The Dirac Delta in Curvilinear Coordinates

The Dirac delta is often defined by the property

$$\int_{\mathrm{V}} f(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}_0)\,dv = \begin{cases} f(\mathbf{r}_0) & \text{if } P_0(x_0,y_0,z_0) \text{ is in V} \\ 0 & \text{if } P_0(x_0,y_0,z_0) \text{ is not in V} \end{cases}$$

There is no restriction in the number of dimensions involved and $f(\mathbf{r})$ can be a scalar function or a vector function. However, it is rather obvious that $f(\mathbf{r})$ must be defined at the point $P_0(x_0, y_0, z_0)$. If the function $f(\mathbf{r})$ is a constant, e.g., unity, then one sees that the delta is normalized. As a result, it is customary to speak of the delta as a symbolic representation for a unit source. However, the source is of a unit magnitude in the sense that the integral of the delta over the coordinates involved is unity. If we consider a three dimensional orthogonal curvilinear coordinate system with coordinates (ξ_1, ξ_2, ξ_3) and scale factors

$$h_i = \left[\left(\frac{\partial x}{\partial \xi_i}\right)^2 + \left(\frac{\partial y}{\partial \xi_i}\right)^2 + \left(\frac{\partial z}{\partial \xi_i}\right)^2\right]^{1/2}$$

then one expresses the Dirac delta $\delta(\mathbf{r}-\mathbf{r}_0)$ as follows:

$$\delta(\mathbf{r}-\mathbf{r}_0) \to \frac{\delta(\xi_1-\xi_{10})}{h_1}\frac{\delta(\xi_2-\xi_{20})}{h_2}\frac{\delta(\xi_3-\xi_{30})}{h_3}$$

In spherical polar coordinates: $\xi_1 = r$, $\xi_2 = \theta$, $\xi_3 = \varphi$, we have

$$x = r\sin\theta\cos\varphi, \quad y = r\sin\theta\sin\varphi, \quad z = r\cos\theta,$$

and

$$h_1 = 1, \quad h_2 = r, \quad h_3 = r\sin\theta, \quad dv = r^2\sin\theta\, dr d\theta d\varphi,$$

and the corresponding Dirac delta is given by

$$\delta(\mathbf{r}-\mathbf{r}_0) = \frac{1}{r^2\sin\theta}\,\delta(r-r_0)\,\delta(\theta-\theta_0)\,\delta(\varphi-\varphi_0)$$

If one considers spherical coordinates with azimuthal symmetry, the φ–integral must be projected out, and the denominator becomes

$$\int_0^{2\pi} r^2\sin\theta\, d\varphi = 2\pi r^2\sin\theta,$$

and consequently

$$\delta(\mathbf{r}-\mathbf{r}_0) = \frac{1}{2\pi r^2\sin\theta}\,\delta(r-r_0)\,\delta(\theta-\theta_0)$$

If the problem involves spherical coordinates, but with no dependence on either φ or θ, the denominator becomes

$$\int_0^{\pi} d\theta \int_0^{2\pi} d\varphi\, r^2\sin\theta = 4\pi r^2,$$

and one has

$$\delta(\mathbf{r}-\mathbf{r}_0) = \frac{1}{4\pi r^2}\,\delta(r-r_0)$$

Similarly, in cylindrical coordinates we have $\xi_1 = \varrho$, $\xi_2 = \varphi$, $\xi_3 = z$, we have

$$h_1 = 1, \quad h_2 = \varrho, \quad h_3 = 1, \quad dv = \varrho d\varrho d\varphi dz,$$

and the corresponding forms for the Dirac delta are given by

$$\begin{aligned} \delta(\mathbf{r}-\mathbf{r}_0) &= \frac{1}{\varrho}\,\delta(\varrho-\varrho_0)\,\delta(\varphi-\varphi_0)\,\delta(z-z_0) \\ &= \frac{1}{2\pi\varrho}\,\delta(\varrho-\varrho_0)\,\delta(z-z_0) \\ &= \frac{1}{2\pi\varrho}\,\delta(\varrho-\varrho_0) \end{aligned}$$

examples:

• The charge density due to a spherical shell with uniform charge Q is given by

$$\rho(\mathbf{r}) = \frac{Q\,\delta(r - r_0)}{4\pi r^2}$$

• Consider a ring of charge Q and radius a oriented to lie in the xy plane with its centre coincident with the origin.

▷ The charge density function expressed in cylindrical coordinates is

$$\rho(\mathbf{r}) = \frac{Q\,\delta(\rho - a)\,\delta(z)}{2\pi\rho}$$

▷ In spherical coordinates the charge density is

$$\rho(\mathbf{r}) = \frac{Q\,\delta(\rho - a)\,\delta(\theta - \pi/2)}{2\pi r^2 \sin\theta}$$

To find the electric potential at an on-axis point: $\mathbf{r} = z\hat{\mathbf{z}}$, one refers to the expression

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{dv'\rho(\mathbf{r}')}{|\,\mathbf{r} - \mathbf{r}'\,|}$$

Using spherical coordinates, and setting

$$\begin{aligned} dv' &= r'^2 \sin\theta' dr' d\theta' d\phi' \\ \mathbf{r}' &= r' \sin\theta' \cos\phi' \hat{\mathbf{x}} + r' \sin\theta' \sin\phi' \hat{\mathbf{y}} + r' \cos\theta' \hat{\mathbf{z}} \end{aligned}$$

one obtains

$$\Phi(\mathbf{r}) = \frac{Q}{4\pi\epsilon_0\,(a^2 + z^2)^{1/2}}$$