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QUADRATIC TRANSFORMATIONS $\mathbb{R}p^2 \to \mathbb{R}p^2$

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This paper is dedicated to the memory of D. A. Gudkov

Abstract. The complete projective classification of 1- and 2-dimensional linear systems of real plane quadrics is given. In other words, this is a classification of quadratic transformations $\mathbb{R}p^2 \to \mathbb{R}p^1$ and $\mathbb{R}p^2 \to \mathbb{R}p^2$ respectively

Introduction

The purpose of this paper is to give the projective classification of quadratic transformations $\mathbb{R}p^2 \to \mathbb{R}p^2$, i.e., transformations given by quadratic polynomials in some homogeneous coordinates. Originally, this problem was raised by O. Viro, who intended to use these transformations in constructing new real algebraic curves. Another origin is the classical problem about the relationship between projective plane algebraic curves and 2-dimensional linear systems of quadratic hypersurfaces, which was solved by Turin [T] in the case of non-singular curves. (Turin's result states that there is a one-to-one correspondence between the set of generic 2-dimensional linear systems of quadrics in $\mathbb{C}p^n$ and a Zariski open subset of the space of pairs (C, φ), where C is a non-singular complex plane curve of degree $n+1$, and φ is a Spin-structure on C with $\operatorname{Arf} \varphi = 0$.)

In this paper, we also exploit the latter relationship: transformation are studied via their *discriminant curves,* which, by definition, are the loci of the singular curves of the corresponding linear systems. For non-singular transformations, which form a one-parameter family, we construct a complete invariant $\varepsilon \in \mathbb{R}$ (see Theorem 3.3), which is a ramification of the classical j-invariant of the discriminant curve (the latter being a non-singular cubic in this case). Given a non-singular real cubic C, transformations with the discriminant curve isomorphic to C are in one-to-one correspondence with the real (i.e., conj-invariant) Spin-structures on the complexification of C. Besides, there are 24 discrete classes of degenerate transformations, which are

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classified by their discriminant curves (which either are singular plane cubics, or coincide with $\mathbb{R}\mathrm{p}^2$), along with some natural stratification of these curves.

In conclusion, I would like to express my gratitude to Professor O. Viro, who inspired this work.

1. Some definitions

Throughout this section F denotes a fixed degree d polynomial transformation $\mathbb{R}\mathrm{p}^n \to \mathbb{R}\mathrm{p}^n$ given in some homogeneous coordinates $(x_0 : \dots : x_n)$, $(y_0 : \dots : y_m)$ by $y_i = f_i(x_0, \dots, x_n)$. The polynomials $f_0, \dots, f_m$ are always assumed *linearly independent.*

1.1. In general, the transformation f may not be defined at some points of $\mathbb{R}\mathrm{p}^n$, namely, at the common points of the zero sets of all the f_i's. These points will be called the *base points* of F, and the set of all the base points will be denoted by $\mathcal{B}$. Sometimes, following Zariski [Z], we will also consider infinitely near base points, i.e., those which lie in the exceptional divisor of some blowing-up of $\mathbb{R}\mathrm{p}^n$.

We define the *Jacobian* $\mathcal{J}$ of F to be the projectivization of the critical point set of a linear lift $\mathbb{R}^{n+1} \to \mathbb{R}^{m+1}$ of F. (In terms of F itself, $\mathcal{J}$ consists of the critical points of the restriction $F\big|_{\mathbb{R}\mathrm{p}^n \setminus \mathcal{B}}$ and, possibly, of some of the base points of F. If $m > n$, then $\mathcal{J} = \mathbb{R}\mathrm{p}^n$. If $m = n$, then $\mathcal{J} \supset \mathcal{B}$.) As usual, Jacobian is given by $\operatorname{rk}(\partial f_i/\partial x_j) < m+1$. If $m = n$, the latter is equivalent to $\det(\partial f_i/\partial x_j) = 0$, so in this case Jacobian is, in general, a hypersurface of degree $(d-1)^n$ in $\mathbb{R}\mathrm{p}^n$ (though it may as well coincide with the whole $\mathbb{R}\mathrm{p}^n$).

Jacobian has a natural stratification defined by the types of the critical points (see, e.g., [AVG]) and multiplicities and hierarchy of base points.

1.2. The transformation F obviously defines an m-dimensional linear system of hypersurfaces of degree d in $\mathbb{R}\mathrm{p}^n$. (This system is spanned by the f_i's.) We will call it the *adjoint linear system* of F and denote by $\mathcal{L}$. In more precise terms, $\mathcal{L}$ is the image of the dual space $(\mathbb{R}\mathrm{p}^m)^\vee$ of $\mathbb{R}\mathrm{p}^m$ under the natural projective map $(\mathbb{R}\mathrm{p}^m)^\vee \to \mathcal{C}_d\mathbb{R}\mathrm{p}^n$ defined by F, where $\mathcal{C}_d\mathbb{R}\mathrm{p}^n$ is the $(C^d_{n+d}-1)$-dimensional projective space of all the hypersurfaces of degree d in $\mathbb{R}\mathrm{p}^n$. The set of all the singular hypersurfaces of $\mathcal{L}$ is called the *discriminant* of F and is denoted by Δ. (Δ either coincides with $\mathcal{L}$, or is a degree $(n+1)(d-1)^n$ hypersurface in $\mathcal{L}$.) The discriminant has a natural stratification defined by the types of degeneration of the hypersurfaces. The isotopy (rigid isotopy, etc.) types of non-singular hypersurfaces define an additional structure on $\pi_0(\mathcal{L} \setminus \Delta)$.

There is the following useful connection between $\mathcal{J}$ and Δ: $\mathcal{J}$ is the locus of the singular points of the singular hypersurfaces of $\mathcal{L}$.

Obviously, the discriminant can be thought of as the intersection $\mathcal{L} \cap \mathrm{Discr}_d$, where $\mathrm{Discr}_d \subset \mathcal{C}_d\mathbb{R}\mathrm{p}^n$ is the set of all the singular hypersurfaces of degree d in $\mathbb{R}\mathrm{p}^n$. We will need the following lemma:

1.3. Lemma (Severy [S]). *Smooth points of* Discr_d *correspond to hypersurfaces with a single point of type* A_1 *(i.e., non-degenerate double point). The tangent hyperplane to* Discr_d *at such a point consists of all the hypersurfaces through the above singular point.*

1.4. Define the *dual transformation* $\check{F}\colon (\mathbb{R}\mathrm{p}^n)^\vee \to \mathbb{R}\mathrm{p}^{D-(m+1)}$, where $D = C^d_{n+d} - 1$, in the following way: The system $\mathcal{L}$, which is an m-plane in $\mathcal{C}_d\mathbb{R}\mathrm{p}^n$, defines its dual $(D-m-1)$-plane $\mathcal{L}^\perp \subset (\mathcal{C}_d\mathbb{R}\mathrm{p}^n)^\vee = \mathcal{C}_d(\mathbb{R}\mathrm{p}^n)^\vee$, which, in turn, defines a transformation $(\mathbb{R}\mathrm{p}^n)^\vee \to (\mathcal{L}^\perp)^\vee \cong \mathbb{R}\mathrm{p}^{D-(m+1)}$. By definition, the latter is $\check{F}$. Obviously, two transformations are projective equivalent if and only if so are their duals.

Note that in the case of quadratic transformations $\mathbb{R}\mathrm{p}^2 \to \mathbb{R}\mathrm{p}^2$ (i.e., $d = 2$, $m = n = 2$), which we are mainly interested in, the dual transformation is of the same type.

2. Main results

Up to projective equivalence, there are one 1-parameter family and 24 discrete types of quadratic transformations $\mathbb{R}\mathrm{p}^2 \to \mathbb{R}\mathrm{p}^2$. These types are presented in Table 1, which gives:

(1) values of the parameters in the canonical representation of a transformation F (see below);
(2) codimension of the corresponding stratum in the 9-dimensional space of all quadratic transformations $\mathbb{R}\mathrm{p}^2 \to \mathbb{R}\mathrm{p}^2$;
(3) the dual transformation $\check{F}$.

Table 1. Quadratic transformations $\mathbb{R}\mathrm{p}^2 \to \mathbb{R}\mathrm{p}^2$

Type	ε	codim	Dual
I1a	$\varepsilon < -8$	0	I1a'
I1b	$-8 < \varepsilon < 0$	0	I1b'
I1a'	$0 < \varepsilon < 1$	0	I1a
I1b'	$\varepsilon > 1$	0	I1b

Type	ε	codim	Dual
I2a	1	1	I2$'a$
I2b	-1	1	I2$'b$
I2$'a$	1	1	I2a
I2$'b$	-1	1	I2b
I3	—	2	I3

Type	$(\varepsilon_0, \varepsilon_1, \varepsilon_2)$	codim	Dual
III	—	5	III
IV	$(1, 0, 0)$	7	IV$'$
IV$'$	$(1, 0, 0)$	7	IV

Type	$(\varepsilon_0, \varepsilon_1, \varepsilon_2)$	codim	Dual
II1a	$(1, 1, 1)$	2	II$'$1a
II1b	$(-1, 1, 1)$	2	II$'$1b
II1c	$(-1, -1, 1)$	2	II$'$1c
II2_1a	$(0, 1, 1)$	3	II$'2_1a$
II2_1b	$(0, -1, 1)$	3	II$'2_1b$
II2_2a	$(1, 0, 1)$	3	II$'2_2a$
II2_2b	$(-1, 0, 1)$	3	II$'2_2b$
II3	$(0, 0, 1)$	4	II$'$3
II$'$1a	$(-1, 1, 1)$	2	II1a
II$'$1b	$(1, 1, 1)$	2	II1b
II$'$1c	$(-1, -1, 1)$	2	II1c
II$'2_1a$	$(0, 1, 1)$	3	II2_1a
II$'2_1b$	$(0, -1, 1)$	3	II2_1b
II$'2_2a$	$(-1, 0, 1)$	3	II2_2a
II$'2_2b$	$(1, 0, 1)$	3	II2_2b
II$'$3	$(0, 0, 1)$	4	II3

Below are given the canonical representations of the transformations.

Type I1: $\begin{cases} y_0 = x_2(x_2 - \varepsilon x_1), \\ y_1 = x_0(x_0 - x_2), \\ y_2 = x_1(x_1 - x_0), \end{cases}$

Type I2: $\begin{cases} y_0 = x_1^2 + \varepsilon x_2^2, \\ y_1 = x_0 x_2, \\ y_2 = x_1^2 - x_0 x_1, \end{cases}$

Type I2′: $\begin{cases} y_0 = (x_1 + x_2)^2, \\ y_1 = x_0 x_2, \\ y_2 = x_0^2 - \varepsilon x_1^2, \end{cases}$

Type I3: $\begin{cases} y_0 = (x_1 + x_2)^2, \\ y_1 = x_0 x_2, \\ y_2 = x_0^2 + x_0 x_1, \end{cases}$

Types II and IV: $\begin{cases} y_0 = \varepsilon_0 x_0^2 + \varepsilon_1 x_1^2 + \varepsilon_2 x_2^2, \\ y_1 = x_0 x_1, \\ y_2 = x_0 x_2, \end{cases}$

Types II′ and IV′: $\begin{cases} y_0 = \varepsilon_2 x_0^2 + \varepsilon_0 x_2^2, \\ y_1 = x_1 x_2, \\ y_2 = x_1^2 - \varepsilon_1 x_2^2, \end{cases}$

Type III: $\begin{cases} y_0 = x_2^2 + x_0 x_1, \\ y_1 = x_1 x_2, \\ y_2 = x_1^2. \end{cases}$

3. Type I1: non-degenerate transformations

The classification of quadratic transformations $F\colon \mathbb{R}p^2 \to \mathbb{R}p^2$ is based upon considering the discriminant curve Δ. If F is given by

$$y_k = \alpha_k^{ij} x_i x_j, \qquad i, j, k = 0, 1, 2, \tag{3.1}$$

then the well-known criterion for a conic to be degenerate gives the following equation for Δ:

$$\det\left(\sum_{k=0}^{2} \check{y}_k \alpha_k^{ij}\right) = 0 \tag{3.2}$$

(where $(x_0 : x_1 : x_2)$ are some coordinates in the domain $\mathbb{R}p^2$, $(y_0 : y_1 : y_2)$ are coordinates in the range $\mathbb{R}p^2$, and $(\check{y}_0 : \check{y}_1 : \check{y}_2)$ are the coordinates in $\mathcal{L}$ dual to $(y_0 : y_1 : y_2)$). So, in general, Δ is a cubic curve in $\mathbb{R}p^2$.

In this section we suppose that Δ is non-singular.

3.3. Theorem. (1) *Every non-degenerate quadratic transformation* $\mathbb{R}p^2 \to \mathbb{R}p^2$ *has a representation of the type*

$$\begin{cases} y_0 = x_2(x_2 - \varepsilon x_1), \\ y_1 = x_0(x_0 - x_2), \\ y_2 = x_1(x_1 - x_0), \end{cases} \tag{3.4}$$

where $\varepsilon \in \mathbb{R} \setminus \{-8, 0, 1\}$*;*

(2) ε *is a complete invariant of the transformation, i.e., two transformations are equivalent if and only if the corresponding values of* ε *coincide;*

(3) *Given a non-singular real cubic curve* C *with one (two) connected components, there is one (resp., three) quadratic transformations with* $\Delta \cong C$*.*

Remark. Straightforward calculation shows that the dual transformation has representation (3.4) with $\check{\varepsilon} = -8/\varepsilon$.

Remark. One can prove that any *complex* non-degenerate quadratic transformation $\mathbb{C}\mathrm{p}^2 \to \mathbb{C}\mathrm{p}^2$ also has a representation (3.4). But in this case ε depends on the choice of a representation, and a complete invariant of a transformation is $\lambda = (\varepsilon - 1)/\varepsilon^3(\varepsilon + 1) \in \mathbb{C} \setminus \{0\}$. The dual transformation has $\check{\lambda} = -16/\lambda$.

Remark. Statement (3) of the theorem agrees with the obvious real version of Turin's result [T]: given a non-singular real cubic C, transformations with $\Delta \cong C$ are in a natural one-to-one correspondence with the real Spin-structures φ on the complexification of C with $\operatorname{Arf} \varphi = 0$.

Proof of Theorem 3.3.

3.5. Lemma. *On any non-singular real plane cubic curve* C *there are exactly two* (*up to cyclic reordering*) *triples of points* (P_0, P_1, P_2) *such that* P_i *lies on the tangent to* C *at* P_{i+1} (*where* $P_3 = P_0$).

Proof. Fix an inflection point of C. This defines a group structure on C such that three points $P, Q, R \in C$ are concurrent if and only if $P + Q + R = 0$. Hence, the above three points must satisfy the system $2P_0 + P_2 = 2P_1 + P_0 = 2P_2 + P_1 = 0$. Since algebraically C is either $\mathbb{R}/\mathbb{Z}$ or $\mathbb{R}/\mathbb{Z} \oplus \mathbb{Z}_2$, this system has 9 solutions. Three of them (with $P_0 = P_1 = P_2$) correspond to the inflections points of C, and the other six form (up to reordering) the desired two triples. $\square$

Fix one of the two triples (P_0, P_1, P_2) of Lemma 3.5 on the discriminant curve Δ. According to Lemma 1.3, P_0, P_1, P_2 are reducible conics with some singular points S_0, S_1, S_2 respectively such that P_i passes through S_{i+1} (where $S_3 = S_0$). Hence, if P_0, P_1, P_2 are the base points of the coordinate system $(\check{y}_0 : \check{y}_1 : \check{y}_2)$ in $\mathcal{L}$ (i.e., the points with coordinates $(1:0:0)$, $(0:1:0)$, $(0:0:1)$ respectively), and S_0, S_1, S_2 are the base points of the coordinate system $(x_0 : x_1 : x_2)$ in $\mathbb{R}\mathrm{p}^2$, then, after a homogeneous coordinate change (i.e., a change of the type $x_i \mapsto \lambda_i x_i$, $y_j \mapsto \mu_j y_j$) the transformation has representation (3.4). Its discriminant has then equation

$$\varepsilon^2 \check{y}_0^2 \check{y}_1 + \check{y}_1^2 \check{y}_2 + \check{y}_2^2 \check{y}_0 + (\varepsilon - 4)\check{y}_0 \check{y}_1 \check{y}_2 = 0, \tag{3.6}$$

and immediate calculation shows that this curve is non-singular for any $\varepsilon \in \mathbb{R} \setminus \{-8, 0, 1\}$. This proves 3.3(1).

By a homogeneous coordinate change (3.6) can be converted to

$$\check{y}_0^2\check{y}_1 + \check{y}_1^2\check{y}_2 + \check{y}_2^2\check{y}_0 + u\check{y}_0\check{y}_1\check{y}_2 = 0, \tag{3.7}$$

where the new parameter u, defined up to multiplication by cubic roots of unity, is given by

$$u^3 = \frac{(\varepsilon - 4)^3}{e^2}. \tag{3.8}$$

3.9. Lemma. *The above u is an invariant of the* (*real*) *curve C given by* (3.7)

Proof. By definition, u is determined by the curve and the choice of one of the two triples (P_0, P_1, P_2) of Lemma 3.5. We will prove that the two triples can be converted to each other by a projective transformation of C. Let P be the inflection point which defines the group structure on C. Then the embedding $C \hookrightarrow \mathbb{R}\mathrm{p}^2$ is the composition of the canonical map $C \hookrightarrow |3P|^\vee$ and some isomorphism $|3P|^\vee \cong \mathbb{R}\mathrm{p}^2$. On the other hand, the two triples (P_0, P_1, P_2) are $\left(\frac{1}{9}, \frac{7}{9}, \frac{4}{9}\right)$ and $\left(\frac{8}{9}, \frac{2}{9}, \frac{5}{9}\right)$ (remind that the unity component of C is $\mathbb{R}/\mathbb{Z}$), and the automorphism $\times(-1)\colon C \to C$ of C induces the desired transformation of $|3P|^\vee$ and $\mathbb{R}\mathrm{p}^2$. $\square$

Now it is clear that, given a curve C, there are at most three quadratic transformations with $\Delta \cong C$ (which correspond to the three solutions of (3.8)). On the other hand, the transformations dual to these three have, in general, distinct values of u, so they, and hence the original transformations, are pairwise distinct.

To prove Statement (3) it suffices to notice that, if a cubic curve has one (two) components, then $u^3 > 27$ (resp., $u^3 < 27$), and (3.8) has one (resp., three) solutions. $\square$

4. Types I2, I3: transformations with irreducible discriminant

4.1. Theorem. *Let the discriminant Δ of a quadratic transformation F*

have a single singular point. Then F has one of the following representations:

$$\left\{\begin{aligned} y_0 &= x_1^2 + \varepsilon x_2^2, \\ y_1 &= x_0 x_2, \\ y_2 &= x_1^2 - x_0 x_1, \end{aligned}\right. \tag{4.2}$$

$$\left\{\begin{aligned} y_0 &= (x_1 + x_2)^2, \\ y_1 &= x_0 x_2, \\ y_2 &= x_0^2 - \varepsilon x_1^2, \end{aligned}\right. \tag{4.3}$$

$$\left\{\begin{aligned} y_0 &= (x_1 + x_2)^2, \\ y_1 &= x_0 x_2, \\ y_2 &= x_0^2 + x_0 x_1, \end{aligned}\right. \tag{4.4}$$

where in the case of (4.2) *and* (4.3) $\varepsilon = +1$ (Δ *has an isolated singular point*) *or* -1 (Δ *has a node*)*, and in the case of* (4.4) Δ *has a cusp.*

Proof. Consider the case when Δ has a non-degenerate double point (i.e., either a node or an isolated singular point). Then it can be given by

$$\check{y}_0(\check{y}_1^2 + \varepsilon \check{y}_2^2) + \check{y}_1^2 \check{y}_2 = 0, \tag{4.5}$$

and in the corresponding coordinate system $P_0 = (1:0:0)$ is the singular point of Δ and $P_1 = (0:1:0)$ lies in the tangent to Δ through $P_2 = (0:0:1)$. Consider the following two cases:

Case 1: $P_0 = (1:0:0)$ *is a tangency point of* $\mathcal{L}$ *and* Discr_2*.* Let the singular points S_0, S_1, S_2 of the curves P_0, P_1, P_2 respectively be the vertices of the coordinate system $(x_0:x_1:x_2)$ in $\mathbb{R}\mathrm{p}^2$. According to Lemma 1.3, every curve of $\mathcal{L}$ should pass through S_0. Besides, P_1 passes through S_2. Hence, the transformation has a representation

$$\left\{\begin{aligned} y_0 &= a x_1^2 + b x_1 x_2 + c x_2^2, \\ y_1 &= x_0 x_2, \\ y_2 &= x_1^2 - x_0 x_1, \end{aligned}\right. \tag{4.6}$$

and Δ is given by

$$\check{y}_0(a\check{y}_1^2 + c\check{y}_2^2) + \check{y}_1^2 \check{y}_2 - 2b\check{y}_0\check{y}_1\check{y}_2 = 0. \tag{4.7}$$

Comparing this and (4.5) one finds $a = 1$, $b = 0$, $c = \varepsilon$, which gives (4.2).

Case 2: $P_0 = (1:0:0)$ *is a singular point of* Discr_2*.* Let S_0, S_1, S_2 and coordinate system in $\mathbb{R}\mathrm{p}^2$ be the same as in Case 1. (Now S_0 is *one* of the singular points of P_0, which is a double line.) Then the transformation has a representation

$$\begin{cases} y_0 = (x_1 + x_2)^2, \\ y_1 = x_0 x_2, \\ y_2 = a x_1^2 + b x_0 x_1 + c x_0^2, \end{cases} \tag{4.8}$$

Δ is given by

$$\check{y}_0[c\check{y}_1^2 + (b^2 - 4ac)\check{y}_2^2] + c\check{y}_1^2\check{y}_2 - 2b\check{y}_0\check{y}_1\check{y}_2 = 0, \tag{4.9}$$

and comparing this to (4.5) gives $a = -\varepsilon/4$, $b = 0$, $c = 1$, i.e., (4.3) (after an appropriate homogeneous coordinate change).

Now consider the case when Δ has a cusp. Then it can be given by

$$\check{y}_0(\check{y}_1 - \check{y}_2)^2 + \check{y}_1^2\check{y}_2 = 0. \tag{4.10}$$

4.11 Lemma. *The cusp of Δ is a singular point of* Discr_2 (*i.e., a double line*).

Proof. Suppose it is not. Then the transformation can be represented by (4.6), and comparing (4.7) and (4.10) gives $a = c = 1$, $b = -2$. Hence, P_0 is given by $(x_1 - x_2)^2 = 0$, i.e., P_0 is a singular point of Discr_2. □

Due to the lemma, the transformation can be represented by (4.8). Then comparing (4.9) and (4.10) gives $a = 0$, $b = c = 1$, i.e., (4.4) □

5. Pencils of conics

In this section we give a projective classification of pencils (i.e., one dimensional linear systems) of plane conics, or, in other words, of quadratic transformations $\mathbb{R}\mathrm{p}^2 \to \mathbb{R}\mathrm{p}^1$.

5.1. Theorem. (1) *Up to projective equivalence a pencil of conics with nonsingular generic curve is determined by its set of base points* (*including complex and infinitely near*). *The nine types are presented in Table 2* (*where the notation $r\mathbb{R}$* (*resp., $r\mathbb{C}$*) *means that a system has a real* (*resp., complex*) *base point along with the infinitely near points up to the* $(r-1)$*st order*).

(2) *Any pencil of singular conics has one of the following representations:*

$$\begin{cases} y_0 = x_0 x_1, \\ y_1 = x_0 x_2, \end{cases} \tag{5.2}$$

$$\begin{cases} y_0 = x_1 x_2, \\ y_1 = x_1^2 - \varepsilon x_2^2, \qquad \varepsilon = \pm 1, 0. \end{cases} \tag{5.3}$$

TABLE 2. Pencils of conics with non-singular generic curve

Type	Base points	codim
I1a	$(\mathbb{R},\mathbb{R},\mathbb{R},\mathbb{R})$	0
I1b	$(\mathbb{R},\mathbb{R},\mathbb{C},\mathbb{C})$	0
I1c	$(\mathbb{C},\mathbb{C},\mathbb{C},\mathbb{C})$	0
I2a	$(2\mathbb{R},\mathbb{R},\mathbb{R})$	1
I2b	$(2\mathbb{R},\mathbb{C},\mathbb{C})$	1

Type	Base points	codim
I3_1	$(3\mathbb{R},\mathbb{R})$	2
I3_2a	$(2\mathbb{R},2\mathbb{R})$	2
I3_2b	$(2\mathbb{C},2\mathbb{C})$	2
I4	$(4\mathbb{R})$	3

A pencil is said to be of type II1_1 *if it has representation* (5.2)*, and of type* II1_2a, II1_2b*, or* II2 *if it has representation* (5.3) *with* $\varepsilon = 1$*,* -1*, or* 0 *respectively*

Proof. Consider first the case when a generic curve of the pencil is non-singular (Statement (1) of the theorem). Then the pencil has four (including complex and infinitely near) base points in general position (i.e., no three of these four points lie in a line), and, for dimension reason, it coincides with the complete linear system through these points. On the other hand, a set of four points in general position in $\mathbb{R}\mathrm{p}^2$ is defined, up to projective equivalence, by its combinatorial characteristics, i.e., the infinite nearness relation and realness. The nine possible types are given in Table 2. (One should keep in mind that, if a pencil has a complex base point, it should also have the conjugate point of the same multiplicity.)

Now suppose that all the conics of the pencil are singular. Consider the following two cases:

Case 1: *the pencil does not have a singular base point.* Then, according to Bertini's theorem, all the conics have a common component, which in this case must be a line, say, $\{x_0 = 0\}$. The reminding components of the conics form a line pencil, whose base point does not belong to $\{x_0 = 0\}$ (since there is no common singular point), and the pencil can obviously be presented by (5.2).

Case 2: *all the conics have a common singular point.* Let $S = (1:0:0)$ be this point. Then eventually we have a pencil of homogeneous quadratic polynomials in $(x_1 : x_2)$, which, in an appropriate coordinate system, can be given by (5.3) □

TYPE II: Δ HAS A LINEAR COMPONENT OF TYPE II1_1

In the following two sections we consider transformations $\mathbb{R}\mathrm{p}^2 \to \mathbb{R}\mathrm{p}^2$ whose discriminant set Δ is reducible. In this case Δ contains at least one line, which is a pencil of singular conics (i.e., a pencil of one of the types $\mathrm{II}1_1$, $\mathrm{II}1_2$, or $\mathrm{II}2$ of Section 5). In this section we suppose that at least one of the line components of Δ is of type $\mathrm{II}1_1$.

6.1. Theorem. *Suppose that the adjoint linear system $\mathcal{L}$ of a transformation F contains a pencil of type $\mathrm{II}1_1$ (see Theorem 5.1), and its discriminant set Δ does not coincide with $\mathbb{R}\mathrm{p}^2$. Then F has a representation*

$$\begin{cases} y_0 = \varepsilon_0 x_0^2 + \varepsilon_1 x_1^2 + \varepsilon_2 x_2^2, \\ y_1 = x_0 x_1, \\ y_2 = x_0 x_2, \end{cases} \tag{6.2}$$

where the value of the parameter triple $(\varepsilon_0, \varepsilon_1, \varepsilon_2)$ is one of those given in Table 1.

Remark. The other values of $\varepsilon_i = \pm 1, 0$ are also possible, but the transformations obtained are equivalent to those given in Table 1.

Proof. Chose a coordinate system $(\check{y}_0 : \check{y}_1 : \check{y}_2)$ in $\mathcal{L}$ so that its base points P_1, P_2 lie in (one of) the components of type $\mathrm{II}1_1$ of Δ and P_0 be an arbitrary point not on this line. Then, in an appropriate coordinate system $(x_0 : x_1 : x_2)$ in $\mathbb{R}\mathrm{p}^2$, the transformation can be given by

$$\begin{cases} y_0 = a x_0^2 + b x_1^2 + c x_2^2 + 2d x_0 x_1 + 2e x_0 x_2 + 2f x_1 x_2, \\ y_1 = x_0 x_1, \\ y_2 = x_0 x_2. \end{cases} \tag{6.3}$$

It is easy to see that there is a projective transformation $x_0 \mapsto \alpha x_0$, $(x_1, x_2) \mapsto (x_1, x_2) \cdot A$ (where $\alpha \neq 0$ is a real, and A is a non-degenerate (2×2)-matrix), $y_0 \mapsto y_0$, $(y_1, y_2) \mapsto (\alpha y_1, \alpha y_2) \cdot A$ which converts the first equation of (6.3) to $y_0 = \varepsilon_0 x_0^2 + \varepsilon_1 x_1^2 + \varepsilon_2 x_2^2 + 2\tilde{d} x_0 x_1 + 2\tilde{e} x_0 x_2$, $\varepsilon_i = \pm 1, 0$, while leaving the two other equations unchanged. Finally, the coordinate change $y_0 \mapsto y_0 + 2\tilde{d} y_1 + 2\tilde{e} y_2$ gives representation (6.2). The discriminant Δ has then equation

$$\check{y}_0 (4\varepsilon_0 \varepsilon_1 \varepsilon_2 \check{y}_0^2 - \varepsilon_2 \check{y}_1^2 - \varepsilon_1 \check{y}_2^2) = 0. \tag{6.4}$$

Now, the fact that

(1) (6.2) is symmetric in $\varepsilon_1, \varepsilon_2$,
(2) ε_1 and ε_2 cannot be both zeros (since otherwise (6.4) would imply $\Delta = \mathbb{R}\mathrm{p}^2$), and
(3) simultaneous change of the signs of all the ε_i's gives an equivalent transformation

shows that there are only eight essentially different values of $(\varepsilon_0, \varepsilon_1, \varepsilon_2)$, which are given in Table 1. □

7. Type II′: Δ has a 1-fold linear component of type $\mathrm{II}1_2$ or II2

7.1. Theorem. *Suppose that the discriminant set Δ of a transformation F has a 1-fold linear component of type $\mathrm{II}1_2$ or $\mathrm{II}2$, and $\Delta \neq \mathcal{L}$. Then F has a representation*

$$\begin{cases} y_0 = \varepsilon_2 x_0^2 + \varepsilon_0 x_2^2, \\ y_1 = x_1 x_2, \\ y_2 = x_1^2 - \varepsilon_1 x_2^2, \end{cases} \tag{7.2}$$

where the value of the parameter triple $(\varepsilon_0, \varepsilon_1, \varepsilon_2)$ is one of those given in Table 1. (*See also remark after Theorem 6.1.*)

Proof. Chose a coordinate system $(\check{y}_0 : \check{y}_1 : \check{y}_2)$ in $\mathcal{L}$ so that its base points P_1, P_2 lie in (one of) the 1-fold components of type $\mathrm{II}1_2$ or II2 of Δ and P_0 be an arbitrary point not on this line. Then, in an appropriate coordinate system $(x_0 : x_1 : x_2)$ in $\mathbb{R}\mathrm{p}^2$, the transformation can be given by

$$\begin{cases} y_0 = a x_0^2 + b x_1^2 + c x_2^2 + 2d x_0 x_1 + 2e x_0 x_2 + 2f x_1 x_2, \\ y_1 = x_1 x_2, \\ y_2 = x_1^2 - \varepsilon_1 x_2^2. \end{cases} \tag{7.3}$$

Prove that $a \neq 0$. Suppose that $a = 0$. Then the coordinate change $y_0 \mapsto y_0 + 2f y_1 + b y_2$ converts the first equation of (7.3) to $y_0 = \tilde{c} x_2^2 + 2d x_0 x_1 + 2e x_0 x_2$, and Δ has equation

$$\check{y}_0^2 [de \check{y}_1 + (d^2 \varepsilon_1 - e^2) \check{y}_2 - \tilde{c} d^2 \check{y}_0] = 0, \tag{7.4}$$

which shows that $\{\check{y}_0 = 0\}$ is an at least 2-fold component of Δ. This contradicts to our choice of the coordinate system.

Hence, $a \neq 0$. In this case the coordinate change $x_0 \mapsto x_0 - \dfrac{d}{a} x_1 - \dfrac{e}{a} x_2$ transforms the first equation to $y_0 = a x_0^2 + \tilde{b} x_1^2 + \tilde{c} x_2^2 + 2\tilde{f} x_1 x_2$, and $y_0 \mapsto y_0 + 2\tilde{f} y_1 + \tilde{b} y_2$, followed by a homogeneous coordinate change, gives (7.2). Δ has then equation

$$\varepsilon_2 \check{y}_0 (\check{y}_1^2 + 4\varepsilon_1 \check{y}_2^2 - 4\varepsilon_0 \check{y}_0 \check{y}_2) = 0. \tag{7.5}$$

Now, taking into account the facts that $\varepsilon_2 \neq 0$ (since otherwise $\Delta = \mathcal{L}$) and simultaneous change of the signs of ε_0 and ε_2 gives an equivalent system, one sees that there are nine essentially different values of $(\varepsilon_0, \varepsilon_1, \varepsilon_2)$. Eight of them are given in Table 1, and the ninth one — $(1, -1, 1)$ — is equivalent to $\mathrm{II}'1c$ (via the coordinate change $x_1 \mapsto x_2$, $x_2 \mapsto x_1$; $y_0 \mapsto y_0 + y_2$). $\square$

8. Other types (III, IV, IV′)

8.1. Theorem. (1) *Any transformation F whose discriminant Δ is a 3-fold line has a representation*

$$\begin{cases} y_0 = x_2^2 + x_0x_1, \\ y_1 = x_1x_2, \\ y_2 = x_1^2. \end{cases} \tag{8.2}$$

(2) *If Δ coincides with the adjoint linear system $\mathcal{L}$, then F has a representation*

$$\begin{cases} y_0 = x_0^2, \\ y_1 = x_0x_1, \\ y_2 - x_0x_2, \end{cases} \quad \text{or} \quad \begin{cases} y_0 = x_2^2, \\ y_1 = x_1x_2, \\ y_2 = x_1^2 \end{cases} \tag{8.3}$$

(which, in fact, are (6.2) *or* (7.2) *with* $(\varepsilon_0, \varepsilon_1, \varepsilon_2) = (1, 0, 0)$ *).*

Proof of Statement (1)*.* We will prove that Δ is a pencil of type $\mathrm{II}1_2$ or $\mathrm{II}2$ (actually, II2), and $a = 0$ in (7.3). Indeed, suppose that Δ is of type $\mathrm{II}1_1$. Then F can be given by (6.2), and Δ, by (6.4), which is never a triple line. Similarly, if Δ is of type $\mathrm{II}1_2$ or $\mathrm{II}2$ (and, hence, F is represented by (7.3)), and $a \neq 0$, then Δ is given by (7.5) (after a coordinate change) and is not a triple line either.

Hence, F is given by (7.3) with $a = 0$, which can be converted to

$$\begin{cases} y_0 = \tilde{c}x_2^2 + 2dx_0x_1 + 2ex_0x_2, \\ y_1 = x_1x_2, \\ y_2 = x_1^2 - \varepsilon x_2^2 \end{cases} \tag{8.4}$$

(see proof of Theorem 7.1), and Δ has equation (7.4). The condition that Δ be a triple line yields then $de = 0$, $d^2\varepsilon_1 - e^2 = 0$, $\tilde{c}d^2 \neq 0$, which implies $\tilde{c} \neq 0$, $d \neq 0$, $e = \varepsilon_1 = 0$, and (8.2) is obtained by a homogeneous coordinate change. □

Proof of Statement (2)*.* If $\mathcal{L}$ contains at least one pencil of type $\mathrm{II}1_1$, then F has representation (6.2), Δ is given by (6.4), and the condition $\Delta = \mathcal{L}$ implies $\varepsilon_1 = \varepsilon_2 = 0$, i.e., the first system of (8.3).

Suppose now that Δ contains a pencil of type $\mathrm{II}1_2$ or $\mathrm{II}2$ and, hence, F is represented by (7.3). If $a \neq 0$, this can be converted to (7.2), (7.5) yields $\varepsilon_2 = 0$, and the coordinate change $y_2 \mapsto y_2 - \varepsilon_1 y_0$ gives the second system of (8.3). If $a = 0$, F is represented by (8.4) with $de = d^2\varepsilon_1 - e^2 = \tilde{c}d^2 = 0$

(see (7.4)). Hence, either $d = e = 0$, or $d \neq 0$, $\tilde{c} = e = \varepsilon_1 = 0$. In the first case $\tilde{c} \neq 0$, and the coordinate change $y_0 \mapsto \tilde{c}y_0$, $y_2 \mapsto y_2 - \varepsilon_1 y_0$ shows that F is equivalent to the previous transformation. In the second case, the pencil $\{y_2 = 0\} \in \mathcal{L}$ is of type $\mathrm{II}1_1$, and, hence, the transformation is equivalent to that given by the first system of (8.3). □

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