**Point Charges and the Delta Function**

In order to express the fact that a charge is located at a given point, it is convenient to introduce a generalization of the Kronecker delta which is called the Dirac delta function. This function has the properties that it vanishes unless its argument vanishes,

\[ \delta(\vec{r} - \vec{r}') \equiv \delta(x - x')\delta(y - y')\delta(z - z') = 0 \quad \text{for} \quad \vec{r} \neq \vec{r}' , \]

and that the integral over all space is unity, i.e.,

\[ \int_{\text{all space}} d\vec{v} \delta(\vec{r} - \vec{r}') \equiv \int_{-\infty}^{\infty} dx \delta(x - x') \int_{-\infty}^{\infty} dy \delta(y - y') \int_{-\infty}^{\infty} dz \delta(z - z') = 1 \]

If we assume that we have made a proper definition of the delta function, consistent with the above requirements, we have

\[ \int_{\text{all space}} d\vec{v} \Psi(\vec{r}) \delta(\vec{r} - \vec{r}') = \Psi(\vec{r}') \]

It can easily be verified by direct differentiation that \( 1/r \) is a solution of Laplace’s equation,

\[ \nabla^2 \left( \frac{1}{r} \right) = 0, \quad r > 0 \]

Since the potential \( \Phi(\vec{r}) \) of a point charge is

\[ \Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{q}{r}, \quad r > 0 \]

we may write the integral of \( \nabla^2 \Phi(\vec{r}) = -\varrho(\vec{r})/\epsilon_0 \) as

\[ \int_{\text{all space}} d\vec{v} \nabla^2 \Phi = \frac{q}{4\pi\epsilon_0} \int_{\text{all space}} d\vec{v} \nabla^2 \left( \frac{1}{r} \right) = -\frac{1}{\epsilon_0} \int_{\text{all space}} d\vec{v} \varrho(\vec{r}) \]

If \( q \) is indeed a point charge, the density \( \varrho(\vec{r}) \) is a delta function:

\[ \int_{\text{all space}} d\vec{v} \varrho(\vec{r}) = \int_{\text{all space}} d\vec{v} q \delta(\vec{r}) = q \]

Combining the last two equations, we have

\[ \int_{\text{all space}} d\vec{v} \nabla^2 \left( \frac{1}{r} \right) = -4\pi \]

Since \( \nabla^2 (1/r) \) vanishes for \( r > 0 \) and has an integral over all space of \(-4\pi\), we can then write

\[ \nabla^2 \left( \frac{1}{r} \right) = -4\pi \delta(\vec{r}) \]

This equation can be expressed more generally as

\[ \nabla^2 \left( \frac{1}{|\vec{r} - \vec{r}'|} \right) = -4\pi \delta(\vec{r} - \vec{r}') \]