

ON PLANE SEXTICS WITH DOUBLE SINGULAR POINTS

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ABSTRACT. We compute the fundamental groups of five maximizing sextics with double singular points only; in four cases, the groups are as expected. The approach used would apply to other sextics as well, given their equations.

1. INTRODUCTION

The fundamental group $\pi_1 := \pi_1(\mathbb{P}^2 \setminus D)$ of a plane curve $D \subset \mathbb{P}^2$, introduced by O. Zariski in [18], is an important topological invariant of the curve. Apart from distinguishing the connected components of the equisingular moduli spaces, this group can be used as a seemingly inexpensive way of studying algebraic surfaces, the curve serving as the branch locus of a projection of the surface onto \mathbb{P}^2 .

At present, the fundamental groups of all curves of degree up to five are known, and the computation of the groups of irreducible curves of degree six (sextics) is close to its completion, see [7] for the principal statements and further references. In higher degrees, little is known: there are a few general theorems, usually bounding the complexity of the group of a curve with sufficiently ‘moderate’ singularities, and a number of sporadic examples scattered in the literature. For further details on this fascinating subject, we refer the reader to the recent surveys [2, 10, 11].

1.1. Principal results. If a sextic $D \subset \mathbb{P}^2$ has a singular point P of multiplicity three or higher, then, projecting from this point, we obtain a trigonal (or, even better, bi- or monogonal) curve in a Hirzebruch surface, see §3.1. By means of the so-called *dessins d’enfants*, such curves and their topology can be studied in purely combinatorial terms, as certain graphs in the plane. The classification of such curves and the computation of their fundamental groups were completed in [7]. If all singular points are double, the best that one can obtain is a tetragonal curve, which is a much more complicated object. (A reduction of tetragonal curves to trigonal curves in the presence of a section is discussed in §3.2, see Remark 3.6. It is the extra section that makes the problem difficult.) At present, I do not know how the group of a tetragonal curve can be computed unless the curve is real and its defining equation is known (and even then, the approach suggested in the paper may still fail, cf. Remark 2.1).

There is a special class of irreducible sextics, the so called \mathbb{D}_{2n} -sextics and, in particular, sextics of *torus type* (see §2.1 for the precise definitions), for which the fundamental group is non-abelian for some simple homological reasons, see [4]. (The fact that a sextic is of torus type is usually indicated by the presence of a pair of parentheses in the notation; their precise meaning is explained in §2.1.) On the other hand, thanks to the special structures and symmetries of these curves, their

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explicit equations are known, see [6, 8, 12]. In this paper, we almost complete the computation of the fundamental groups of \mathbb{D}_{2n} -sextics (with one pair of complex conjugate sextics of torus type left). Our principal results can be stated as follows.

Theorem 1.1. *The fundamental group of the \mathbb{D}_{14} -special sextic with the set of singularities $3\mathbf{A}_6 \oplus \mathbf{A}_1$, line 37 in Table 1 is $\mathbb{Z}_3 \times \mathbb{D}_{14}$.*

Theorem 1.2. *The fundamental groups of the irreducible sextics of torus type with the sets of singularities $(\mathbf{A}_{14} \oplus \mathbf{A}_2) \oplus \mathbf{A}_3$, line 8, $(\mathbf{A}_{14} \oplus \mathbf{A}_2) \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$, line 9, and $(\mathbf{A}_{11} \oplus 2\mathbf{A}_2) \oplus \mathbf{A}_4$, line 17 in Table 1 are isomorphic to $\Gamma := \mathbb{Z}_2 * \mathbb{Z}_3$. The group of the curve with the set of singularities $(\mathbf{A}_8 \oplus 3\mathbf{A}_2) \oplus \mathbf{A}_4 \oplus \mathbf{A}_1$, line 33 is*

$$(1.3) \quad \pi_1 = \langle \alpha_2, \alpha_3, \alpha_4 \mid [\alpha_3, \alpha_4] = \{\alpha_2, \alpha_3\}_3 = \{\alpha_2, \alpha_4\}_9 = 1, \\ \alpha_4 \alpha_2 \alpha_3^{-1} \alpha_4 \alpha_2 \alpha_4 (\alpha_4 \alpha_2)^{-2} \alpha_3 = (\alpha_2 \alpha_4)^2 \alpha_3^{-1} \alpha_2 \alpha_4 \alpha_3 \alpha_2 \rangle,$$

where $\{\alpha, \beta\}_{2k+1} := (\alpha\beta)^k \alpha (\alpha\beta)^{-k} \beta^{-1}$.

Theorem 1.1 is proved in §4.3, and Theorem 1.2 is proved in §4.5–§4.8, one curve at a time. I do not know whether the last group (1.3) is isomorphic to Γ : all ‘computable’ invariants seem to coincide, see Remark 4.7, but the presentations obtained resist all simplification attempt. The quotient of (1.3) by the extra relation $\{\alpha_2, \alpha_4\}_3 = 1$ is Γ .

The next proposition is proved in §4.9. (The perturbation $3\mathbf{A}_6 \oplus \mathbf{A}_1 \rightarrow 3\mathbf{A}_6$ excluded in the statement results in a \mathbb{D}_{14} -special sextic and the fundamental group equals $\mathbb{Z}_3 \times \mathbb{D}_{14}$, see [8].)

Proposition 1.4. *Let D' be a nontrivial perturbation of a sextic as in Theorems 1.1 or 1.2. Unless the set of singularities of D' is $3\mathbf{A}_6$, the group $\pi_1(\mathbb{P}^2 \setminus D')$ is Γ or \mathbb{Z}_6 , depending on whether D' is or, respectively, is not of torus type.*

With Theorem 1.1 in mind, the fundamental groups of all \mathbb{D}_{2n} -special sextics, $n \geq 5$, are known, see [7]. Modulo the feasible conjecture that any sextic of torus type degenerates to a maximizing one, the only such sextic whose group remains unknown is $(\mathbf{A}_8 \oplus \mathbf{A}_5 \oplus \mathbf{A}_2) \oplus \mathbf{A}_4$, line 32 in Table 1. (This conjecture has been proved, and all groups except the one just mentioned are indeed known; details will appear elsewhere.) Most of these groups are isomorphic to Γ , see [7] for details and further references.

I would like to mention an alternative approach, see [1], reducing a plane sextic with large Milnor number to a trigonal curve equipped with a number of sections, all but one splitting in the covering elliptic surface. It was used in [1] to handle the curves in lines 1–6 in Table 1. This approach is also used in a forthcoming paper to produce the defining equations of most sextics listed in Table 1; then, the fundamental groups of most *real* ones can be computed using Theorem 3.16. All groups that could be found are abelian. Together with the classification of sextics, which is also almost completed, this fact implies that, with very few exceptions, the fundamental group of a non-special irreducible simple sextic is abelian.

1.2. Idea of the proof (see §4.1 for more details). We use the classical Zariski–van Kampen method, cf. Theorem 3.16, expressing the fundamental group of a curve in terms of its braid monodromy with respect to an appropriate pencil of lines. The curves and pencils considered are *real*, and the braid monodromy in a neighborhood of the real part of the pencil is computed in terms of the real part of the curve.

(This approach originates in topology of real algebraic curves; historically, it goes back to Viro, Fiedler, Kharlamov, Rokhlin, and Klein.) Our main contribution is the description of the monodromy along a real segment where all four branches of the curve are non-real, see [Proposition 3.12](#). Besides, the curves are not required to be *strongly real*, *i.e.*, non-real singular fibers are allowed. Hence, we follow Orevkov [\[13\]](#) and attempt to extract information about such non-real fibers from the real part of the curve. The outcome is [Theorem 3.16](#), which gives us an ‘upper bound’ on the fundamental group in question. The applicability issues and a few other common tricks are discussed in [§4.1](#).

1.3. Contents of the paper. In [§2](#), we introduce the terminology related to plane sextics, list the sextics that are still to be investigated, and discuss briefly the few known results. In [§3](#), we outline an approach to the (partial) computation of the braid monodromy of a real tetragonal curve and state an appropriate version of the Zariski–van Kampen theorem. Finally, in [§4](#) the results of [§3](#) and known equations are used to prove [Theorems 1.1 and 1.2](#) and [Proposition 1.4](#).

1.4. Conventions. All group actions are *right*. Given a right action $X \times G \rightarrow X$ and a pair of elements $x \in X$, $g \in G$, the image of (x, g) is denoted by $x \uparrow g \in X$. The same postfix notation and multiplication convention is often used for maps: it is under this convention that the monodromy $\pi_1(\text{base}) \rightarrow \text{Aut}(\text{fiber})$ of a locally trivial fibration is a homomorphism rather than an anti-homomorphism.

The assignment symbol $:=$ is used as a shortcut for ‘is defined as’.

We use the conventional symbol \square to mark the ends of the proofs. Some statements are marked with \triangleleft or \triangleright : the former means that the proof has already been explained (for example, most corollaries), and the latter indicates that the proof is not found in the paper and the reader is directed to the literature, usually cited at the beginning of the statement.

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2. PRELIMINARIES

2.1. Special classes of sextics. A plane sextic $D \in \mathbb{P}^2$ is called *simple* if all its singularities are simple, *i.e.*, those of type **A–D–E**. The total Milnor number μ of a simple sextic D does not exceed 19, see [\[14\]](#); if $\mu = 19$, then D is called *maximizing*. Maximizing sextics are always defined over algebraic number fields and their moduli spaces are discrete: two such sextics are equisingular deformation equivalent if and only if they are related by a projective transformation of \mathbb{P}^2 .

A sextic D is said to be of *torus type* if its equation can be represented in the form $f_2^3 + f_3^2 = 0$, where f_2 and f_3 are some polynomials of degree 2 and 3, respectively. The points of intersection of the conic $\{f_2 = 0\}$ and cubic $\{f_3 = 0\}$ are always singular for D . These singular points play a very special rôle; they are called the *inner* singularities (with respect to the given torus structure). For the vast majority

of curves, a torus structure is unique, and in this case it is common to parenthesize the inner singularities in the notation.

An irreducible sextic D is called \mathbb{D}_{2n} -special if its fundamental group $\pi_1(\mathbb{P}^2 \setminus D)$ admits a dihedral quotient $\mathbb{D}_{2n} := \mathbb{Z}_n \rtimes \mathbb{Z}_2$. According to [4], only \mathbb{D}_6 -, \mathbb{D}_{10} -, and \mathbb{D}_{14} -special sextics exist, and an irreducible sextic is of torus type if and only if it is \mathbb{D}_6 -special. (In particular, torus type is a topological property.)

Any sextic D of torus type is a degeneration of Zariski's six-cuspidal sextic, which is obtained from a generic pair (f_2, f_3) . It follows that the fundamental group of D factors to the modular group $\Gamma := SL(2, \mathbb{Z}) = \mathbb{Z}_2 * \mathbb{Z}_3 = \mathbb{B}_3 / (\sigma_1 \sigma_2 \sigma_1)^2$, see [18]; in particular, this group is infinite. Conjecturally, the fundamental groups of all other irreducible simple sextics are finite.

2.2. Sextics to be considered. It is expected that, with few explicit exceptions (e.g., $9\mathbf{A}_2$), any simple sextic degenerates to a maximizing one. (The proof of this conjecture, which relies upon the theory of $K3$ -surfaces, is currently a work in progress. In fact, most curves degenerate to one of those whose group is already known.) Hence, it is essential to compute the fundamental groups of the maximizing sextics; the others would follow. The groups of all irreducible sextics with a singular point of multiplicity three or higher are known, see [7] for a summary of the results, and those with \mathbf{A} type singularities only are still to be investigated.

A list of irreducible maximizing sextics with \mathbf{A} type singular points only can be compiled using the results of [17] (a list of the sets of singularities realized by such sextics) and [15] (a description of the moduli spaces). We represent the result in Table 1, where the column (r, c) shows the number of classes: r is the number of real sextics, and c is the number of pairs of complex conjugate ones. The approach developed further in the paper lets one compute (or at least estimate) the fundamental group of a sextic with \mathbf{A} type singularities, provided that its equation is known. In the literature, I could find explicit equations for lines 1–6, 8, 9, 17, 28, 29, 32, 33, and 37. With the results of this paper (Theorems 1.1 and 1.2) taken into account, the groups of all these sextics except $(\mathbf{A}_8 \oplus \mathbf{A}_5 \oplus \mathbf{A}_2) \oplus \mathbf{A}_4$, line 32 (which is not real) are known.

Remark 2.1. Unfortunately, our approach does not always work even if the curve is real. Thus, each of the two sextics with the set of singularities \mathbf{A}_{19} , line 1 has a single real point (the isolated singular point of type \mathbf{A}_{19} ; see [1] for the equations) and Theorem 3.16 does not provide enough relations to compute the group.

2.3. Known results. The fundamental group of the \mathbb{D}_{10} -special sextic with the set of singularities $\mathbf{A}_9 \oplus 2\mathbf{A}_4 \oplus \mathbf{A}_2$, line 28 in Table 1, can be described as follows, see [6] (where ' temporarily stands for the commutant of a group):

$$(2.2) \quad \pi_1/\pi_1'' = \mathbb{Z}_3 \times \mathbb{D}_{10}, \quad \pi_1'' = SL(2, \mathbb{k}_9),$$

where \mathbb{k}_9 is the field of nine elements. The fundamental groups of the first twelve sextics, lines 1–6, have been found in [1]: with the exception of $(\mathbf{A}_{17} \oplus \mathbf{A}_2)$, line 3 (sextic of torus type, $\pi_1 = \Gamma$), they are all abelian. To my knowledge, the groups not mentioned in Table 1 have not been computed yet.

3. THE BRAID MONODROMY

3.1. Hirzebruch surfaces. A Hirzebruch surface Σ_d , $d > 0$, is a geometrically ruled rational surface with a (unique) exceptional section E of self-intersection $-d$.

TABLE 1. Irreducible maximizing sextics with \mathbf{A} type singularities

#	Singularities	(r, c)	Equation, π_1 , remarks
1.	\mathbf{A}_{19}	$(2, 0)$	$\pi_1 = \mathbb{Z}_6$, see [1]
2.	$\mathbf{A}_{18} \oplus \mathbf{A}_1$	$(1, 1)$	$\pi_1 = \mathbb{Z}_6$, see [1]
3.	$(\mathbf{A}_{17} \oplus \mathbf{A}_2)$	$(1, 0)^*$	$\pi_1 = \Gamma$, see [1, 5] (torus type)
4.	$\mathbf{A}_{16} \oplus \mathbf{A}_3$	$(2, 0)$	$\pi_1 = \mathbb{Z}_6$, see [1]
5.	$\mathbf{A}_{16} \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$	$(1, 1)$	$\pi_1 = \mathbb{Z}_6$, see [1]
6.	$\mathbf{A}_{15} \oplus \mathbf{A}_4$	$(0, 1)^*$	$\pi_1 = \mathbb{Z}_6$, see [1]
7.	$\mathbf{A}_{14} \oplus \mathbf{A}_4 \oplus \mathbf{A}_1$	$(0, 3)$	
8.	$(\mathbf{A}_{14} \oplus \mathbf{A}_2) \oplus \mathbf{A}_3$	$(1, 0)$	$\pi_1 = \Gamma$, see §4.5 (torus type)
9.	$(\mathbf{A}_{14} \oplus \mathbf{A}_2) \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$	$(1, 0)$	$\pi_1 = \Gamma$, see §4.6 (torus type)
10.	$\mathbf{A}_{13} \oplus \mathbf{A}_6$	$(0, 2)$	
11.	$\mathbf{A}_{13} \oplus \mathbf{A}_4 \oplus \mathbf{A}_2$	$(2, 0)$	
12.	$\mathbf{A}_{12} \oplus \mathbf{A}_7$	$(0, 1)$	
13.	$\mathbf{A}_{12} \oplus \mathbf{A}_6 \oplus \mathbf{A}_1$	$(1, 1)$	
14.	$\mathbf{A}_{12} \oplus \mathbf{A}_4 \oplus \mathbf{A}_3$	$(1, 0)$	
15.	$\mathbf{A}_{12} \oplus \mathbf{A}_4 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$	$(1, 1)$	
16.	$\mathbf{A}_{11} \oplus 2\mathbf{A}_4$	$(2, 0)$	
17.	$(\mathbf{A}_{11} \oplus 2\mathbf{A}_2) \oplus \mathbf{A}_4$	$(1, 0)$	$\pi_1 = \Gamma$, see §4.7 (torus type)
18.	$\mathbf{A}_{10} \oplus \mathbf{A}_9$	$(2, 0)^*$	
19.	$\mathbf{A}_{10} \oplus \mathbf{A}_8 \oplus \mathbf{A}_1$	$(1, 1)$	
20.	$\mathbf{A}_{10} \oplus \mathbf{A}_7 \oplus \mathbf{A}_2$	$(2, 0)$	
21.	$\mathbf{A}_{10} \oplus \mathbf{A}_6 \oplus \mathbf{A}_3$	$(0, 1)$	
22.	$\mathbf{A}_{10} \oplus \mathbf{A}_6 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$	$(1, 1)$	
23.	$\mathbf{A}_{10} \oplus \mathbf{A}_5 \oplus \mathbf{A}_4$	$(2, 0)$	
24.	$\mathbf{A}_{10} \oplus 2\mathbf{A}_4 \oplus \mathbf{A}_1$	$(1, 1)$	
25.	$\mathbf{A}_{10} \oplus \mathbf{A}_4 \oplus \mathbf{A}_3 \oplus \mathbf{A}_2$	$(1, 0)$	
26.	$\mathbf{A}_{10} \oplus \mathbf{A}_4 \oplus 2\mathbf{A}_2 \oplus \mathbf{A}_1$	$(2, 0)$	
27.	$\mathbf{A}_9 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4$	$(1, 1)^*$	
28.	$\mathbf{A}_9 \oplus 2\mathbf{A}_4 \oplus \mathbf{A}_2$	$(1, 0)^*$	$\pi_1 = (2.2)$, see [6] (\mathbb{D}_{10} -sextic)
29.	$(2\mathbf{A}_8) \oplus \mathbf{A}_3$	$(1, 0)$	$\pi_1 = \Gamma$, see [5] (torus type)
30.	$\mathbf{A}_8 \oplus \mathbf{A}_7 \oplus \mathbf{A}_4$	$(0, 1)$	
31.	$\mathbf{A}_8 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_1$	$(1, 1)$	
32.	$(\mathbf{A}_8 \oplus \mathbf{A}_5 \oplus \mathbf{A}_2) \oplus \mathbf{A}_4$	$(0, 1)$	nt104 in [12] (torus type)
33.	$(\mathbf{A}_8 \oplus 3\mathbf{A}_2) \oplus \mathbf{A}_4 \oplus \mathbf{A}_1$	$(1, 0)$	$\pi_1 = (1.3)$, see §4.8 (torus type)
34.	$\mathbf{A}_7 \oplus 2\mathbf{A}_6$	$(0, 1)$	
35.	$\mathbf{A}_7 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_2$	$(2, 0)$	
36.	$\mathbf{A}_7 \oplus 2\mathbf{A}_4 \oplus 2\mathbf{A}_2$	$(1, 0)$	
37.	$3\mathbf{A}_6 \oplus \mathbf{A}_1$	$(1, 0)$	$\pi_1 = \mathbb{Z}_3 \times \mathbb{D}_{14}$, see §4.3 (\mathbb{D}_{14} -sextic)
38.	$2\mathbf{A}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$	$(2, 0)$	
39.	$\mathbf{A}_6 \oplus \mathbf{A}_5 \oplus 2\mathbf{A}_4$	$(2, 0)$	

Marked with a * are the sets of singularities realized by reducible sextics as well
There are 42 real and 20 pairs of complex conjugate curves

Typically, we use affine coordinates (x, y) in Σ_d such that E is given by $y = \infty$; then, x can be regarded as an affine coordinate in the base of the ruling. (The line $\{x = \infty\}$ plays no special rôle; usually, it is assumed sufficiently generic.) The fiber of the ruling over a point x in the base is denoted by F_x , and the *affine fiber* over x is $F_x^\circ := F_x \setminus E$. This is an affine space over \mathbb{C} ; in particular, one can speak about convex hulls in F_x° .

An *n-gonal curve* is a reduced curve $C \subset \Sigma_d$ intersecting each fiber at n points, *i.e.*, such that the restriction to C of the ruling $\Sigma_d \rightarrow \mathbb{P}^1$ is a map of degree n . A *singular fiber* of an *n-gonal curve* C is a fiber F of the ruling intersecting $C + E$ geometrically at fewer than $(n + 1)$ points. A singular fiber F is *proper* if C does not pass through $F \cap E$. The curve C is *proper* if so are all its singular fibers. In other words, C is proper if it is disjoint from E .

In affine coordinates (x, y) as above an *n-gonal curve* $C \subset \Sigma_d$ is given by a polynomial of the form $\sum_{i=0}^n a_i(x)y^i$, where $\deg a_i \leq m + d(n - i)$ for some $m \geq 0$ (in fact, $m = C \cdot E$) and at least one polynomial a_i does have the prescribed degree (so that C does not contain the fiber $\{x = \infty\}$). The curve is proper if and only if $m = 0$; in this case $a_n(x) = \text{const}$.

A proper *n-gonal curve* $C \subset \Sigma_d$ defines a distinguished *zero section* $Z \subset \Sigma_d$, sending each point $x \in \mathbb{P}^1$ to the barycenter of the n points of $F_x^\circ \cap C$. Certainly, this section does not need to coincide with $\{y = 0\}$, which depends on the choice of the coordinates.

3.2. The cubic resolvent. Consider a reduced *real* quartic polynomial

$$(3.1) \quad f(x, y) := y^4 + p(x)y^2 + q(x)y + r(x),$$

so that its roots y_1, y_2, y_3, y_4 (at each point x) satisfy $y_1 + y_2 + y_3 + y_4 = 0$, and consider the (*modified*) *cubic resolvent* of f

$$(3.2) \quad y^3 - 2p(x)y^2 + b_1(x)y + q(x)^2, \quad b_1 := p^2 - 4r,$$

and its reduced form

$$(3.3) \quad \bar{y}^3 + g_2(x)\bar{y} + g_3(x)$$

obtained by the substitution $y = \bar{y} + \frac{2}{3}p$. The discriminants of (3.1)–(3.3) are equal:

$$(3.4) \quad D = 16p^4r - 4p^3q^2 - 128p^2r^2 + 144pq^2r - 27q^4 + 256r^3.$$

Recall that $D = 0$ if and only if (3.1) or, equivalently, (3.2) or (3.3) has a multiple root. Otherwise, $D < 0$ if and only if exactly two roots of (3.1) are real. The roots of (3.2) are

$$(3.5) \quad \begin{aligned} \alpha &:= (y_1 + y_2)(y_3 + y_4) = -(y_1 + y_2)^2, \\ \beta &:= (y_1 + y_3)(y_2 + y_4) = -(y_1 + y_3)^2, \\ \gamma &:= (y_1 + y_4)(y_2 + y_3) = -(y_1 + y_4)^2, \end{aligned}$$

and those of (3.3) are obtained from (3.5) by shifting the barycenter $\frac{1}{3}(\alpha + \beta + \gamma)$ to zero.

Remark 3.6. If $\{f(x, y) = 0\}$ is a proper tetragonal curve in a Hirzebruch surface Σ_d , then (3.2) defines a proper trigonal curve $C' \subset \Sigma_{2d}$ and a distinguished section $S := \{y = 0\}$ (in general, *other* than the zero section) which is tangent (more precisely, has even intersection index at each intersection point) to C' . Conversely,

(3.1) can be recovered from (3.2) (together with the section $S = \{y = 0\}$) uniquely up to the automorphism $y \mapsto -y$, which takes q to $-q$.

Remark 3.7. One has

$$q = -(y_1 + y_2)(y_1 + y_3)(y_1 + y_4);$$

hence, q vanishes if and only if two of the roots of (3.1) are opposite. If all roots are non-real, $y_{1,2} = \alpha \pm \beta' i$, $y_{3,4} = -\alpha \pm \beta'' i$, $\alpha, \beta', \beta'' \in \mathbb{R}$, then

$$q = 2\alpha(\beta'^2 - \beta''^2), \quad b_1 = -8\alpha^2(\beta'^2 + \beta''^2) + (\beta'^2 - \beta''^2)^2.$$

Hence, $q(x) = 0$ if and only if either $\alpha = 0$ (and then $b_1(x) > 0$, assuming that $D(x) \neq 0$) or $\beta' = \pm\beta''$ (and then $b_1(x) < 0$). If $y_1 = y_2$, *i.e.*, $\beta' = 0$, then $q(x) > 0$ if and only if one has the inequality $y_1 < \operatorname{Re} y_3 = \operatorname{Re} y_4$ equivalent to $y_1 < 0$.

Remark 3.8. Observe also that, if $y_1 = y_2$, then g_3 takes the form

$$g_3 = \frac{2}{27}(y_1 - y_4)^3(y_1 - y_3)^3.$$

Hence, $g_3(x) < 0$ if and only if the two other roots are real and separated by the double root $y_1 = y_2$. Otherwise, either $y_1 < \operatorname{Re} y_3, \operatorname{Re} y_4$ or $y_1 > \operatorname{Re} y_3, \operatorname{Re} y_4$, and, in view of Remark 3.7, the former holds if and only if $q(x) > 0$.

3.3. The real monodromy. Choose affine coordinates (x, y) in the Hirzebruch surface Σ_d so that the exceptional section E is $\{y = \infty\}$. Consider a *real* proper tetragonal curve $C \subset \Sigma_d$; it is given by a real polynomial $f(x, y)$ as in (3.1). Over a generic real point $x \in \mathbb{R}$, the four points y_1, \dots, y_4 of the intersection $C \cap F_x^\circ$ can be ordered lexicographically, according to the *decreasing* of $\operatorname{Re} y$ first and $\operatorname{Im} y$ second. We always assume this ordering. Then, choosing a real reference point $y \gg 0$, we have a *canonical* geometric basis $\{\alpha_1, \dots, \alpha_4\}$ for the fundamental group $\pi(x) := \pi_1(F_x^\circ \setminus C, y)$, see Figure 1.

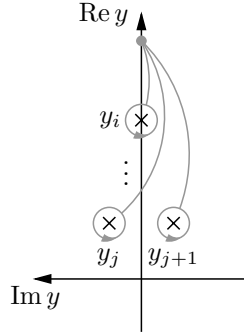
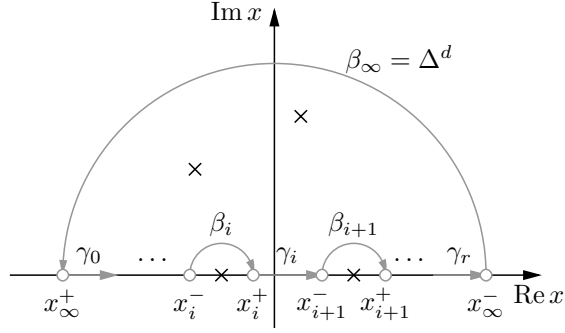


FIGURE 1. The canonical basis

Let x_1, \dots, x_r be all *real* singular fibers of C , ordered by increasing. For each i , consider a pair of nonsingular fibers $x_i^- := x_i - \epsilon$ and $x_i^+ := x_i + \epsilon$, where ϵ is a sufficiently small positive real number, see Figure 2. Denote $x_0 = x_{r+1} = \infty$ and, assuming the fiber $x = \infty$ nonsingular, pick also a pair of real nonsingular fibers $x_{r+1}^- = x_\infty^- := R \gg 0$ and $x_0^+ = x_\infty^+ := -R$. Identify all groups $\pi(x_i^\pm)$ with the free group \mathbb{F}_4 by means of their respective canonical bases. (All reference points are chosen in a real section $y = \text{const} \gg 0$, which is assumed disjoint from the fiberwise

FIGURE 2. The monodromies β_i and γ_j

convex hull of C over the disk $|x| \leq R$.) Consider the semicircles $t \mapsto x_i + \epsilon e^{i\pi(1-t)}$, $t \in [0, 1]$, and the line segments $t \mapsto t$, $t \in [x_j^+, x_{j+1}^-]$, cf. Figure 2. These paths give rise to the *monodromy isomorphisms*

$$\beta_i: \pi(x_i^-) \rightarrow \pi(x_i^+), \quad \gamma_j: \pi(x_j^+) \rightarrow \pi(x_{j+1}^-),$$

$i = 1, \dots, r$, $j = 0, \dots, r$. In addition, we also have the monodromy $\beta_0 = \beta_\infty = \beta_{r+1}: \pi(x_\infty^-) \rightarrow \pi(x_\infty^+)$ along the semicircle $t \mapsto Re^{i\pi t}$, $t \in [0, 1]$, and the *local monodromies*

$$\mu_i: \pi(x_i^+) \rightarrow \pi(x_i^+), \quad i = 1, \dots, r$$

along the circles $t \mapsto x_i + \epsilon e^{2\pi i t}$, $t \in [0, 1]$. Using the identifications $\pi(x_i^\pm) = \mathbb{F}_4$ fixed above, all β_i , μ_i , γ_j can be regarded as elements of the automorphism group $\text{Aut } \mathbb{F}_4$, and as such they belong to the braid group \mathbb{B}_4 . Recall, see [3], that Artin's *braid group* $\mathbb{B}_4 \subset \text{Aut}\langle \alpha_1, \dots, \alpha_4 \rangle$ is the subgroup consisting of the automorphisms taking each generator α_i to a conjugate of a generator and preserving the product $\alpha_1 \alpha_2 \alpha_3 \alpha_4$. It is generated by the three braids

$$\sigma_i: \alpha_i \mapsto \alpha_i \alpha_{i+1} \alpha_i^{-1}, \quad \alpha_{i+1} \mapsto \alpha_i, \quad i = 1, 2, 3,$$

the defining relations being $\{\sigma_1, \sigma_2\}_3 = \{\sigma_2, \sigma_3\}_3 = [\sigma_1, \sigma_3] = 1$.

3.4. The computation. The braids β_i , μ_i , and γ_j introduced in the previous section are easily computed from the real part $C_{\mathbb{R}} \subset \mathbb{R}^2$ of the curve. In the figures, we use the following notation:

- real branches of C are represented by solid bold lines;
- pairs y_i, y_{i+1} of complex conjugate branches are represented by dotted lines (showing the common real part $\text{Re } y_i = \text{Re } y_{i+1}$);
- relevant fibers of Σ_d are represented by vertical dotted grey lines.

Certainly, the dotted lines are not readily seen in the figures; however, in most cases, it is only the intersection indices that matter, and the latter are determined by the indexing of the branches at the starting and ending positions.

We summarize the results in the next three statements. The first one is obvious: essentially, one speaks about the link of the singularity $y^4 - x^{4d}$.

Lemma 3.9. *Assume that $R \gg 0$ is so large that the disk $\{|x| < R\}$ contains all singular fibers of C . Then one has $\beta_\infty = \Delta^d$, where $\Delta := \sigma_1 \sigma_2 \sigma_3 \sigma_1 \sigma_2 \sigma_1 \in \mathbb{B}_4$ is the Garside element. \triangleleft*

The following lemma is easily proved by considering the local normal forms of the singularities. (In the simplest case of a vertical tangent, the circumventing braids β are computed, *e.g.*, in [13]; the general case is completely similar.) For the statement, we extend the standard notation \mathbf{A}_m , $m \geq 1$, to \mathbf{A}_0 to designate a simple tangency of C and the fiber.

Lemma 3.10. *The braids β_j and μ_j about a singular fiber x_j of type \mathbf{A}_m , $m \geq 0$, depend only on m and the pair $(i, i+1)$ of indices of the branches that merge at the singular point. They are as shown in Figure 3. \triangleleft*

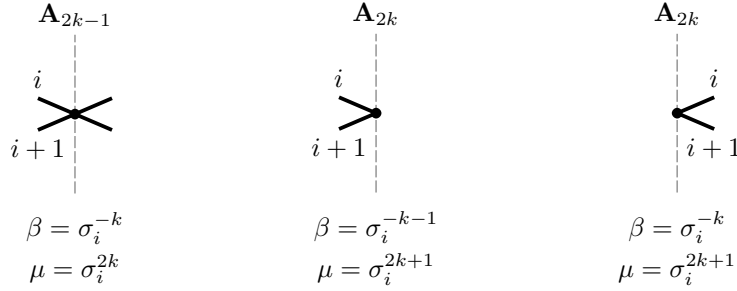


FIGURE 3. The braids β and μ

Remark 3.11. At a point of type \mathbf{A}_{2k-1} , it is *not* important whether the two branches of C at this point are real or complex conjugate. On the other hand, at a point of type \mathbf{A}_{2k} it *does* matter whether the number of real branches increases or decreases. If a fiber contains two double points, with indices $(1, 2)$ and $(3, 4)$, then the powers of σ_1 and σ_3 contributed to β or μ by each of the points are multiplied; since σ_1 and σ_3 commute, the order is not important.

The following statement is our principal technical tool, most important being Figure 4, right, describing the behaviour of the ‘invisible’ branches. (Note that the two dotted lines in the figure may cross; the permutation of the branches depends on the parity of the twist parameter t introduced in the statement.)

Proposition 3.12. *Let I be a real segment in the x -axis free of singular fibers of C . Then the monodromy γ over I is*

- *identity, if all four branches of C over I are real, and*
- *as shown in Figure 4 otherwise.*

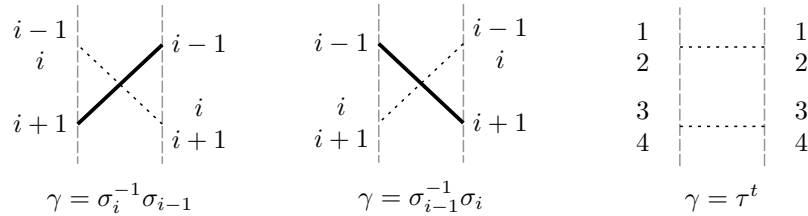


FIGURE 4. The braids γ

Here, $\tau := \sigma_2^{-1} \sigma_3 \sigma_1^{-1} \sigma_2$ and the twist parameter t in Figure 4, right is the number of roots $x' \in I$ of the coefficient $q(x)$, see (3.1), such that $b_1(x') > 0$, see (3.2), and

q changes sign at x' ; each root x' contributes $+1$ or -1 depending on whether q is increasing or decreasing at x' , respectively.

Proof. The only case that needs consideration, *viz.* that of four non-real branches, see Figure 4, right, is given by Remark 3.7. Indeed, the canonical basis in the fiber F_x° over $x \in I$ changes when the real parts of all four branches vanish, and this happens when $q(x) = 0$ and $b_1(x) > 0$. This change contributes $\tau^{\pm 1}$ to γ , and the sign ± 1 (the direction of rotation) depends on whether q increases or decreases. \square

Remark 3.13. A longer segment I with exactly two real branches of C over it can be divided into smaller pieces I_1, I_2, \dots , each containing a single crossing point as in Figure 4; then, the monodromy γ over I is the product of the contributions of each piece. In fact, as explained above, the precise position and number of crossings is irrelevant; what only matters is the final permutation between the endpoints of I . For example, to minimize the number of elementary pieces, one can always assume the branches, both bold and dotted, monotonous.

3.5. The Zariski–van Kampen theorem. We are interested in the fundamental group $\pi_1 := \pi_1(\Sigma_{\tilde{d}} \setminus (\tilde{C} \cup E))$, where $\tilde{C} \subset \Sigma_{\tilde{d}}$ is a real tetragonal curve, possibly improper, and $E \subset \Sigma_{\tilde{d}}$ is the exceptional section. To compute π_1 , we consider the proper model $C \subset \Sigma_d$, obtained from \tilde{C} by blowing up all points of intersection $\tilde{C} \cap E$ and blowing down the corresponding fibers. In addition to the braids β_i, μ_i , and γ_j introduced in §3.3, to each real singular fiber x_i of C we assign its local slope $\varkappa_i \in \pi(x_i^+)$, which depends on the type of the corresponding singular fiber of the original curve \tilde{C} . Roughly, consider a small analytic disk $\Phi \subset \Sigma_d$ transversal to the fiber F_{x_i} and disjoint from C and E , and a similar disk $\tilde{\Phi} \subset \Sigma_{\tilde{d}}$ with respect to \tilde{C} . Let $\tilde{\Phi}' \subset \Sigma_d$ be the image of $\tilde{\Phi}$, and assume that the boundaries $\partial\Phi$ and $\partial\tilde{\Phi}'$ have a common point in the fiber over x_i^+ . Then the loop $[\partial\tilde{\Phi}'] \cdot [\partial\Phi]^{-1}$ is homotopic to a certain class $\varkappa_i \in \pi(x_i^+)$, well defined up to a few moves irrelevant in the sequel. This class is the slope.

Roughly, the slope measures (in the form of the twisted monodromy, see the definitions prior to Theorem 3.16) the deviation of the braid monodromy of an improper curve \tilde{C} from that of its proper model C . Slopes appear in the relation at infinity as well, compensating for the fact that, near improper singular fibers, the curve intersects any section of $\Sigma_{\tilde{d}}$. Details and further properties are found in [7, §5.1.3]; in this paper, slopes are used in Theorem 3.16.

Remark 3.14. In all examples considered below, $\tilde{C} \subset \Sigma_{d-1}$ has a single improper fiber F , where \tilde{C} has a singular point of type $\tilde{\mathbf{A}}_m$, $m \geq 1$, maximally transversal to both E and F . If $F = \{x = 0\}$, such a curve \tilde{C} is given by a polynomial \tilde{f} of the form $\sum_{i=0}^4 y^i a_i(x)$ with $a_4(x) = x^2$ and $x \mid a_3(x)$, and the defining polynomial of its transform $C \subset \Sigma_d$ is $f_{\text{nr}}(x, y) := x^2 \tilde{f}(x, y/x)$. The corresponding singular fiber of C has a node \mathbf{A}_1 at $(0, 0)$ and another double point \mathbf{A}_{m-2} (assuming $m \geq 2$).

Thus, the only nontrivial example relevant in the sequel is the one described below. (By the very definition, at each singular fiber x_i proper for \tilde{C} the slope is $\varkappa_i = 1$.) A great deal of other examples of both computing the slopes and using them in the study of the fundamental group are found in [7].

Example 3.15. At the only improper fiber $x_i = 0$ described in Remark 3.14 the slope is the class of $\alpha_j \alpha_{j+1}$, where $(j, j+1)$ are the two branches merging at the

node, cf. [Figure 3](#). This fact can easily be seen using a local model. In a small neighborhood of $x = 0$, one can assume that \tilde{C} is given by $(y - a)(y - b) = 0$. Let $\tilde{\Phi} \subset \Sigma_{\tilde{d}}$ and $\Phi \subset \Sigma_d$ be the disk $\{y = c, |x| \leq 1\}$, $c \in \mathbb{R}$ and $c \gg |a|, |b|$. Then, the relevant part of C is the node $(y - ax)(y - bx) = 0$, and $\tilde{\Phi}$ projects onto the disk $\tilde{\Phi}' = \{y = cx, |x| \leq 1\}$, which meets Φ at $(1, c)$. Now, consider one full turn $x = \exp(2\pi it)$, $t \in [0, 1]$, and follow the point (x, cx) in $\partial\tilde{\Phi}'$: it describes the circle $y = c \exp(2\pi it)$ encompassing once the two points of the intersection $C \cap F_1^\circ$. The class $\alpha_j \alpha_{j+1}$ of this circle is the slope. Even more precisely, one should start with the constant path $[0, 1] \rightarrow (1, c)$ and homotope this path in $F_x^\circ \setminus C$, keeping one end in Φ and the other, in $\tilde{\Phi}'$. In the terminal position, the path is a loop again, and its class $\alpha_j \alpha_{j+1}$ is the slope.

Define the *twisted local monodromy* $\tilde{\mu}_i := \mu_i \cdot \text{inn } \varkappa_i$, where $\text{inn}: G \rightarrow \text{Aut } G$ is the homomorphism sending an element g of a group G to the inner automorphism $\text{inn } g: h \mapsto g^{-1}hg$. Thus, $\tilde{\mu}_i: \pi(x_i^+) \rightarrow \pi(x_i^+)$ is the map $\alpha \mapsto \varkappa_i^{-1}(\alpha \uparrow \mu_i)\varkappa_i$. In general, $\tilde{\mu}_i$ is not a braid. Take $x_0^+ = x_\infty^+$ for the reference fiber and consider the braids

$$\rho_i := \prod_{j=1}^i \gamma_{j-1} \beta_j: \pi(x_0^+) \rightarrow \pi(x_i^+), \quad i = 1, \dots, r+1 = \infty$$

(left to right product), the (global) slopes $\bar{\varkappa}_i := \varkappa_i \uparrow \rho_1^{-1} \in \pi(x_0^+)$, $i = 1, \dots, r$, and the twisted monodromy homomorphisms

$$\tilde{\mathfrak{m}}_i := \rho_i \tilde{\mu}_i \rho_i^{-1}: \pi(x_0^+) \rightarrow \pi(x_0^+), \quad i = 1, \dots, r.$$

The following theorem is essentially due to Zariski and van Kampen [16], and the particular case of improper curves in Hirzebruch surfaces, treated by means of the slopes, is considered in details in [7, §5.1.3]. Here, we state and outline the proof of a very special case of this approach, incorporating the (partial) computation of the braid monodromy of a real tetragonal curve in terms of its real part.

We use the following common convention: given an automorphism β of the free group $\langle \alpha_1, \dots, \alpha_4 \rangle$, the *braid relation* $\beta = \text{id}$ stands for the quadruple of relations $\alpha_j \uparrow \beta = \alpha_j$, $j = 1, \dots, 4$. Note that, since β is an automorphism, this is equivalent to the infinitely many relations $\alpha = \alpha \uparrow \beta$, $\alpha \in \langle \alpha_1, \dots, \alpha_4 \rangle$.

Theorem 3.16. *In the notation above, the inclusion of a the reference fiber induces an epimorphism $\pi(x_0^+) = \langle \alpha_1, \dots, \alpha_4 \rangle \twoheadrightarrow \pi_1$, and the relations $\tilde{\mathfrak{m}}_i = \text{id}$, $i = 1, \dots, r$, hold in π_1 . If the fiber $x = \infty$ is nonsingular and all non-real singular fibers are proper for \tilde{C} , then one also has the relations at infinity $\rho_\infty = \text{id}$ and $(\alpha_1 \dots \alpha_4)^d = \bar{\varkappa}_r \dots \bar{\varkappa}_1$. If, in addition, C has at most one pair of conjugate non-real singular fibers, then the relations listed define π_1 .*

Proof. The assertion is a restatement of the classical Zariski–van Kampen theorem modified for the case of improper curves, see [7, Theorem 5.50]. The relation at infinity $(\alpha_1 \dots \alpha_4)^d = \bar{\varkappa}_r \dots \bar{\varkappa}_1$ holds in π_1 whenever all slopes not accounted for, namely those at the non-real fibers, are known to be trivial. The automorphism $\rho_{r+1}: \pi(x_0^+) \rightarrow \pi(x_{r+1}^+) = \pi(x_0^+)$ is the monodromy along the ‘boundary’ of the upper half-plane $\text{Im } x > 0$, see [Figure 2](#), i.e., the product of the monodromies about all singular fibers in this half-plane; if the slopes at these fibers are all trivial, then $\rho_{r+1} = \text{id}$ in π_1 . Finally, if \tilde{C} has at most one pair of conjugate non-real singular fibers, then all but possibly one braid relations are present and hence they define the group, see [7, Lemma 5.59]. \square

4. THE COMPUTATION

4.1. The strategy. We start with a plane sextic $D \subset \mathbb{P}^2$ and choose homogeneous coordinates $(z_0 : z_1 : z_2)$ so that D has a singular point of type \mathbf{A}_m , $m \geq 3$, at $(0 : 0 : 1)$ tangent to the axis $\{z_1 = 0\}$. Then, in the affine coordinates $x := z_1/z_0$, $y := z_2/z_0$, the curve D is given by a polynomial \tilde{f} as in [Remark 3.14](#), and the same polynomial \tilde{f} defines a certain tetragonal curve $\tilde{C} \subset \Sigma_1$, *viz.* the proper transform of D under the blow-up of $(0 : 0 : 1)$. The common fundamental group

$$\pi_1 := \pi_1(\mathbb{P}^2 \setminus D) = \pi_1(\Sigma_1 \setminus (\tilde{C} \cup E))$$

is computed using [Theorem 3.16](#) applied to \tilde{C} and its transform $C \subset \Sigma_2$, with the only nontrivial slope $\varkappa = \alpha_1\alpha_2$ or $\alpha_3\alpha_4$ over $x = 0$ given by [Example 3.15](#). (Here, $E \subset \Sigma_1$ is the exceptional section, *i.q.* the exceptional divisor over the point $(0 : 0 : 1)$ blown up.) *A priori*, [Theorem 3.16](#) may only produce a certain group \mathfrak{g} that surjects onto π_1 rather than π_1 itself; however, in most cases this group \mathfrak{g} is ‘minimal expected’ (*cf.* [§4.4](#) below) and we do obtain π_1 .

The assumption that the fiber $x = \infty$ is nonsingular is not essential as long as the singularity over ∞ is taken into consideration: one can always move ∞ to a generic point by a real projective change of coordinates. To keep the defining equations as simple as possible, we assume such a change of coordinates implicitly. Furthermore, it is only the cyclic order of the singular fibers in the circle $\mathbb{P}_{\mathbb{R}}^1$ that matters, and sometimes we reorder the fibers by applying a cyclic permutation to their ‘natural’ indices. In other words, the braid $\beta_{\infty} = \Delta^2$ is in the center of \mathbb{B}_4 and, hence, it can be inserted at any place in the relation $\gamma_0\beta_1\gamma_1 \dots \gamma_r\beta_{\infty} = \text{id}$.

To compute the braids, we outline the real (bold lines) and imaginary (dotted lines) branches of C in the figures. Recall that it is only the mutual position of the real branches and their intersection indices with the imaginary ones that matters, see [Remark 3.13](#). The ‘special’ node that contributes the only non-trivial slope (the blow-up center in the passage from C to \tilde{C} , see [Remark 3.14](#)) is marked with a white dot; the other singular points of C (including those of type \mathbf{A}_0) are marked with black dots. The shape of the curve can mostly be recovered using [Remarks 3.7](#) and [3.8](#); however, it is usually easier to determine the mutual position of the roots directly *via* [Maple](#). The braids β_i , μ_i , and γ_j are computed from the figures as explained in [§3.4](#).

Warning 4.1. The polynomial f_{nr} given by [Remark 3.14](#) is used to determine the slope and mutual position of the two singular points over $x = 0$: the ‘special’ node is always at $(0, 0)$. For all other applications, *e.g.*, for [Proposition 3.12](#), this polynomial should be converted to the reduced form [\(3.1\)](#).

4.2. Relations. Recall that a braid relation $\tilde{\mathbf{m}}_i = \text{id}$ stands for a quadruple of relations $\alpha_j \uparrow \tilde{\mathbf{m}}_i = \alpha_j$, $j = 1, \dots, 4$. Alternatively, this can be regarded as an infinite sequence of relations $\alpha \uparrow \tilde{\mathbf{m}}_i = \alpha$, $\alpha \in \mathbb{F}_4$, or, equivalently, as a quadruple of relations $\alpha'_j \uparrow \tilde{\mathbf{m}}_i = \alpha'_j$, $j = 1, \dots, 4$, where $\alpha'_1, \dots, \alpha'_4$ is *any* basis for \mathbb{F}_4 . For this reason, in the computation below we start with the braid relations $\alpha'_j \uparrow \tilde{\mu}_i = \alpha'_j$ in the canonical basis over x_i^+ and translate them to x_0^+ *via* ρ_i^{-1} . In the most common case $\tilde{\mu}_i = \sigma_r^p$, $r = 1, 2, 3$, $p \in \mathbb{Z}$, the whole quadruple is equivalent to the single relation $\{\alpha'_r, \alpha'_{r+1}\}_p = 1$, where

$$\{\alpha, \beta\}_{2k} := (\alpha\beta)^k(\beta\alpha)^{-k}, \quad \{\alpha, \beta\}_{2k+1} := (\alpha\beta)^k\alpha(\alpha\beta)^{-k}\beta^{-1}.$$

Remark 4.2. The braid relations about the fiber $x_k = 0$ with the only nontrivial slope, see [Example 3.15](#), can also be presimplified. Let $\alpha'_1, \dots, \alpha'_4$ be the canonical basis in x_k^+ . If $\varkappa_k = \alpha'_1 \alpha'_2$ and $\mu_k = \sigma_1^2 \sigma_3^p$, the braid relations $\tilde{\mu}_k = \text{id}$ and relation at infinity $(\alpha'_1 \dots \alpha'_4)^2 = \varkappa_k$ together are equivalent to

$$\alpha'_1 \alpha'_2 (\alpha'_3 \alpha'_4)^2 = \{\alpha'_3, \alpha'_4\}_{p+4} = 1.$$

Similarly, if $\varkappa_k = \alpha'_3 \alpha'_4$ and $\mu_k = \sigma_1^p \sigma_3^2$, we obtain

$$(\alpha'_1 \alpha'_2)^2 \alpha'_3 \alpha'_4 = \{\alpha'_1, \alpha'_2\}_{p+4} = 1.$$

Certainly, these relations should be translated back to x_0^+ via ρ_k^{-1} . Note, though, that we do not use this simplification in the sequel.

Remark 4.3. In some cases, simpler relations are obtained if another point x_i^+ , $i > 0$, is taken for the reference fiber. To do so, one merely replaces the braids ρ_j , $j = 1, \dots, r+1 = \infty$, with $\rho'_j := \rho_i^{-1} \rho_j$.

All computations below were performed using [GAP](#) [9], with the help of the simple braid manipulation routines contained in [7]. The [GAP](#) code can be downloaded from <http://www.fen.bilkent.edu.tr/~degt/papers/papers.htm>. The processing is almost fully automated, the input being the braids β_i, μ_i, γ_j and the only nontrivial slope $\varkappa_k = \alpha_1 \alpha_2$ or $\alpha_3 \alpha_4$, which are read off from the diagrams depicting the curves.

4.3. The set of singularities $3\mathbf{A}_6 \oplus \mathbf{A}_1$, [line 37](#). Any sextic with this set of singularities is \mathbb{D}_{14} -special, see [4], and, according to [8], any \mathbb{D}_{14} -special sextic can be given by an equation of the form

$$\begin{aligned} & 2t(t^3 - 1)(z_0^4 z_1 z_2 + z_1^4 z_2 z_0 + z_2^4 z_0 z_1) \\ & + (t^3 - 1)(z_0^4 z_1^2 + z_1^4 z_2^2 + z_2^4 z_0^2) + t^2(t^3 - 1)(z_0^4 z_2^2 + z_1^4 z_0^2 + z_2^4 z_1^2) \\ & + 2t(t^3 + 1)(z_0^3 z_1^3 + z_1^3 z_2^3 + z_2^3 z_0^3) + 4t^2(t^3 + 2)(z_0^3 z_1^2 z_2 + z_1^3 z_2^2 z_0 + z_2^3 z_0^2 z_1) \\ & + 2(t^6 + 4t^3 + 1)(z_0^3 z_1 z_2^2 + z_1^3 z_2 z_0^2 + z_2^3 z_0 z_1^2) + t(t^6 + 13t^3 + 10)z_0^2 z_1^2 z_2^2, \end{aligned}$$

$t^3 \neq 1$. The set of singularities of this curve is $3\mathbf{A}_6 \oplus \mathbf{A}_1$ if and only if $t^3 = -27$; we use the real value $t = -3$. After the substitution $z_0 = 1$, $z_1 = x + \frac{1}{3}$, and $z_2 = y/x$ the equation is brought to the form considered in [Remark 3.14](#). Up to a positive factor, the discriminant (3.4) with respect to y is

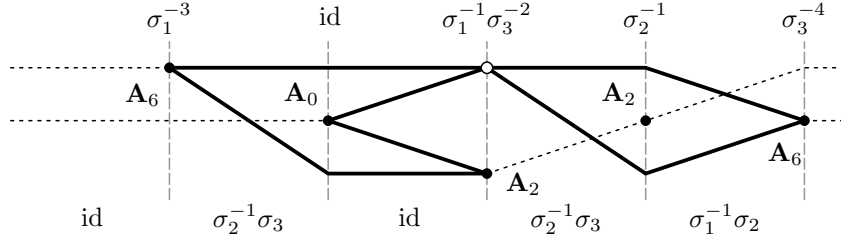
$$-x^5(27x^3 - 648x^2 + 6363x + 7)(3x - 2)^2(3x + 1)^7,$$

which has real roots

$$x_1 = -\frac{1}{3}, \quad x_2 \approx -0.001, \quad x_3 = 0, \quad x_4 = \frac{2}{3}, \quad x_5 = \infty$$

and two simple imaginary roots. Hence, [Theorem 3.16](#) does compute the group.

The only root of q on the real segment $[-\infty, x_1]$ is $x' \approx -3.48$, and $b_1(x') < 0$; hence, one has $\gamma_0 = \text{id}$, see [Proposition 3.12](#). The other braids β_i, γ_j are easily found from [Figure 5](#), and, using [Theorem 3.16](#) and [GAP](#), we obtain a group of order 42. This concludes the proof of [Theorem 1.1](#). \square

FIGURE 5. The set of singularities $3\mathbf{A}_6 \oplus \mathbf{A}_1$, [line 37](#)

4.4. Sextics of torus type. All maximal, in the sense of degeneration, sextics of torus type are described in [12], where a sextic D is represented by a pair of polynomials $f_2(x, y)$, $f_3(x, y)$ of degree 2 and 3, respectively, so that the defining polynomial of D is $f_{\text{tor}} := f_2^3 + f_3^2$. (Below, these equations are cited in a slightly simplified form: I tried to clear the denominators by linear changes of variables and appropriate coefficients.) Each curve (at least, each of those considered below) has a type \mathbf{A}_m , $m \geq 3$, singularity at $(0, 0)$ tangent to the y -axis. Hence, we start with the substitution $\tilde{f}(x, y) := y^6 f_{\text{tor}}(x/y, 1/y)$ to obtain a polynomial \tilde{f} as in [Remark 3.14](#); then we proceed as in [§4.1](#).

To identify the group \mathfrak{g} given by [Theorem 3.16](#) as Γ , we use the following GAP code, which was suggested to me by E. Artal:

```
(4.4) P := PresentationNormalClosure(g, Subgroup(g, a));
      SimplifyPresentation(P);
```

here, \mathbf{a} is an appropriate ratio $\alpha_i \alpha_j^{-1}$ which normally generates the commutant of \mathfrak{g} . If the resulting presentation has two generators and no relations, we conclude that $\mathfrak{g} = \pi_1 = \Gamma$, *even when the statement of [Theorem 3.16](#) does not guarantee a complete set of relations*. Indeed, *a priori* we have epimorphisms $\mathfrak{g} \twoheadrightarrow \pi_1 \twoheadrightarrow \Gamma$ (the latter follows from the fact that the curve is assumed to be of torus type), which induce epimorphisms $[\mathfrak{g}, \mathfrak{g}] \twoheadrightarrow [\pi_1, \pi_1] \twoheadrightarrow [\Gamma, \Gamma] = \mathbb{F}_2$ of the commutants. If $[\mathfrak{g}, \mathfrak{g}] = \mathbb{F}_2$, both these epimorphisms are isomorphisms (since \mathbb{F}_2 is Hopfian) and the 5-lemma implies that $\mathfrak{g} \twoheadrightarrow \pi_1 \twoheadrightarrow \Gamma$ are also isomorphisms.

In fact, in some cases (*e.g.*, in [§4.5](#) and [§4.6](#)), the call `SimplifiedFpGroup(g)` returns a recognizable presentation of Γ .

4.5. The set of singularities $(\mathbf{A}_{14} \oplus \mathbf{A}_2) \oplus \mathbf{A}_3$, [line 8](#). The curve in question is `nt139` in [12]:

$$f_2 = 80(-36y^2 + 120xy - 82x^2 + 2x),$$

$$f_3 = 100(-1512y^3 + 7794y^2x - 18y^2 - 11664yx^2 + 144xy + 5313x^3 - 194x^2 + x).$$

Up to a positive coefficient, the discriminant of f_{nr} is

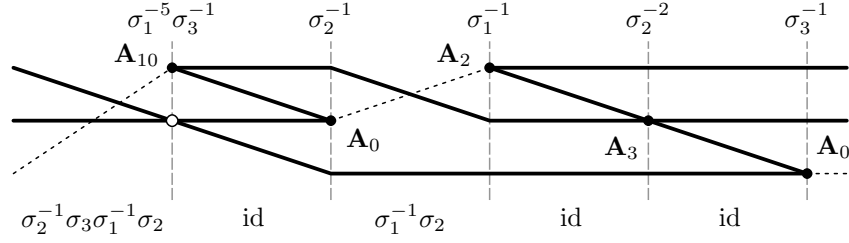
$$x^{13}(5120x^4 + 36864x^3 + 3456x^2 - 2160x - 405)(x - 1)^3.$$

It has five real roots, which we reorder cyclically as follows:

$$x_1 = 0, \quad x_2 \approx 0.27, \quad x_3 = 1, \quad x_4 = \infty, \quad x_5 \approx -7.1.$$

Besides, there are two conjugate imaginary singular fibers, which are of type \mathbf{A}_0 .

The curve is depicted in [Figure 6](#), from which all braids β_i , γ_j are easily found. Taking x_0^+ for the reference fiber and using $\mathbf{a} = \alpha_1 \alpha_2^{-1}$ in (4.4), we obtain $\pi_1 = \Gamma$.

FIGURE 6. The set of singularities $(\mathbf{A}_{14} \oplus \mathbf{A}_2) \oplus \mathbf{A}_3$, line 8

4.6. **The set of singularities $(\mathbf{A}_{14} \oplus \mathbf{A}_2) \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$, line 9.** The curve is nt142 in [12]:

$$f_2 = -45y^2 - 240yx - 106x^2 + 90x,$$

$$f_3 = 1025y^3 + 6045y^2x - 375y^2 + 5490yx^2 - 4050yx + 1354x^3 - 2040x^2 + 750x.$$

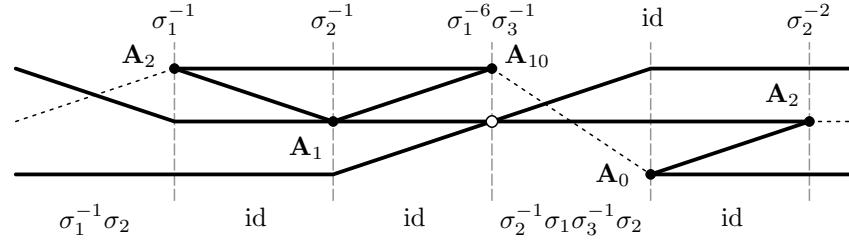
Up to a positive coefficient, the discriminant of f_{nr} is

$$x^{13}(8x^3 - 10720x^2 + 14250x - 5625)(x+1)^2(14x+15)^3,$$

and all its roots are real:

$$x_1 = -\frac{15}{14}, \quad x_2 = -1, \quad x_3 = 0, \quad x_4 \approx 1338, \quad x_5 = \infty.$$

The braids β_i, γ_j are found from Figure 7 and, using x_0^+ as the reference fiber and

FIGURE 7. The set of singularities $(\mathbf{A}_{14} \oplus \mathbf{A}_2) \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$, line 9

$\mathbf{a} = \alpha_1 \alpha_2^{-1}$ in (4.4), we conclude that $\pi_1 = \Gamma$.

4.7. **The set of singularities $(\mathbf{A}_{11} \oplus 2\mathbf{A}_2) \oplus \mathbf{A}_4$, line 17.** This is nt118 in [12]:

$$f_2 = \frac{1}{5}(-3456y^2 + 1200yx - 3005x^2 + 240x),$$

$$f_3 = \frac{1}{5}(-89856y^3 + 130464y^2x - 6912y^2 - 112680yx^2 + 8640yx + 91345x^3 - 13320x^2 + 480x).$$

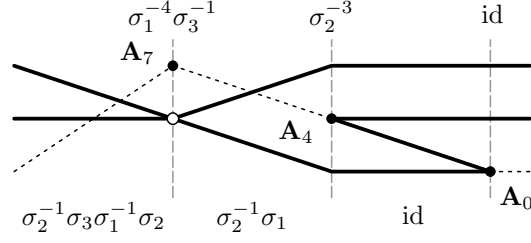
Up to a positive coefficient, the discriminant of f_{nr} is

$$-x^{10}(25x^3 + 290x^2 + 360x + 162)(35x^2 - 384x + 1152)^3.$$

It has three real roots, which we reorder cyclically as follows:

$$x_1 = 0, \quad x_2 = \infty, \quad x_3 \approx -10.26.$$

In addition, there are two pairs of complex conjugate singular fibers, of types \mathbf{A}_2

FIGURE 8. The set of singularities $(\mathbf{A}_{11} \oplus 2\mathbf{A}_2) \oplus \mathbf{A}_4$, line 17

and \mathbf{A}_0 . Thus, *a priori* Theorem 3.16 only gives us a certain epimorphism $\mathfrak{g} \twoheadrightarrow \pi_1$. However, using $\mathfrak{a} = \alpha_1\alpha_2^{-1}$ in (4.4), we conclude that $\mathfrak{g} = \pi_1 = \Gamma$. (All braids are found from Figure 8 and the reference fiber is x_1^+ , see Remark 4.3.)

4.8. **The set of singularities $(\mathbf{A}_8 \oplus 3\mathbf{A}_2) \oplus \mathbf{A}_4 \oplus \mathbf{A}_1$, line 33.** This curve is nt83 in [12]:

$$(4.5) \quad \begin{aligned} f_2 &= -565y^2 - 14yx + 176y - 5x^2 + 104x - 16, \\ f_3 &= 13321y^3 + 3135y^2x - 6294y^2 + 207yx^2 - 3516yx + 1056y + \\ &\quad 25x^3 - 558x^2 + 624x - 64. \end{aligned}$$

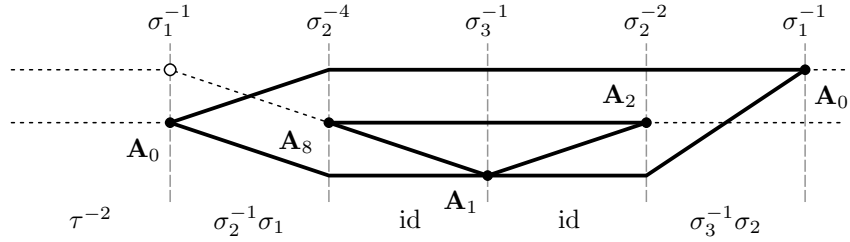
Up to a positive coefficient, the discriminant of f_{nr} is

$$x^3(x+3)(x+9)^2(11915x^3 + 96579x^2 - 14823x + 729)^3(x-9)^9.$$

It has five real roots, which we reorder cyclically as follows:

$$x_1 = 0, \quad x_2 = 9, \quad x_3 = -9, \quad x_4 \approx -8.26, \quad x_5 = -3.$$

We conclude that the curve has only two non-real singular fibers, which are cusps. Hence, Theorem 3.16 gives us a complete presentation of π_1 .

FIGURE 9. The set of singularities $(\mathbf{A}_8 \oplus 3\mathbf{A}_2) \oplus \mathbf{A}_4 \oplus \mathbf{A}_1$, line 33, projected from \mathbf{A}_4

In the interval (x_5, x_1) , where f has four imaginary branches, q has four roots

$$x'_1 \approx -2.93, \quad x'_2 = -1.92, \quad x'_3 \approx -0.79, \quad x'_4 \approx -0.14,$$

with b_1 negative at x'_1, x'_3 and positive at x'_2, x'_4 ; at the latter two points one also has $q' < 0$. Hence, $\gamma_0 = \tau^{-2}$, see Proposition 3.12. All other braids are easily found from Figure 9.

Remark 4.6. For a further simplification, observe that the braid ρ_∞ appearing in [Theorem 3.16](#) equals

$$\sigma_2^{-1}\sigma_1\sigma_3^{-1}\sigma_1\sigma_3^{-1}\sigma_2 \cdot \sigma_1^{-1} \cdot \sigma_2^{-1}\sigma_1 \cdot \sigma_2^{-4} \cdot \sigma_3^{-1} \cdot \sigma_2^{-2} \cdot \sigma_3^{-1}\sigma_2 \cdot \sigma_1^{-1} \cdot (\sigma_3\sigma_1\sigma_2)^4,$$

and one can check that $\rho_\infty = \rho_{\text{im}}^{-1}\sigma_1^3\rho_{\text{im}}$, where $\rho_{\text{im}} := \sigma_2\sigma_1^{-1}\sigma_3^2\sigma_2$. (Note that ρ_∞ represents the monodromy about a single imaginary cusp of the curve; hence, it is expected to be conjugate to σ_1^3 .) Thus, we can replace the quadruple of relations $\rho_\infty = \text{id}$ with a single relation $\{\alpha_1, \alpha_2\}_3 \uparrow \rho_{\text{im}} = 1$, cf. [§4.2](#).

Now, taking x_3^+ for the reference fiber, see [Remark 4.3](#), using [Remark 4.6](#), and applying `SimplifiedFpGroup(g)`, we arrive at [\(1.3\)](#). This presentation has three generators and four relations of total length 48. Together with the previous sections, this concludes the proof of [Theorem 1.2](#). \square

Remark 4.7. The Alexander module of the group π_1 considered in this section is $\mathbb{Z}[t, t^{-1}]/(t^2 - t + 1)$, and the finite quotients π_1/α_2^p , $p = 2, 3, 4$, are isomorphic to the similar quotients of Γ . My laptop failed to compute the order of π_1/α_2^5 .

Remark 4.8. In [\(4.5\)](#), the singular point at the origin is of type \mathbf{A}_4 . One can start with a change of variables $x \mapsto y + 9$, $y \mapsto x + 1$ and resolve the type \mathbf{A}_8

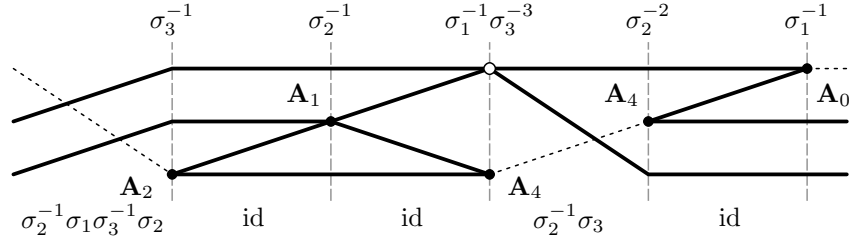


FIGURE 10. The set of singularities $(\mathbf{A}_8 \oplus 3\mathbf{A}_2) \oplus \mathbf{A}_4 \oplus \mathbf{A}_1$, [line 33](#), projected from \mathbf{A}_8

point instead. The tetragonal model is depicted in [Figure 10](#), and the computation becomes slightly simpler, but the resulting presentation is of the same complexity, even with the additional observation that $\rho_\infty = \rho_{\text{im}}^{-1}\sigma_1^3\rho_{\text{im}}$, where $\rho_{\text{im}} := \sigma_2\sigma_1^{-1}\sigma_3\sigma_2$, cf. [Remark 4.6](#).

4.9. Proof of Proposition 1.4. For the sets of singularities $(\mathbf{A}_{14} \oplus \mathbf{A}_2) \oplus \mathbf{A}_3$, [line 8](#), $(\mathbf{A}_{14} \oplus \mathbf{A}_2) \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$, [line 9](#), and $(\mathbf{A}_{11} \oplus 2\mathbf{A}_2) \oplus \mathbf{A}_4$, [line 17](#), the statement is an immediate consequence of [7, Theorem 7.48]. For $3\mathbf{A}_6 \oplus \mathbf{A}_1$, [line 37](#), the only proper quotient of the commutant $[\pi_1, \pi_1] = \mathbb{Z}_7$ is trivial; hence, the group π_1' of any perturbation D' is either abelian, $\pi_1' = \mathbb{Z}_6$, or isomorphic to π_1 , the latter being the case if and only if D' is \mathbb{D}_{14} -special, see [4].

For the remaining set of singularities $(\mathbf{A}_8 \oplus 3\mathbf{A}_2) \oplus \mathbf{A}_4 \oplus \mathbf{A}_1$, [line 33](#), we proceed as follows. Any proper perturbation factors through a maximal one, where a single singular point P of type \mathbf{A}_m splits into two points $\mathbf{A}_{m'}$, $\mathbf{A}_{m''}$, so that $m' + m'' = m - 1$. Assume that $P \neq (0 : 0 : 1)$, see [§4.1](#). Then this point corresponds to a certain singular fiber x_i of the tetragonal model C and gives rise to a braid relation $\{\alpha_k, \alpha_{k+1}\}_{m+1} \uparrow \rho_i^{-1} = 1$, see [§4.2](#). For the new curve D' , this relation changes to $\{\alpha_k, \alpha_{k+1}\}_s \uparrow \rho_i^{-1} = 1$, where $s := \text{g.c.d.}(m' + 1, m'' + 1)$.

For any perturbation of any point P , we have $s = 3$ if P is of type \mathbf{A}_8 or \mathbf{A}_2 and the result is still of torus type, and $s = 1$ otherwise. Now, the statement is easily proved by repeating the computation with the braid $\mu_i = \sigma_k^{m+1}$ replaced with σ_k^3 . (If it is the type \mathbf{A}_4 point that is perturbed, one can use the alternative tetragonal model given by Remark 4.8.) \square

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