 \to 0$ is exact, then $V_G(M_2) \subseteq V_G(M_1) \bigcup V_G(M_3)$. In particular $V_G(M_1 \oplus M_2) = V_G(M_1) \bigcup V_G(M_2)$.

(iv) $V_G(N \uparrow^G_H) = \mathrm{res}^*_{G,H}(V_H(N))$.

(v) $(\mathrm{res}^*_{G,H})^{-1}(V_G(M)) = V_H(M \downarrow^G_H)$.

(vi) $V_G(M_1 \otimes M_2) = V_G(M_1) \cap V_G(M_2)$.

(vii) Let $\zeta \in \mathrm{Ext}^n_{kG}(k,k)$. Then $V_G(L_\zeta) = V_G(\zeta)$ where $V_G(\zeta)$ is the variety of the ideal generated by ζ.

Proofs of these statements can be found in [2], [4], [7], and [8]. Note that the above properties hold for an arbitrary field if and only if they hold for an algebraically closed field. To see this, observe that the ring homomorphism $\phi : H^\bullet(G,k) \to H^\bullet(G,K) \cong H^\bullet(G,k) \otimes_k K$ defined by $\phi(\zeta) = 1 \otimes \zeta$ induces a map $\phi^* : V_G(K) \to V_G(k)$ on varieties which is finite to one. For a kG-module M, we have $J_G(K \otimes M) \cong K \otimes J_G(M)$, which gives $V_G(K \otimes M)) = (\phi^*)^{-1}(V_G(M))$. Thus proving these results for $V_G(M \otimes K)$ will give corresponding results for $V_G(M)$. For details of this argument, we refer the reader to Theorem 10.4.2 and Remark 10.4.3 in [7].

Lemma 5.2. *Let E be an elementary abelian p-group, and $\mathcal{M}$ be a collection of the maximal subgroups of E. Suppose M is a finitely generated kG-module such that*

$$V_E(M) = \bigcup_{D \in \mathcal{M}} \mathrm{res}^*_{E,D}(V_D(M \downarrow^E_D)).$$

Then, $J_E(M)$ includes a product of Bocksteins of one dimensional classes in $H^1(E, \mathbb{F}_p)$ whose kernels are in $\mathcal{M}$.

Proof. Applying $(\phi^*)^{-1}$ to the given equality, we obtain

$$V_E(K \otimes M) \subseteq \bigcup_{D \in \mathcal{M}} \mathrm{res}^*_{E,D}(V_D(K \otimes M \downarrow^E_D)) \subseteq \bigcup_{D \in \mathcal{M}} \mathrm{res}^*_{E,D}(V_D(K)).$$

For each $D \in \mathcal{M}$, choose a one dimensional class $x_D \in H^1(E, \mathbb{F}_p)$ such that the kernel of x_D is D. Then, it is clear that $\beta(x_D)$ is in the kernel of the restriction map $\mathrm{res}_{E,D}$, so we have

$$\mathrm{res}^*_{E,D}(V_D(K)) = V_E(\ker \mathrm{res}_{E,D}) \subseteq V_E(\beta(x_D))$$

for each D. This gives

$$V_E(K \otimes M) \subseteq V_E(\prod_{D \in \mathcal{M}} \beta(x_D)).$$

By Hilbert's Nullstellensatz,

$$u = (\prod_{D \in \mathcal{M}} \beta(x_D))^r \in J_E(K \otimes M)$$

for some $r > 0$. Since $J_E(K \otimes M) = K \otimes J_E(M)$ and $u \in H^\bullet(E, \mathbb{F}_p)$, u lies in $J_E(M)$. $\square$

Now, we are ready to prove Theorem 1.4, which is our main result in this section.

Proof of Theorem 1.4. $(ii) \Rightarrow (i)$ Suppose that there exists a kG-module V such that $M \oplus V$ has a filtration

$$0 = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_n = M \oplus V$$

where for each $i = 1, \ldots, n$, the ith section L_i/L_{i-1} is isomorphic to $W_i \uparrow_{H_i}^G$ for some subgroup $H_i \in \mathcal{H}$ and some kH_i-module W_i. Applying the properties $(ii) - (iv)$ listed in Lemma 5.1, we obtain

$$V_G(M) \subseteq \bigcup_i V_G(W_i \uparrow_{H_i}^G) = \bigcup_i \mathrm{res}^*_{G,H_i}(V_{H_i}(W_i))$$

which gives

$$V_G(M) \subseteq \bigcup_{H \in \mathcal{H}} \mathrm{res}^*_{G,H} V_H(k).$$

Note that by property (v), we have

$$V_G(M) \cap \mathrm{res}^*_{G,H}(V_H(k)) = \mathrm{res}^*_{G,H}(V_H(M \downarrow_H^G))$$

for all $H \in \mathcal{H}$, so we obtain

$$V_G(M) = \bigcup_{H \in \mathcal{H}} \mathrm{res}^*_{G,H}(V_H(M \downarrow_H^G))$$

as desired.

$(i) \Rightarrow (ii)$ Note that by Theorem 3.2, there exists a finitely generated kG-module V such that $k \oplus V$ has a filtration

$$0 = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_n = k \oplus V$$

where for every $i = 1, \ldots, n$, the sections L_i/L_{i-1} are isomorphic to $\Omega^{n_i}(k) \uparrow_{E_i}^G$ for some integer n_i and some maximal elementary abelian p-subgroup E_i of G. Tensoring this system with M, we obtain a filtration

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M \oplus (M \otimes V)$$

where $M_i = M \otimes L_i$, and hence

$$M_i/M_{i-1} \cong (L_i/L_{i-1}) \otimes M \cong \Omega^{n_i}(k) \uparrow_{E_i}^G \otimes M \cong (\Omega^{n_i}(M \downarrow_{E_i}^G) \oplus P_i) \uparrow_{E_i}^G$$

where P_i is a projective kE_i-module. Since E_i is a p-group, P_i is a free kE_i-module. Thus,

$$M_i/M_{i-1} \cong \Omega^{n_i}(M \downarrow_{E_i}^G) \uparrow_{E_i}^G \oplus F_i$$

for some free kG-module F_i.

If for some $i \in \{1, \ldots, n\}$, the ideal $J_{E_i}(M \downarrow_{E_i}^G)$ includes a product of Bocksteins of one dimensional classes, then by Corollary 4.4 there is a finitely generated kE_i-module W such that $\Omega^{n_i}(M \downarrow_{E_i}^G) \oplus W$ has a filtration

$$0 = T_0 \subseteq T_1 \subseteq \cdots \subseteq T_m = \Omega^{n_i}(M \downarrow_{E_i}^G) \oplus W$$

with

$$T_j/T_{j-1} \cong \Omega^{m_j}(M \downarrow_{D_j}^G) \uparrow_{D_j}^{E_i}$$

where for $j = 1, \ldots, m$, we have $m_j = n_i + j - 1$ and D_j is a maximal subgroup of E_i. Inducing the entire filtration to G, we obtain a filtration

$$0 = S_0 \subseteq S_1 \subseteq \cdots \subseteq S_m = \Omega^{n_i}(M \downarrow_{E_i}^G) \uparrow_{E_i}^G \oplus W \uparrow_{E_i}^G$$

where $S_j = T_j \uparrow_{E_i}^G$ and

$$S_j/S_{j-1} \cong \Omega^{m_j}(M \downarrow_{D_j}^G) \uparrow_{D_j}^G .$$

Finally, adding the free summand F_i to the last section, we obtain a filtration

$$0 = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_m = (M_i/M_{i-1}) \oplus W \uparrow_{E_i}^G$$

where

$$U_j/U_{j-1} \cong \Omega^{m_j}(M \downarrow_{D_j}^G) \uparrow_{D_j}^G \oplus F_j$$

for some free summand F_j (only the last section has a nontrivial free summand).

We can use this filtration to refine the original filtration for $M \oplus (M \otimes V)$ as follows: Let $\hat{U}_j$ denote the pre-image of U_j under the quotient map $M_i \oplus W' \to (M_i \oplus W')/M_{i-1}$ where $W' = W \uparrow_{E_i}^G$. We have

$$M_{i-1} = \hat{U}_0 \subseteq \hat{U}_1 \subseteq \cdots \subseteq \hat{U}_m = M_i \oplus W'$$

with

$$\hat{U}_j/\hat{U}_{j-1} \cong U_j/U_{j-1} \cong \Omega^{m_j}(M \downarrow_{D_j}^G) \uparrow_{D_j}^G \oplus F_j.$$

Splicing this into the filtration

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{i-1} \subseteq M_i \oplus W' \subseteq M_{i+1} \oplus W' \subseteq \cdots \subseteq M_n \oplus W' = M \oplus V'$$

where $V' = (M \otimes V) \oplus W'$, we obtain a new filtration for $M \oplus V'$ where the ith section is replaced by a sequence of modules with quotients induced from maximal subgroups of E_i. Applying this process to every section, we can assume that there is a kG-module V such that $M \oplus V$ has a filtration

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M \oplus V$$

with

$$M_i/M_{i-1} \cong \Omega^{n_i}(M \downarrow_{E_i}^G) \uparrow_{E_i}^G \oplus F_i$$

where F_i is a free kG-module and E_i is an elementary abelian p-group such that the ideal $J_{E_i}(M \downarrow_{E_i}^G)$ does not have any element which is a product of Bocksteins of one dimensional classes. Note that by Proposition 5.2, this implies that for any collection of maximal subgroups $\mathcal{M}_i$ of E_i

$$V_{E_i}(M \downarrow_{E_i}^G) \neq \bigcup_{D \in \mathcal{M}_i} \mathrm{res}_{E_i,D}^*(V_D(M \downarrow_D^{E_i})).$$

We claim that this forces E_i to be conjugate to a subgroup H in $\mathcal{H}$. To see this, consider the following calculation. We have

$$V_{E_i}(M \downarrow_{E_i}^G) = (\mathrm{res}_{G,E_i}^*)^{-1}(V_G(M)) \subseteq \bigcup_{H \in \mathcal{H}} (\mathrm{res}_{G,E_i}^*)^{-1}\Big(\mathrm{res}_{G,H}^*(V_H(k))\Big)$$

which gives

$$V_{E_i}(M \downarrow_{E_i}^G) \subseteq \bigcup_{H \in \mathcal{H}} V_{E_i}(k \uparrow_H^G \downarrow_{E_i}^G) = \bigcup_{H \in \mathcal{H}} \Big(\bigcup_{H\backslash G/E_i} V_{E_i}(k \uparrow_{{}^gH \cap E_i}^{E_i}) \Big).$$

From this we obtain

$$V_{E_i}(M \downarrow_{E_i}^G) = \bigcup_L \mathrm{res}_{E_i,L}^*(V_L(M \downarrow_L^{E_i}))$$

where L runs over the subgroups of the form ${}^gH \cap E_i$ over the set of double coset $H\backslash G/E_i$ and over $H \in \mathcal{H}$. But, no such equality exists for a collection of proper subgroups, so we must have $E_i = {}^gH \cap E_i \leq {}^gH$ for every $g \in H\backslash G/E_i$ and $H \in \mathcal{H}$. Thus E_i is conjugate to a subgroup of some $H \in \mathcal{H}$ for all $i = 1, \ldots, n$.

To complete the proof, we observe that $\Omega^{n_i}(M\downarrow^G_{E_i})\uparrow^G_{E_i}\cong\Omega^{n_i}(M\downarrow^G_{E_i^g})\uparrow^G_{E_i^g}$ for all $g\in G$, so by replacing E_i's with their conjugates, we can assume E_i's are subgroups of H. Thus $M\oplus V$ has a filtration with sections induced from subgroups in $\mathcal{H}$. □

From the above proof it is clear that the following is also true.

Corollary 5.3. *Let M be a kG-module, and $\mathcal{H}$ a collection of subgroups in G. Suppose that there is a kG-module V such that $M\oplus V$ has a filtration*

$$0=L_0\subseteq L_1\subseteq\cdots\subseteq L_{m-1}\subseteq L_m=M\oplus V$$

such that for each $i=1,2,\dots,m$, the quotient module L_i/L_{i-1} is induced from subgroups in $\mathcal{H}$. Then there exists a kG-module U such that $M\oplus U$ has a filtration

$$0=M_0\subseteq M_1\subseteq\cdots\subseteq M_n=M\oplus U$$

such that for $i=1,\dots,n$,

$$M_i/M_{i-1}\cong\Omega^{n_i}(M\downarrow^G_{E_i})\uparrow^G_{E_i}\oplus F_i$$

where F_i is a free kG-module, and E_i is an elementary abelian p-group included in a subgroup H in $\mathcal{H}$.

In particular this tells us that if we have a filtration for a kG-module M where the sections are induced modules, then there is a kG module U such that $M\oplus U$ has a filtration whose sections are given in terms of M up to a free module. This could be useful for proving theorems by induction for modules with complexity strictly smaller than the p-rank of the group.

We conclude the paper with the proof of Corollary 1.5 stated in the introduction.

Proof of Corollary 1.5. It is enough to prove the result under the assumption that k is algebraically closed. By Theorem 1.3, we have a filtration for L_ζ with sections isomorphic to Heller shifts of M_i's. Thus

$$V_G(L_\zeta)\subseteq\bigcup_{i=0}^{n-1}V_G(M_i).$$

Since the modules $M_0,\dots,M_{n-1}$ are direct sums of modules induced from proper subgroups, there is a collection $\mathcal{M}=\{H_1,\dots,H_m\}$ of maximal subgroups of G such that

$$V_G(L_\zeta)\subseteq\bigcup_j\mathrm{res}^*_{G,H_j}(V_{H_j}(k)).$$

As in the proof of Lemma 5.2, we can choose one dimensional classes $x_1,\dots,x_m\in H^1(G,\mathbb{F}_2)$ such that the kernel of x_j is H_j for each $j=1,\dots,m$, and replace each $\mathrm{res}^*_{G,H_j}(V_{H_j}(k))$ with $V_G(x_j)$. Note that by property (vii), we have $V_G(\zeta)=V_G(L_\zeta)$, so we obtain

$$V_G(\zeta)\subseteq V_G(x_1x_2\cdots x_m).$$

Since $H^\bullet(G,k)=H^*(G,k)$ is a polynomial algebra, we can apply Hilbert's Nullstellensatz, and conclude that there exists an integer $t>0$, such that $(x_1x_2\cdots x_n)^t\in(\zeta)$. This means $(x_1x_2\cdots x_n)^t=a\cdot\zeta$ for some $a\in H^*(G,k)$. Thus, $\zeta=x_1^{t_1}\cdots x_n^{t_n}$ for some integers $t_i\geq 0$.

Let X_i be a two point G-set with isotropy H_i. Now, consider the G-sphere X defined as the join of G-spheres X_i where we take t_i copies of X_i for each i. The cellular homology of X with coefficients in k gives an extension

$$0\to H_m(X,k)\to C_m(X,k)\to\cdots\to C_1(X,k)\to C_0(X,k)\to H_0(X,k)\to 0$$

where each $C_m(X,k)$ is a kG-module and $m = \sum_i t_i$. Note that both $H_0(X,k)$ and $H_m(X,k)$ are isomorphic to k. We say a k by k extension E comes from a geometric group action on a sphere if after choosing appropriate isomorphism for $H_0(X,k) \cong k$ and $H_m(X,k) \cong k$, the resulting extension is equivalent to E. Since the Euler class of join of spheres is the product of Euler classes, we can choose these isomorphisms to make the extension class equal to ζ. $\square$

For $p > 2$, the situation is more complicated. In this case $H^\bullet(G,k)$ is not a polynomial algebra, so we do not have unique factorization. For example, if x_1 and x_2 are two one dimensional classes then, we have $(\beta(x_1)+x_1x_2)^p = \beta(x_1)^p$. On the other hand, if we assume that ζ is a class lying in the subalgebra generated by Bocksteins of one dimensional classes, then there is a similar conclusion for ζ, namely, ζ is of the form $\lambda\beta(x_1)\cdots\beta(x_n)$ for some scalar $\lambda \in k$ and one dimensional classes $x_1,\ldots,x_n$ in $H^1(G,\mathbb{F}_p)$. A linear action on a sphere such that the associated k-invariant is this product can easily be constructed using a direct sum of one dimensional complex representations. Note that since for every $\zeta \in H^\bullet(G,k)$ the p-th power ζ^p lies in the polynomial subalgebra, we can also conclude that for every ζ satisfying the conditions of Corollary 1.5, the class ζ^p is a scalar multiple of a product of Bocksteins of one dimensional classes in $H^1(G,\mathbb{F}_p)$.

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