electrostatic energy of charge distributions

Suppose we wish to build up a charge density \( \rho(\vec{r}) \) in some region of space with relevant electric potential value \( V(\vec{r}) \). The work needed to assemble such a charge configuration will be stored eventually in the form of electrostatic potential energy. The procedure is to start with an initial state where we have complete empty space with whatsoever no charges and no electric potentials, i.e.,

\[
\rho(\vec{r}) = 0 \quad \text{and} \quad V(\vec{r}) = 0, \quad \text{everywhere}
\]

Bringing incremental amounts of charges from infinity and distributing those appropriately we aim to reach a final state where the charge density and the potential have their final forms, given as \( \rho(\vec{r}) \) and \( V(\vec{r}) \).

During the procedure we imagine that we have obtained an intermediate configuration (described under primed notation) where the charge density and the potential lie somewhere between their initial and final values, i.e.,

\[
0 < \rho'(\vec{r}) < \rho(\vec{r}) \quad \text{and} \quad 0 < V'(\vec{r}) < V(\vec{r})
\]

Carrying further incremental charges \( \Delta q \) and placing into infinitesimal sized volume element \( \Delta v \) we perform and additional work

\[
\Delta W = V'(\vec{r}) \Delta q, \quad \text{where} \quad \Delta q = \Delta \rho' \Delta v
\]

We require that all parts of the charge distribution are brought to their final charge values at the same pace and uniformity. On this purpose we introduce a gauge (ayar) parameter \( g \) that can be tuned from from zero to unity and serves to keep trace of the transition from the initial to the final charge configuration. Increasing \( g \) will make \( \rho' \) and \( V' \) to increase and grow to their final values \( \rho \) and \( V \). The linkage between the intermediate and final values is therefore

\[
\rho' = g\rho, \quad V' = gV, \quad \Delta \rho' = \Delta g \rho
\]

Under an additional amount of charge \( \Delta q \) put at \( \vec{r} \) where the potential is \( V'(\vec{r}) \) the increase in the energy is

\[
\Delta W = V'(\vec{r}) \Delta q \\
= V'(\vec{r}) \Delta \rho' \Delta v \\
= gV(\vec{r}) \Delta g \rho(\vec{r}) \Delta v
\]

To find the total work \( W = \int \Delta W \) we must sum over infinitely many small contributions attained through the variation of parameter \( g \) from 0 to its full value 1, as well as contributions coming from infinitesimal volume elements to yield the overall energy in the overall space.
We thus have
\[
W = \int_0^1 dg \int dv V(\vec{r}) \rho(\vec{r}) = \frac{1}{2} \int dv V(\vec{r}) \rho(\vec{r})
\]  
(1)

In the expression given by eq.(1), using Gauss’ law
\[
\rho(\vec{r}) = \epsilon_0 \vec{\nabla} \cdot \vec{E}(\vec{r})
\]
we obtain
\[
W = \frac{\epsilon_0}{2} \int dv V(\vec{r}) \vec{\nabla} \cdot \vec{E}(\vec{r})
\]

This expression can be cast into a more convenient form if one uses the identity
\[
\vec{\nabla} \cdot (V \vec{E}) = \partial_i (VE_i) = V \partial_i E_i + E_i \partial_i V
\]

Hence we equivalently write
\[
W = \frac{\epsilon_0}{2} \int dv \vec{\nabla} \cdot (V \vec{E}) - \frac{\epsilon_0}{2} \int dv \vec{E} \cdot \vec{\nabla} V
\]

Using the divergence theorem we conform the first volume integral into a surface integral, and in the second volume integral replace \(\vec{\nabla} V\) by \(-\vec{E}\) and hence obtain
\[
W = \frac{\epsilon_0}{2} \oint_S V \vec{E} \cdot \hat{n} da + \frac{\epsilon_0}{2} \int dv \vec{E} \cdot \vec{E}
\]

Clearly, surface \(S\) which is supposed to bound the entire space should have boundaries extending far out to infinity where the electric potential has zero value. Thus omitting the first integral we reach the following alternative expression for the total energy, expressed solely in terms the electric field in the overall space:
\[
W = \frac{\epsilon_0}{2} \int dv | \vec{E}(\vec{r}) |^2,
\]  
(2)

or simply \(W = \int dv u(\vec{r})\) in which
\[
u(\vec{r}) = \frac{\epsilon_0}{2} | \vec{E}(\vec{r}) |^2
\]
is the electrostatic energy density.
As a specific example consider a spherical conductor of radius $R$ bearing uniform charge $Q$ on its surface. The corresponding electric potential is spherically symmetric having dependence only on the radial distance $r$ from the centre, given in the form $V(r)$. The volume charge density is described by the Dirac delta as

$$\rho(r) = \frac{Q}{4\pi r^2} \delta(r - R)$$

The energy of such a sphere is therefore provided by the integral in eq.(1)

$$W = \frac{1}{2} \int_0^\infty \left\{ 4\pi r^2 dr \right\} \left\{ \frac{Q}{4\pi r^2} \delta(r - R) \right\} V(r)$$

$$= \frac{1}{2} \int_0^\infty dr Q \delta(r - R)V(r)$$

$$= \frac{1}{2} Q V(R)$$

where $V(R)$ is the potential on the surface, given by

$$V(R) = \frac{1}{4\pi \varepsilon_0} \frac{Q}{R}$$

Substitution yields

$$W = \frac{Q^2}{8\pi \varepsilon_0 R} = \frac{Q^2}{2C}$$

where $C = 4\pi \varepsilon_0 R$ is the capacitance of the sphere.

An alternative approach which leads to the same result can be attained by using eq.(2), rather than eq.(1). Setting

$$E(r) = \begin{cases} 
0 & \text{if } r < R; \\
\frac{Q}{4\pi \varepsilon_0 r^2} & \text{if } r > R.
\end{cases}$$

Substitution yields

$$W = \frac{\varepsilon_0}{2} \int_R^\infty \left\{ 4\pi r^2 dr \right\} \left\{ \frac{Q}{4\pi \varepsilon_0 r^2} \right\}^2$$

$$= \frac{Q^2}{8\pi \varepsilon_0 R}$$