# SLOPES AND CONCORDANCE OF LINKS

ALEX DEGTYAREV, VINCENT FLORENS, AND ANA G. LECUONA

Abstract. The slope is an isotopy invariant of colored links with a distinguished component, initially introduced by the authors to describe an extra correction term in the computation of the signature of the splice. It appeared to be closely related to several classical invariants, such as the Conway potential function or the Kojima $\eta$-function (defined for two-components links). In this paper, we prove that the slope is invariant under colored concordance of links. Besides, we present a formula to compute the slope in terms of C-complexes and generalized Seifert forms.

## 1. Introduction

The slope is an isotopy invariant defined for so-called $(1, \mu)$-colored links $K \cup L$ (with a distinguished component $K$ given color 0) in rational homology spheres. It is closely related to several classical invariants [DFL17, DFL22, DFL21], such as the Conway potential and Kojima-Yamasaki $\eta$-function (defined for two-components links [KY79, Jin88, Coc85]). To certain $\mathbb{C}^\times$-valued characters of the group $\pi_1(S \subset L)$, *viz.* those trivial on $[K]$, see (2.2), the slope associates a complex number (possibly infinite). The torus of characters preserving the coloring is naturally identified with the complex torus $(\mathbb{C}^\times)^\mu$, and the slope is a function on (a Zariski open subset of) the variety $\mathcal{A}(K/L) \subset (\mathbb{C}^\times)^\mu$ of admissible characters. This function is rational away from a certain singular locus determined by the Alexander module of $K \cup L$; however, in general, the *values* of the slope are not determined by the Alexander module.

The aim of this paper is to show that the slope is invariant under colored concordance of links, see Theorem 3.2, and to present a method to compute the slope in terms of the Seifert forms of the colored link $L$ with an extra piece of data, see Theorem 4.3. In the case of algebraically split links of two components, the invariance of the slope under colored concordance was known for certain values, *viz.* those where it: coincides with the $\eta$-function [DFL22, Corollary 3.24]. In this paper we show that, outside a certain subset of $(\mathbb{C}^\times)^\mu$, the *Knottennullstellen* [NP17, CNT17], concordant links have the same slope. More generally, for algebraically split links with an arbitrary number of components, our result implies that a certain quotient of the Conway functions of $K \cup L$ and $L$ is invariant under colored concordance of $K \cup L$ (see Corollary 3.4), whereas the Conway functions themselves are *not* concordance invariants (see [Kaw96]).

One can compute the slope directly from the definition using the Fox calculus; this is explained in [DFL22, Section 3.2]. While allowing for easy computer assisted computations, this approach is not particularly useful when dealing with families of examples. In

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certain cases, the slope can also be computed as a quotient of the Conway polynomials (see [DFL22, Theorem 3.1]), but this formula is inconclusive at the common roots of the numerator and denominator (l'Hôpital's rule does not work); in particular, it leaves wide open the most interesting case where both polynomials vanish identically. In this paper we suggest yet another method of computing the slope, using C-complexes. These were introduced by Cooper [Coo82] and extended, in very recent years, by different groups to compute many link invariants ([Cim04, CF08, CFT18, Mer21] among others) and to study their properties ([DR17, DMO21, AADG20] among others).

The computation of the slope using C-complexes is particularly powerful when dealing with families of examples as in [DFL22, Example 3.28] and [DFL21, Example 5.5]. For the moment, our formula only works in the special case of $K$ algebraically unlinked from each monochrome component $L_i$. For an algebraically split two component link, the C-complex used in the computation is merely a Seifert surface.

The paper is organized as follows. In Section 2 we recall the construction and the basic properties of the slope. Section 3 is devoted to the proof of the concordance invariance. In Section 4 the computation of the slope in terms of (generalized) Seifert forms is given, and the main formula is proved in Section 5.

## 2. Slopes

A $\mu$-*colored link* is an oriented link $L$ in $S^3$ equipped with a surjective map $\pi_0(L) \twoheadrightarrow \{1, \ldots, \mu\}$, called *coloring*. The union of the components of $L$ given the same color $i = 1, \ldots, \mu$ is denoted by $L_i$. Each link has a canonical *maximal coloring*, where each component is given a separate color. In this special case, each $L_i$ is a knot.

We denote by $X := S^3 \smallsetminus T_L$ the complement of a small open tubular neighborhood of $L$. The group $H_1(X)$ is free abelian, generated by the classes $m_C$ of the meridians of the components $C \subset L$. By convention, $m_C$ is oriented so that $m_C \circ \ell_C = 1$ in $\partial T_C$, where $\ell_C$ is a longitude and the orientation on $\partial T_C$ is that induced from $X$. The coloring induces an epimorphism

$$\varphi\colon \pi_1(X) \twoheadrightarrow H := \bigoplus_{i=1}^{\mu} \mathbb{Z} t_i$$

sending $m_C$ to $t_i$ whenever $C \subset L_i$. A multiplicative character $\omega\colon \pi_1(X) \to \mathbb{C}^\times$ is determined by its values on the meridians, and the torus of characters preserving the coloring (*i.e.*, those that factor through $\varphi$) is naturally identified with the complex torus $(\mathbb{C}^\times)^\mu$. Through this identification, we set $\omega_i := \omega(\varphi(t_i))$ and, with a certain abuse of the language, speak about a character $\omega = (\omega_1, \ldots, \omega_\mu)$. We define

$$\omega^{-1} := (\omega_1^{-1}, \ldots, \omega_\mu^{-1}), \qquad \bar\omega := (\bar\omega_1, \ldots, \bar\omega_\mu), \qquad \omega^* := \bar\omega^{-1}.$$

A character $\omega$ is called *unitary* if $\omega^* = \omega$, *i.e.*, $|\omega_i| = 1$ for all $i = 1, \ldots, \mu$. Unitary characters constitute a torus $(S^1)^\mu \subset (\mathbb{C}^\times)^\mu$.

Given a topological space $X$ and a multiplicative character $\omega\colon \pi_1(X) \to \mathbb{C}^\times$, we denote by $H_*(X; \mathbb{C}(\omega))$ the homology of $X$ with coefficient in the local system $\mathbb{C}(\omega)$ twisted by $\omega$. See [DFL22, Section 2] for more details.

In this paper, we consider mainly colored links with a distinguished component. They are $(1,\mu)$-*colored links*, defined as $(1+\mu)$-colored links of the form

$$K \cup L = K \cup L_1 \cup \ldots \cup L_\mu,$$

where the *knot* $K$ is the only component given the distinguished color 0. The *linking vector* of a $(1,\mu)$ colored link is $\overline{\ell k}(K,L) := (\lambda_1,\ldots,\lambda_\mu) \in \mathbb{Z}^\mu$, where $\lambda_i := \ell k(K,L_i)$.

**Definition 2.1.** A character $\omega\colon \pi_1(X) \longrightarrow \mathbb{C}^\times$ on a $(1,\mu)$-colored link $K \cup L$ is called *admissible* if $\omega([K]) = 1$; it is called *non-vanishing* if $\omega_i \neq 1$ for all $i = 1,\ldots,\mu$.

The variety of admissible characters is denoted $\mathcal{A}(K/L)$, and $\mathcal{A}^\circ(K/L) \subset \mathcal{A}(K/L)$ is the subvariety of admissible non-vanishing characters. Letting $\lambda := \overline{\ell k}(K,L)$, we have

$$\mathcal{A}(K/L) = \left\{\omega \in (\mathbb{C}^\times)^\mu \;\middle|\; \omega^\lambda = 1\right\}, \quad \mathcal{A}^\circ(K/L) = \mathcal{A}(K/L) \cap (\mathbb{C}^\times \smallsetminus 1)^\mu, \tag{2.2}$$

where $\omega^\lambda := \prod \omega_i^{\lambda_i}$. In particular, if $\lambda = 0$, then $\mathcal{A}^\circ(K/L) = (\mathbb{C}^\times \smallsetminus 1)^\mu$.

Let $X_K = S^3 \smallsetminus T_{K\cup L}$ be the complement of an open tubular neighborhood of $K \cup L$. We abbreviate $m := m_K$ and $\ell := \ell_K$, where $\ell_K$ is the *preferred* (*viz.* unlinked with $K$) longitude, also called *Seifert longitude*.

**Remark 2.3.** Any character $\omega \in (\mathbb{C}^\times)^\mu$ extends to a natural character $\pi_1(X_K) \to \mathbb{C}^\times$ sending $m$ to 1; for short, this extension is also denoted $\omega$. In this language, the original character $\omega$ is admissible if and only if $\omega(\ell) = 1$.

We denote by $\partial_K X_K = \partial T_K$ the intersection of $\partial X_K$ with the closure of $T_K$ and consider the inclusion

$$i\colon \partial_K X_K \hookrightarrow \partial X_K \hookrightarrow X_K.$$

If $\omega \in \mathcal{A}^\circ(K/L)$, the induced homomorphism

$$i_*\colon H_1(\partial_K X_K;\mathbb{C}(\omega)) \xrightarrow{\simeq} H_1(\partial X_K;\mathbb{C}(\omega)) \longrightarrow H_1(X_K;\mathbb{C}(\omega)) \tag{2.4}$$

can be regarded as that induced by the inclusion $\partial X_K \hookrightarrow X_K$ of the boundary and the space $H_1(\partial_K X_K;\mathbb{C}(\omega)) \simeq \mathbb{C}^2$ is generated by the meridian $m$ and Seifert longitude $\ell$.

**Definition 2.5** (see [DFL22])**.** If $\operatorname{Ker} i_*$ in (2.4) has dimension one, it is generated by a single vector $am + b\ell$ for some $[a:b] \in \mathbb{P}^1(\mathbb{C})$, and the *slope* of $K \cup L$ at $\omega \in \mathcal{A}^\circ(K/L)$ is defined as the quotient

$$(K/L)(\omega) := -\frac{a}{b} \in \mathbb{C} \cup \infty.$$

This notion is extended to all characters $\omega \in \mathcal{A}(K/L)$ by "patching" the components $L_i$ on which $\omega_i = 1$. (This operation results in patching with solid tori the corresponding boundary components of the manifold $X := S^3 \smallsetminus T_L$.)

**Proposition 2.6** (see [DFL22])**.** *The slope at a character $\omega \in \mathcal{A}^\circ(K/L)$ is well defined if and only if the two inclusion homomorphisms $H_1(K;\mathbb{C}(\zeta)) \to H_1(S^3 \smallsetminus L;\mathbb{C}(\zeta))$, $\zeta = \omega$ or $\omega^*$, are either both trivial or both nontrivial. The slope is finite, $(K/L)(\omega) \in \mathbb{C}$, if and only if both homomorphisms are trivial.* ◁

Note also (see [DFL22, Section 2.4] for details) that the slope is always defined on a *unitary* character $\omega \in (S^1)^\mu$: in this case, by twisted Poincaré duality, $\operatorname{Ker} i_*$ is a Lagrangian subspace of

$$H_1(\partial_K X_K; \mathbb{C}(\omega)) = H_1(\partial X_K; \mathbb{C}(\omega)),$$

*cf.* (2.4), with respect to the twisted intersection form and, hence, $\dim \operatorname{Ker} i_* = 1$.

Recall (see, *e.g.*, [Lib01]) that the *characteristic varieties* associated with a $\mu$-colored link $L$ are the jump loci

$$\mathcal{V}_r(L) := \bigl\{\omega \in (\mathbb{C}^\times)^\mu \bigm| \dim H_1(X; \mathbb{C}(\omega)) \geqslant r\bigr\}, \quad r \geqslant 0.$$

They are indeed nested algebraic subvarieties:

A: We have never introduced $\Delta_L$. We decide it's "classic enough"? Who are we to explain $\Delta$ to knot theorists? :)

$$(2.7) \qquad (\mathbb{C}^\times)^\mu = \mathcal{V}_0 \supset \mathcal{V}_1 \supset \mathcal{V}_2 \supset \ldots, \qquad \text{with } \mathcal{V}_1(L) = \bigl\{\omega \bigm| \Delta_L(\omega) = 0\bigr\}.$$

The first *proper* characteristic variety, *i.e.*, the first member $\mathcal{V}_r$ of the sequence (2.7) such that $\mathcal{V}_r \neq (\mathbb{C}^\times)^\mu$, is denoted by $\mathcal{V}_{\max} := \mathcal{V}_{\max}(L)$. This variety depends on $L$ only and, if $\lambda := \overline{\ell k}(K, L) = 0$, it is a proper algebraic subvariety of the torus $\mathcal{A}(K/L) = (\mathbb{C}^\times)^\mu$ of admissible characters.

Is it a divisor by any chance?

good idea to explain the relation to $K$; extended a little

**Remark 2.8.** If $\lambda := \overline{\ell k}(K, L) \neq 0$, the situation is slightly more involved. Let $\lambda = n\lambda'$, where $\lambda' \in \mathbb{Z}^\mu$ is a primitive vector. In view of (2.2), the variety $\mathcal{A}(K/L)$ of admissible characters (depending on $\lambda$ only) splits over $\mathbb{Q}$ into irreducible components

$$\mathcal{A}_d := \bigl\{\Phi_d(\omega^{\lambda'}) = 0\bigr\}, \quad d \mid n,$$

where $\Phi_d$ stands for the cyclotomic polynomial, and we should speak about a separate *first proper characteristic variety* $\mathcal{V}^{\lambda,d}_{\max}(L) \subsetneq \mathcal{A}_d$ for each component $\mathcal{A}_d$. In general, $\mathcal{V}^{\lambda,d}_{\max}(L) \neq \mathcal{V}_{\max}(L) \cap \mathcal{A}_d$ as $\mathcal{V}_{\max}(L)$ may contain $\mathcal{A}_d$. To keep the notation uniform, we occasionally extend it to the case $\lambda = 0$ *via* $\mathcal{A}_0 := \mathcal{A}(K/L)$ and $\mathcal{V}^{0,0}_{\max}(L) := \mathcal{V}_{\max}(L)$.

notation fixed throughout

**Theorem 2.9** (see [DFL22, Theorems 3.19 and 3.21])**.** *Let $\lambda := \overline{\ell k}(K, L)$. For each rational component $\mathcal{A}_d \subset \mathcal{A}(K/L)$, the slope restricts to a rational function, possibly identical $\infty$, on the complement $\mathcal{A}^\circ_d \smallsetminus \mathcal{V}^{\lambda,d}_{\max}(L)$. In other words, the slope gives rise to an element of the extended function field $\mathbb{Q}(\mathcal{A}_d) \cup \infty$.*

*If $\mathcal{V}^{\lambda,d}_{\max}(L) = \mathcal{V}_1(L) \cap \mathcal{A}_d$,* i.e., *$\Delta_L$ does not vanish identically on $\mathcal{A}_d$, one has*

$$(K/L)(\omega) = -\frac{\nabla'(1, \sqrt{\omega})}{2\nabla_L(\sqrt{\omega})} \in \mathbb{C} \cup \infty,$$

*where $\nabla'$ is the derivative of $\nabla_{K\cup L}(t, \cdot)$ with respect to $t$.* ◁

## 3. Concordance of links

Two oriented $\mu$-colored links $L^0$ and $L^1$ are *concordant* if there exists a collection of properly embedded disjoint locally flat cylinders $A := A_1 \sqcup \ldots \sqcup A_\mu$ in $S^3 \times [0, 1]$ such that

$$\partial A_i \cap (S^3 \times 0) = -L^0_i \text{ and } \partial A_i \cap (S^3 \times 1) = L^1_i$$

for all $i = 1, \ldots, \mu$. (In general, each $A_i$ is a union of cylinders.)

3.1. **The concordance invariance.** In the study of knot and link concordance, there is a subset of the complex numbers of particular relevance, the so-called *Knotennullstellen*. This was first introduced in [NP17] for knots and extended to the multi-component link case in [CNT17]. For our purposes, we only need the following definition. Consider the subset of Laurent polynomials

$$U := \big\{p \in \mathbb{Z}[t_1^{\pm 1}, \dots, t_\mu^{\pm 1}] \;\big|\; p(1, \dots, 1) = \pm 1\big\}.$$

An element $\omega \in \mathcal{A}(K/L)$ is called a *concordance root* if there is a polynomial $p \in U$ such that $p(\omega) = 0$. We denote by $\mathcal{A}_c(K/L) \subset \mathcal{A}(K/L)$ the subset of admissible characters that are *not* concordance roots and abbreviate $\mathcal{A}_c^\circ(K/L) := \mathcal{A}_c(K/L) \cap \mathcal{A}^\circ(K/L)$. Note that these sets are larger than the set $\mathbb{T}_!$ used in [CNT17], since we allow for non-unitary characters.

**Remark 3.1.** If $\overline{\ell k}(K, L) = 0$, the set $\mathcal{A}_c(K/L)$ is dense in $\mathcal{A}(K/L) = (\mathbb{C}^\times)^\mu$, as it is a countable intersection of Zariski open sets. In general, $\mathcal{A}_c(K/L)$ is only dense in the components $\mathcal{A}_d$ (see Remark 2.8) for which $d$ is a prime power (or $d = 1$ as a special case). Indeed, if $d$ is *not* a prime power, then $\Phi_d(\cdot) \in U$ and, hence, each point of $\mathcal{A}_d$ is a concordance root.

**Theorem 3.2.** *Let $K^0 \cup L^0$ and $K^1 \cup L^1$ be two concordant $(1, \mu)$-colored links. Then $\mathcal{A}_c(K^0/L^0)$ and $\mathcal{A}_c(K^1/L^1)$ coincide as subsets of $(\mathbb{C}^\times)^\mu$ and one has*

$$(K^0/L^0)(\omega) = (K^1/L^1)(\omega)$$

*for any character $\omega \in \mathcal{A}_c(K^0/L^0)$.*

The proof of Theorem 3.2 is postponed till §3.2. The next few corollaries are direct consequences of Theorem 3.2 and Theorem 4.3 below.

**Corollary 3.3.** *Let $K^0 \cup L^0$ and $K^1 \cup L^1$ be two concordant $(1, \mu)$-colored links such that $\overline{\ell k}(K^s, L^s) = 0$, $s = 0, 1$. Then the slopes $K^0/L^0$ and $K^1/L^1$ are equal as elements of the extended function field $\mathbb{Q}\big((\mathbb{C}^\times)^\mu\big) \cup \infty$. In particular, $(K^0/L^0)(\omega) = (K^1/L^1)(\omega)$ for each character $\omega$ in the complement of the (common) first proper characteristic variety $\mathcal{V}_{\max}(L^0) = \mathcal{V}_{\max}(L^1)$.* ◁

*Proof.* If $L^0$ and $L^1$ are concordant, their nullities coincide (see [CF08, Theorem 7.1]); hence, so do their first proper characteristic varieties. Therefore, the statement is an immediate consequence of Theorem 3.2, the rationality of the slope given by Theorem 2.9, and the density of $\mathcal{A}_c(K/L)$ discussed in Remark 3.1. □

**Corollary 3.4** (of Corollary 3.3 and Theorem 2.9)**.** *Let $K^0 \cup L^0$ and $K^1 \cup L^1$ be two concordant $(1, \mu)$-colored links such that $\overline{\ell k}(K^s, L^s) = 0$ and $\Delta_{L^s} \not\equiv 0$, $s = 0, 1$. Then*

$$\frac{\nabla'_{K^0 \cup L^0}(1, \bar t)}{\nabla_{L^0}(\bar t)} = \frac{\nabla'_{K^1 \cup L^1}(1, \bar t)}{\nabla_{L^1}(\bar t)}, \quad \bar t := (t_1, \dots, t_\mu). \qquad ◁$$

**Remark 3.5.** *A priori*, the conclusions of Corollaries 3.3 or 3.4 do not need to hold if $\lambda := \overline{\ell k}(K^s, L^s) \neq 0$: it is not even obvious that the first proper varieties $\mathcal{V}^{\lambda,d}_{\max}(L^s)$ or even their indices in (2.7) should coincide if $d$ is not a prime power. (Note though that we do not know any counterexample, as that would require going far beyond the known link tables.) The precise statements, based on Remarks 2.8 and 3.1 and Theorems 3.2 and 2.9, are left to the reader.

Recall that a link is *slice* if it is concordant to an unlink. It is a *boundary link* if the components bound a collection of mutually disjoint Seifert surfaces in $S^3$.

**Corollary 3.6.** *If $K \cup L$ is a slice link, then $(K/L)(\omega) = 0$ for all $\omega$ in $\mathcal{A}_c(K/L)$.* $\triangleleft$

**Corollary 3.7.** *If $K \cup L$ is concordant to a boundary link, then $(K/L)(\omega) = 0$ for all $\omega$ in $\mathcal{A}_c(K/L)$ (and for any coloring used to define the slope).* $\triangleleft$

Corollary 3.7 is in fact a particular case of the following statement (see [CF08] or §4.1 below for the definition of $C$-complex).

**Corollary 3.8.** *If $K \cup L$ is concordant to a $(1, \mu)$-colored link $K' \cup L'$ admitting a $C$-complex $F$ for $L$ and a Seifert surface $S$ for $K$ disjoint from $F$, then $(K/L)(\omega) = 0$ for all $\omega \in \mathcal{A}_c(K/L)$.* $\triangleleft$

The following example illustrates that the values of the slope at concordance roots, that is outside the set $\mathcal{A}_c(K/L)$, might not be invariant under concordance. We observe a similar pattern with knot signatures: Knotennullstelle unitary characters are precisely where they fail to be concordance invariants [CL04, NP17]. See [CNT17] for the case of colored links.

**Example 3.9.** Let $K \cup L$ be the $(1, 1)$-colored two-component slice link L10n36, where $K$ is the unknotted component. One has $\nabla_{K \cup L}(t, t_1) = 0$ and $\nabla_L(t_1) = (t_1 - 1 + t_1^{-1})^2$; hence, by Theorem 3.21 in [DFL22], $(K/L)(\omega) = 0$ unless $\omega$ is one of the two roots $\alpha_\pm$ of $\nabla_L$, which agrees with Theorem 3.2 and Corollary 3.4. (By definition, $\alpha_\pm \notin \mathcal{A}_c(K/L)$.) A computation using Fox calculus (see §3.2 in [DFL22]) gives us $(K/L)(\alpha_\pm) = \infty$.

In the proof of Theorem 3.2 we will need the following lemma. We state it in our more general setting of arbitrary, not necessarily unitary, characters, but the proof found in [CNT17] extends literally as it relies on simple homological algebra.

**Lemma 3.10** (Lemma 2.16 in [CNT17])**.** *Let $k \geqslant 0$ be an integer. If $(X, Y)$ is a CW-pair over $B\mathbb{Z}^\mu$ such that $H_i(X, Y; \mathbb{Z}) = 0$ for all $0 \leqslant i \leqslant k$, then also $H_i(X, Y; \mathbb{C}(\omega)) = 0$ for all $0 \leqslant i \leqslant k$ and any character $\omega \in (\mathbb{C}^\times)^\mu$ that is not a concordance root.* $\triangleleft$

3.2. **Proof of Theorem 3.2.** To save space, we abbreviate $H^\omega_*(-) := H_*(-; \mathbb{C}(\omega))$.

Let $D \cup A \subset S^3 \times [0, 1]$ be the concordance, $\partial D = -K^0 \sqcup K^1$, and consider an open tubular neighborhood $T_{D\cup A}$ of $D \cup A$ with a fixed trivialisation extending Seifert framings (in the tubular neighborhoods $T_{K^s \cup L^s} := T_{D\cup A} \cap (S^3 \times s)$, $s = 0, 1$) of the links. Denote

$$U := S^3 \times [0, 1] \smallsetminus T_A, \qquad U_K := S^3 \times [0, 1] \smallsetminus T_{D\cup A}$$

and let

$$X^s := U \cap (S^3 \times s), \qquad X^s_K := U_K \cap (S^3 \times s)$$

for $s = 0, 1$. The inclusions $X^s_K \hookrightarrow U_K$ send the meridians of $K^s \cup L^s$ to those of $D \cup A$. The relative Mayer–Vietoris exact sequences applied to

$$(S^3 \times I, S^3 \times s) = (U_K, X^s_K) \cup (\bar{T}_{D\cup A}, \bar{T}_{K^s\cup L^s}) = (U, X^s) \cup (\bar{T}_A, \bar{T}_{L^s})$$

(where $\bar{T}_*$ stands for the closure of a tubular neighborhood $T_*$) give us

$$H_*(U_K, X^s_K) = H_*(U, X^s) = 0 \tag{3.11}$$

for $s = 0, 1$. In particular, the inclusions $X^s_K \hookrightarrow U_K$ induce isomorphisms

$$H_1(X^0_K) \xrightarrow{\simeq} H_1(U_K) \xleftarrow{\simeq} H_1(X^1_K) \tag{3.12}$$

preserving the meridians and, thus, identify the three character tori. Since the trivialization of $T_D$ homotopes $\ell^0$ to $\ell^1$, we have $\mathcal{A}_c(K^0/L^0) = \mathcal{A}_c(K^1/L^1)$ (*cf.* Remark 2.3).

From now on, patching, if necessary, a few components of both links (and the concordance), we can assume the character $\omega$ non-vanishing, $\omega \in \mathcal{A}^\circ_c(K^0/L^0)$. Referring to Remark 2.3 and using the above identification of the character tori, we can regard $\omega$ as a homomorphism $\pi_1(U_K) \to \mathbb{C}^\times$. The twisted Mayer–Vietoris sequence applied to the pairs

$$(U, X^s) = (U_K, X^s_K) \cup (\bar{T}_D, \bar{T}_{K^s})$$

gives us, for all $i$,

$$\to H^\omega_i(D \times S^1, K^s \times S^1) \to H^\omega_i(U_K, X^s_K) \oplus H^\omega_i(\bar{T}_D, \bar{T}_{K^s}) \to H^\omega_i(U, X^s) \to,$$

where $\{\cdot\} \times S^1$ are the meridians of $K^s$ and $D$, on which $\omega$ is trivial. Since

$$H^\omega_*(D \times S^1, K^s \times S^1) = 0 \quad \text{and} \quad H^\omega_*(U_K, X^s_K) = H^\omega_*(U, X^s) = 0,$$

the latter by Lemma 3.10 and (3.11), we obtain $H^\omega_*(U_K, X^s_K) = 0$ and the inclusions $X^s_K \hookrightarrow U_K$ induce isomorphisms

$$H^\omega_1(X^0_K) \xrightarrow{\simeq} H^\omega_1(U_K) \xleftarrow{\simeq} H^\omega_1(X^1_K)$$

preserving the meridians and, similar to (3.12), taking the class of $\ell^0$ to that of $\ell^1$. It follows that $am^0 + b\ell^0 = 0 \in H^\omega_1(X^0_K)$ if and only if $am^1 + b\ell^1 = 0 \in H^\omega_1(X^1_K)$. □

## 4. Computation with Seifert forms

In this section, unless specified otherwise, we abbreviate

$$H_*(-) := H_*(-;\mathbb{C}), \quad H^*(-) := H^*(-;\mathbb{C}), \quad H^\omega_*(-) = H_*(-;\mathbb{C}(\omega)).$$

For a character $\omega \in (\mathbb{C}^\times \smallsetminus 1)^\mu$, we also abbreviate $\widetilde{\omega}_i := (1 - \omega_i^{-1})$, $1 \leqslant i \leqslant \mu$.

4.1. **Seifert forms.** Let $L = L_1 \cup \ldots \cup L_\mu \subset$ be an oriented $\mu$-colored link in $S^3$. A *C-complex* $F$ for $L$ is a collection of Seifert surfaces $F_1, \ldots, F_\mu$ for the sublinks $L_1, \ldots, L_\mu$ that intersect only along (a finite number of) *clasps*. Each class in $H_1(F;\mathbb{Z})$ can be represented by a collection of *proper loops*, *i.e.*, loops $\alpha\colon S^1 \to F$ such that the pull-back of each clasp is a single segment (possibly empty). We routinely identify classes, loops, and their images.

Given a vector $\varepsilon \in \{\pm 1\}^\mu$, the *push-off* $\alpha^\varepsilon$ of a proper loop $\alpha$ is the loop in $S^3 \smallsetminus F$ obtained by a slight shift of $\alpha$ off each surface $F_i$ in the direction of $\varepsilon_i$. (If $\alpha$ runs along a clasp $\mathfrak{c} \subset F_i \cap F_j$, the shift respects both directions $\varepsilon_i$ and $\varepsilon_j$.) Due to [CF08], this operation gives rise to a well-defined homomorphism

$$\Theta^\varepsilon\colon H_1(F;\mathbb{Z}) \to H_1(S^3 \smallsetminus F;\mathbb{Z}) = H^1(F;\mathbb{Z})$$

(we use Alexander duality), which can be computed by means of the *Seifert forms*

$$\theta^\varepsilon\colon H_1(F;\mathbb{Z})\otimes H_1(F;\mathbb{Z})\to\mathbb{Z},\qquad \alpha\otimes\beta\mapsto \ell k(\alpha,\beta^\varepsilon).$$

Now, given a character $\omega\in(\mathbb{C}^\times\smallsetminus 1)^\mu$, we define

$$\Pi(\omega):=\prod_{i=1}^{\mu}(1-\omega_i)\in\mathbb{C}^\times,\quad A(\omega):=\sum_{\varepsilon\in\{\pm1\}^\mu}\prod_{i=1}^{\mu}\varepsilon_i\omega_i^{(1-\varepsilon_i)/2}\Theta^\varepsilon\colon H_1(F)\to H^1(F)$$

and let

$$E(\omega):=\Pi(\omega^{-1})^{-1}A(\omega^{-1})\colon H_1(F)\to H^1(F). \tag{4.1}$$

Throughout the text we will use the shortcut notation $\operatorname{Ker}E(\omega)^\perp$ to denote the subset of $H^1(F)$ defined as $\operatorname{Ann}\operatorname{Ker}E(\omega)$. It is straightforward that

$$E^*(\omega)=E(\omega^{-1}),\quad \bar{E}(\omega)=E(\bar\omega),$$

where: $E^*$ is the adjoint in the sense of linear algebra over an arbitrary field, and for a linear map $L\colon U\otimes\mathbb{C}\to V\otimes\mathbb{C}$ between two complexified real vector spaces, we let $\bar{L}\colon u\mapsto\overline{L(\bar u)}$. In particular, if $\omega\in(S^1\smallsetminus 1)^\mu$ is unitary, the operator $E(\omega)$ is Hermitian, *i.e.*, $\bar{E}^*(\omega)=E(\omega)$; thus, it has a well-defined signature. Furthermore, if $\omega$ is unitary, the operator $E(\omega^{-1})$ differs from $H(\omega)$ considered in [CF08] by the positive real constant $\Pi(\omega)^{-1}\Pi(\bar\omega)^{-1}$; hence, the two have the same signature and nullity and $E$ can be used instead of $H$ in the following theorem.

**Theorem 4.2** (see [CF08])**.** *If $\omega\in(S^1\smallsetminus 1)^\mu$ is a non-vanishing unitary character, then one has $\sigma_L(\omega)=\operatorname{sign}E(\omega)$ and $\eta_L(\omega)=\dim\operatorname{Ker}E(\omega)+b_0(F)-1$.*

In the case of a 1-colored link $L$, the $C$-complex reduces to a single Seifert surface $F$, so that $\theta:=\theta^+$ and $\Theta:=\Theta^+$ are the classical Seifert form and operator, respectively. Since, in this case, we obviously have $\theta^-=\theta^*$ and, hence, $\Theta^-=\Theta^*$, the operator $E$ takes the classical form

$$E(\omega^{-1})=(1-\omega)^{-1}(\Theta-\omega\Theta^*).$$

4.2. **The statement.** Let $K\cup L$ be a $(1,\mu)$-colored link. Assume that $\lambda$, the linking vector between $K$ and $L$, vanishes and fix a $C$-complex $F$ for $L$ disjoint from $K$. By Alexander duality $H_1(S^3\smallsetminus F;\mathbb{Z})=H^1(F;\mathbb{Z})$, there is a well-defined cohomology class

$$\kappa:=[K]\in H^1(F;\mathbb{Z})\subset H^1(F),\qquad \kappa\colon\alpha\mapsto\ell k(\alpha,K).$$

**Theorem 4.3.** *Under the above assumptions, for any character $\omega\in\mathcal{A}^\circ(K/L)$, consider the operator $E(\omega)\colon H_1(F)\to H^1(F)$, see (4.1). Then*

$$(K/L)(\omega)=\begin{cases}-\langle\alpha,\kappa\rangle, & \text{if } \kappa\in\operatorname{Im}E(\omega)\cap\operatorname{Ker}E(\omega)^\perp,\\ \infty, & \text{if } \kappa\notin\operatorname{Im}E(\omega)\cup\operatorname{Ker}E(\omega)^\perp,\\ \text{undefined}, & \text{otherwise},\end{cases}$$

*where, in the first case, $\alpha\in H_1(F)$ is any class such that $E(\omega)(\alpha)=\kappa$.*

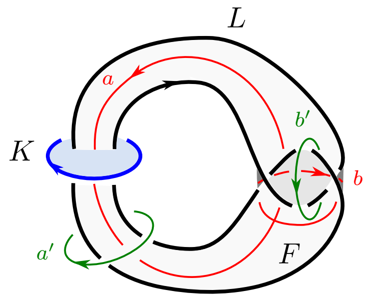


FIGURE 1. The Whitehead link $K \cup L$ with a C-complex $F$ for $L$ (a Seifert surface in this case) and chosen bases $\{a, b\}$ and $\{a', b'\}$ of $H_1(F)$ and $H_1(S^3 \smallsetminus F) = H^1(F)$ respectively.

**Example 4.4.** Consider the Whitehead link $K \cup L$ with the C-complex $F$ depicted in Figure 1, which is simply a genus one Seifert surface for the knot $L$. We want to compute the slope $(K/L)(\omega)$ using Theorem 4.3 and to this end we fix the basis $\{a, b\}$ of $H_1(F)$ and $\{a', b'\}$ of $H_1(S^3 \smallsetminus F) = H^1(F)$ which are illustrated in Figure 1. With respect to these bases we have:

$$\theta^+ = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \quad A(\omega) = \begin{bmatrix} 0 & -\omega \\ 1 & 1-\omega \end{bmatrix}, \quad E(\omega) = \begin{bmatrix} 0 & (1-\omega)^{-1} \\ (1-\omega^{-1})^{-1} & 1 \end{bmatrix}.$$

It is evident from the figure that $\kappa$ is the same class as $a'$. One can easily compute a class $\alpha \in H_1(F)$ such that $E(\omega)(\alpha) = \kappa$:

$$E(\omega) \begin{bmatrix} (1-\omega^{-1})(\omega - 1) \\ 1-\omega \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \kappa$$

Finally, we calculate the slope as $-\langle \alpha, \kappa \rangle$, that is,

$$(K/L)(\omega) = (1-\omega)(1-\omega^{-1}),$$

which coincides with previous computations using Fox calculus (see [DFL22]).

## 5. Proof of Theorem 4.3

5.1. **Geometry of $C$-complexes.** The notation and maps introduced in this section are illustrated in Figure 2. Let $L$ be a $\mu$-colored link and $F$, a $C$-complex for $L$. Given a pair $i \neq j$ of indices, let $C_{ij} := F_i \cap F_j$ and $\mathfrak{C}_{ij} := \pi_0(C_{ij})$ be the set of clasps in the intersection of the surfaces $F_i$ and $F_j$. Denote also $C := \bigcup C_{ij}$ and $\mathfrak{C} := \bigcup \mathfrak{C}_{ij}$.

By convention, each clasp $\mathfrak{c} \in \mathfrak{C}_{ij}$ is oriented from $\mathfrak{c} \cap L_i$ to $\mathfrak{c} \cap L_j$, if $i < j$. The *sign* of $\mathfrak{c}$, denoted by $\operatorname{sg} \mathfrak{c} \in \{\pm 1\}$, is the local intersection index $L_i \circ F_j = L_j \circ F_i$ at the corresponding endpoint of $\mathfrak{c}$.

Fix a regular open neighborhood $V \subset F$ of the union of all clasps, denote by $\bar{V}$ its closure, and let $F_i^\circ := F_i \smallsetminus V$ for all $i$. Then, we have $\partial F_i^\circ = \partial_L F_i^\circ \cup \partial_{\mathfrak{C}} F_i^\circ$, where

$$\partial_L F_i^\circ := \partial F_i^\circ \cap L, \qquad \partial_{\mathfrak{C}} F_i^\circ := \partial F_i^\circ \cap \bar{V}.$$

Given a clasp $\mathfrak{c} \in \mathfrak{C}_{ij}$, let $\bar{V}_{\mathfrak{c}}$ be the connected component of $\bar{V}$ containing $\mathfrak{c}$, and let $\mathfrak{c}_i \in H_1(F_i^\circ, \partial_L F_i^\circ)$ be the arc $F_i^\circ \cap \bar{V}_{\mathfrak{c}}$, with its boundary orientation induced from $V$, as well as the class realized by this arc.

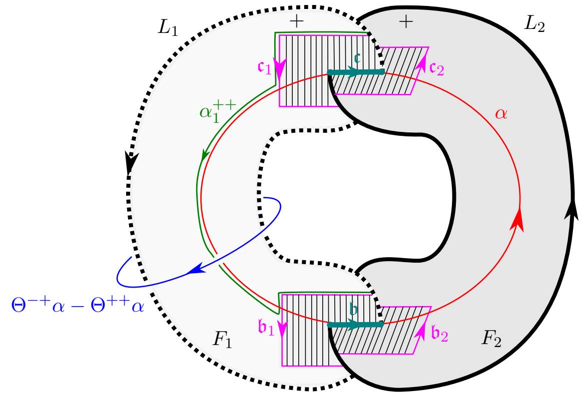


FIGURE 2. This minimal example shows a two colored link $L = L_1 \cup L_2$ bounding a C-complex with two positive claps. In this example $\mathfrak{C} = \mathfrak{C}_{12} = \{\mathfrak{c}, \mathfrak{b}\}$. The lined subset is the open set $V$ with two connected components $V_\mathfrak{c}$ and $V_\mathfrak{b}$. The relative class $\alpha_1^{++} \in H_1(F_1^\circ, \partial_L F_1^\circ)$ and the element $\Theta^{-+}\alpha - \Theta^{++}\alpha = \operatorname{rel}_1^{++}\alpha \in H^1(F)$ are identified through the isomorphism in Lemma 5.1.

The following statement is a formalization of the intuitive fact that any class in $H^1(F)$ can be represented as the intersection index with a certain surface $S \subset S^3$ such that $\partial S \cap F = \varnothing$; on the other hand, any such surface can be made disjoint from $C$ and, when doing so, each clasp can be "circumvented" in two ways.

**Lemma 5.1.** *The intersection pairing establishes an isomorphism*

$$H^1(F) = \bigoplus_{i=1}^{\mu} H_1(F_i^\circ, \partial_L F_i^\circ) \Big/ \big\{\mathfrak{c}_i + \mathfrak{c}_j = 0 \bigm| \mathfrak{c} \in \mathfrak{C}_{ij},\ 1 \leqslant i < j \leqslant \mu\big\}.$$

*Proof.* Since all groups involved are torsion free, the statement follows from the exact sequence of the pair $(F, \bar{V})$:

$$0 \longrightarrow H_1(F) \longrightarrow H_1(F, \bar{V}) \longrightarrow H_0(\bar{V}) \longrightarrow H_0(F),$$

where $H_1(F, \bar{V}) = \bigoplus_i H_1(F_i^\circ, \partial_\mathfrak{C} F_i^\circ)$. Then, there remains to apply Poincaré–Lefschetz duality $H^1(F_i^\circ, \partial_\mathfrak{C} F_i^\circ) = H_1(F_i^\circ, \partial_L F_i^\circ)$. □

Let $\varepsilon \in \{\pm 1\}^\mu$. Pick a class $\alpha \in H_1(F)$, represent it by a proper loop, and denote by $\alpha_i^\varepsilon \in H_1(F_i^\circ, \partial_L F_i^\circ)$ the class realized by the arc $\alpha \cap F_i$ pushed off each clasp $\mathfrak{c} \in \mathfrak{C}_{ij}$ in the direction prescribed by $\varepsilon_j$. Passing further to the image in $H^1(F)$, see Lemma 5.1, we obtain a well-defined homomorphism $\operatorname{rel}_i^\varepsilon\colon H_1(F) \to H^1(F)$. It is easily seen that $\operatorname{rel}_i^\varepsilon$ is independent of $\varepsilon_i$. In fact,

$$\operatorname{rel}_i^\varepsilon \alpha = \Theta^{\varepsilon[-i]}\alpha - \Theta^{\varepsilon[+i]}\alpha,$$

where $\varepsilon[\pm i]$ is obtained from $\varepsilon$ by replacing the $i$-th component by $\pm 1$. Furthermore,

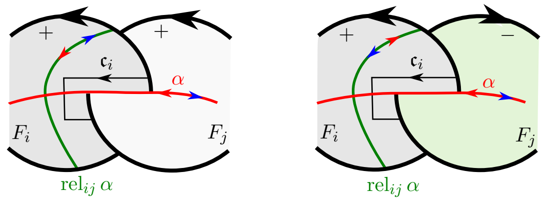


FIGURE 3. In this figure, the element $\alpha \in H_1(F)$ is depicted with both possible orientations. The orientation of the element $\mathrm{rel}_{ij}\,\alpha$ depends on the sign of the clasp, as illustrated. Remark that the element $\mathrm{rel}_{ij}\,\alpha$ is by definition in $H^1(F)$: the green curve depicted is a representative of that element *via* Lemma 5.1.

for an index $j \neq i$, we have

$$\mathrm{rel}_i^{\varepsilon[+j]}\,\alpha - \mathrm{rel}_i^{\varepsilon[-j]}\,\alpha = \mathrm{rel}_{ij}\,\alpha := \sum_{\mathfrak{c}\in\mathfrak{C}_{ij}} \mathrm{sg}\,\mathfrak{c}\cdot\langle\alpha,\mathfrak{c}_i\rangle\mathfrak{c}_i. \tag{5.2}$$

For the reader's convenience a local illustration is presented in Figure 3. (Note that $\langle\alpha,\mathfrak{c}_i\rangle\mathfrak{c}_i = \langle\alpha,\mathfrak{c}_j\rangle\mathfrak{c}_j$ for each clasp $\mathfrak{c} \in \mathfrak{C}_{ij}$ and, hence, $\mathrm{rel}_{ij}\,\alpha = \mathrm{rel}_{ji}\,\alpha$ as elements of $H^1(F)$.) Let $- := [-1,\ldots,-1] \in \{\pm1\}^\mu$. Then, applying the last two equations inductively, for each $\varepsilon \in \{\pm1\}^\mu$ we get

$$\Theta^\varepsilon\alpha - \Theta^-\alpha = -\sum_{i\,:\,\varepsilon_i>0}\mathrm{rel}_i^-\,\alpha - \sum_{i<j\,:\,\varepsilon_i=\varepsilon_j>0}\mathrm{rel}_{ij}\,\alpha. \tag{5.3}$$

**Remark 5.4.** It follows from (5.3) that, as in the classical case of a single Seifert surface, all operators $\Theta^\varepsilon$ are almost determined by any one of them, as the relativization homomorphisms $\mathrm{rel}_i^\varepsilon$ and $\mathrm{rel}_{ij}$ are intrinsic to the abstract $C$-complex $F$ *with prescribed signs* $\mathrm{sg}\,\mathfrak{c}$ of the clasps. In the classical case, (5.3) takes the well-known form

$$\Theta^* - \Theta = \mathrm{rel}\colon H_1(F) \to H_1(F,\partial F) = H^1(F),$$

which explains the notation rel.

Now, given a character $\omega \in (\mathbb{C}^\times \smallsetminus 1)^\mu$, observe that

$$A(\omega) = \Pi(\omega)\Theta^- + \sum_{\varepsilon\in\{\pm1\}^\mu}\prod_{i=1}^{\mu}\varepsilon_i\omega_i^{(1-\varepsilon_i)/2}(\Theta^\varepsilon - \Theta^-).$$

Hence, using (5.3), rearranging the terms, and using the definition $\widetilde{\omega}_i = 1 - \omega_i^{-1}$, we arrive at

$$E(\omega) = \Theta^- - R(\omega), \quad R(\omega) := \sum_{i=1}^{\mu}\widetilde{\omega}_i^{-1}\,\mathrm{rel}_i^- + \sum_{1\leqslant i<j\leqslant\mu}\widetilde{\omega}_i^{-1}\widetilde{\omega}_j^{-1}\,\mathrm{rel}_{ij}\,. \tag{5.5}$$

5.2. **Reference sheets.** We briefly recall how twisted homology can be computed *via* coverings. Consider a connected $CW$-complex $X$, an abelian group $G$, and an epimorphism $\varphi\colon \pi_1(X) \twoheadrightarrow H_1(X;\mathbb{Z}) \twoheadrightarrow G$. The kernel of $\varphi$, which is a normal subgroup of $\pi_1(X)$, gives rise to a Galois $G$-covering $\tilde{X} \to X$, where the deck transformation $g \in G$

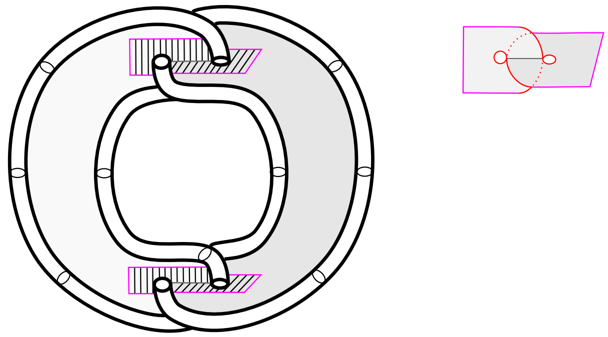

Figure 4. A minimal example of the set $F_L = (F \cup \bar{T}) \smallsetminus T$ consisting of the gray shaded surface together with the two depicted tori. The lined subset is $\bar{V}_L$. To the right we have a copy of a connected component of $\bar{V}_L$ with the subset $\partial_L \bar{V}_L$ highlighted in red.

sends a point $\tilde{x} \in \tilde{X}$ to the other endpoint of the arc that begins at $\tilde{x}$ and covers a loop representing an element of $\varphi^{-1}(g)$. This model induces a structure of $\mathbb{Z}[G]$-module on $C_*(\tilde{X})$ and, for each multiplicative character $\omega\colon G \to \mathbb{C}^\times$, there is a canonical chain isomorphism of complexes of $\mathbb{C}(\omega)$-modules

$$C_*(X; \mathbb{C}(\omega)) \simeq C_*(\tilde{X}) \otimes_{\mathbb{Z}G} \mathbb{C}(\omega).$$

Occasionally, the homomorphism $\varphi\colon H_1(X;\mathbb{Z}) \to G$ might not necessarily be surjective. (Typically, this situation occurs when we restrict the construction to a subcomplex $Y \subset X$.) Then, letting $G' := \operatorname{Im}\varphi$, the $G$-covering $\tilde{X}$ consists of $[G : G']$ connected components, each isomorphic to the $G'$-covering $\tilde{X}'$, and we have

$$C_*(\tilde{X}) \simeq C_*(\tilde{X}') \otimes_{\mathbb{Z}G'} \mathbb{Z}G.$$

However, this isomorphism is no longer canonical: to make it such, we need to fix a *reference component* $\tilde{X}' \subset \tilde{X}$. An important special case is that where the restriction of $\omega$ to $X$ is trivial. Then we have an isomorphism

$$H_*\big(C_*(\tilde{X}) \otimes_{\mathbb{Z}G} \mathbb{C}(\omega)\big) \simeq H^\omega_*(X) = H_*(X),$$

canonical *provided that a* reference sheet $X$ *in the trivial covering* $\tilde{X} \to X$ *is fixed.*

Back to the original set-up, when dealing with the twisted homology, we need to avoid the ramification locus $L$. Hence, we fix pairwise disjoint tubular neighborhoods $T_i \supset L_i$ and, denoting by $\bar{T}_i$ the closure of $T_i$ and letting $T := \bigcup_i T_i$, $\bar{T} := \bigcup_i \bar{T}_i$, introduce

$$S_L := S^3 \smallsetminus T, \qquad F_L := (F \cup \bar{T}) \smallsetminus T \subset S_L, \qquad C_L := C \smallsetminus T,$$
$$\bar{V}_L := \bar{V} \smallsetminus T, \qquad \partial_L \bar{V}_L := \bar{V}_L \cap \bar{T},$$

see Figure 4. Here, $V \supset C$ is the neighborhood introduced in §5.1, and we assume the radius of $T$ so small that $F_i \cap \bar{T}_j \subset V$ for each $i \neq j$.

Formally, we also need to shrink the surfaces $F_i^\circ$ to $F_i^\circ \smallsetminus T$, changing the boundary $\partial_L F_i^\circ$ to $(F_i^\circ \smallsetminus T) \cap \bar{T}$; however, using the obvious isomorphisms in (co-)homology, we keep the notation $(F_i^\circ, \partial_L F_i^\circ)$ for these new pairs.

We make use of the isomorphisms

$$H^{\omega}_*(S_L, F_L) \simeq H_*(S_L, F_L) = H_*(S, F), \tag{5.6}$$

$$H^{\omega}_*(F_i^{\circ}, \partial_L F_i^{\circ}) \simeq H_*(F_i^{\circ}, \partial_L F_i^{\circ}), \tag{5.7}$$

$$H^{\omega}_*(\bar V_L, \partial_L \bar V_L) = H^{\omega}_*(C_L, \partial C_L) \simeq H_*(C_L, \partial C_L) = H_*(C, \partial C), \tag{5.8}$$

*etc.*, and, in order to fix the (not quite canonical in the context of a common $G$-covering) isomorphisms denoted by $\simeq$, we need a coherent choice of reference sheets, upon which we change the notation to $=$. (The other isomorphisms are standard combinations of excision and homotopy equivalences and, thus, are canonical.) To this end, we consider a "negative" collar (trace of the push-off in the negative direction) $N := (-\delta, 0) \times (F \smallsetminus T)$, $\delta \ll \operatorname{radius}(\bar T)$, and, letting $S'_L := S_L \smallsetminus N$, use excision to identify

$$H_*(S_L, F_L) = H_*(S'_L, \partial S'_L), \qquad H^{\omega}_*(S_L, F_L) = H^{\omega}_*(S'_L, \partial S'_L).$$

Since the covering is obviously trivial over $S'_L$, we can choose and fix a reference sheet $S'_L \subset \tilde S_L$ and use it for (5.6). There remains to observe that this sheet contains a single copy of each of $F_i^{\circ}$ and $C_L$, which are used for (5.7) and (5.8), respectively.

**Convention 5.9.** We have then $H^{\omega}_2(S_L, F_L) = H_2(S_L, F_L)$ and $H_1(F_L) = H^{\omega}_1(F_L)$. For the twisted boundary operators like

$$H_2(S_L, F_L) \to H_1(F_L),$$

we assume that $\partial^{\omega} = \sum_i (\partial^- + \omega_i^{-1}\partial^+)$, where $\partial^+$ is the *lower* boundary (the + superscript is related to the orientation conventions.)

**Convention 5.10.** The "reference lift" of a loop is the loop in the covering whose *end point* is in the reference sheet.

5.3. **The homology of $F$.** Throughout this section, *we assume that $F$ is connected and that $\kappa \neq 0$.* Recall from Lemma 5.1 that $H^1(F)$ is a quotient of $\bigoplus H_1(F_i^{\circ}, \partial_L F_i^{\circ})$ by relations of the form $\mathfrak{c}_i + \mathfrak{c}_j = 0$. We deduce the following description of the twisted homology of $F$.

**Lemma 5.11.** *The assignment $\tau\colon H^1(F) \to H^{\omega}_1(F_L, \partial \bar T) = H^{\omega}_1(F_L)$*

$$\sum_{i=1}^{\mu} \alpha_i \longmapsto \text{inclusion}_* \bigoplus_{i=1}^{\mu} \widetilde{\omega}_i \alpha_i, \quad \alpha_i \in H_1(F_i^{\circ}, \partial_L F_i^{\circ}),$$

*is a well-defined isomorphism.*

*Proof.* The isomorphisms $H^{\omega}_*(F_L, \partial \bar T) = H^{\omega}_*(F_L)$ follow from the assumption $\omega_i \neq 1$ for each $i$ and, hence, $H^{\omega}_*(\partial \bar T) = 0$. We compute $H^{\omega}_1(F_L, \partial \bar T)$ using the relative Mayer–Vietoris sequence associated to the decomposition $F \smallsetminus T = \bar V_L \cup \bigl(\bigcup_{i=1}^{\mu} F_i^{\circ}\bigr)$:

$$H^{\omega}_1(\partial \bar V_L, \partial_L \bar V_L) \longrightarrow H^{\omega}_1(\bar V_L, \partial_L \bar V_L) \oplus \bigoplus_{i=1}^{\mu} H^{\omega}_1(F_i^{\circ}, \partial_L F_i^{\circ}) \xrightarrow{\ p\ } H^{\omega}_1(F_L, \partial \bar T) \to 0, \tag{5.12}$$

the last term being $H^{\omega}_0(\partial \bar V_L, \partial_L \bar V_L) = 0$, see (5.8) and Figure 4. By (5.8), we also have $H^{\omega}_1(\partial \bar V_L, \partial_L \bar V_L) = \bigoplus \mathbb{C}\mathfrak{c}_i$, the summation running over all $\mathfrak{c} \in \mathfrak{C}_{ij}$ and all pairs

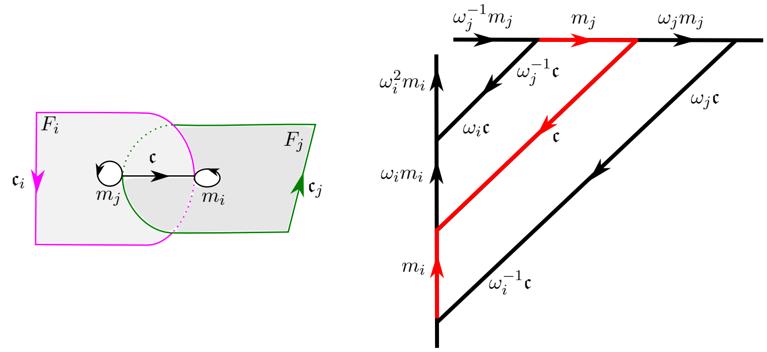


FIGURE 5. To the left is a local picture of a positive clasp with $i < j$. To the right, the schematics of the behavior of the lifted curves on a covering space. Shown in red are the chosen reference lifts.

$1 \leqslant i \neq j \leqslant \mu$. The inclusions induce the homomorphisms

$$\begin{aligned} &\mathfrak{c}_i \mapsto \mathfrak{c}_i && \in H_1^\omega(F_i^\circ, \partial_L F_i^\circ) = H_1(F_i^\circ, \partial_L F_i^\circ), \text{ see } (5.7), \\ &\mathfrak{c}_i \mapsto \operatorname{sg}(j-i) \cdot \operatorname{sg}\mathfrak{c} \cdot \widetilde{\omega}_j \mathfrak{c} && \in H_1^\omega(\bar{V}_L, \partial_L \bar{V}_L) = \textstyle\bigoplus_{\mathfrak{c}\in\mathfrak{C}} \mathbb{C}\mathfrak{c}. \end{aligned} \tag{5.13}$$

(To follow the above formulas, the reader might find helpful the schematics of the behavior of the twisted homology in Figure 5.) Identifying the two images of each generator $\mathfrak{c}_i$, we conclude that the inclusions $F_i^\circ \hookrightarrow F_L$ induce an isomorphism

$$\bigoplus_{i=1}^{\mu} H_1(F_i^\circ, \partial_L F_i^\circ) \big/ \big\{ \widetilde{\omega}_i \mathfrak{c}_i + \widetilde{\omega}_j \mathfrak{c}_j = 0 \bigm| \mathfrak{c} \in \mathfrak{C}_{ij} \big\} = H_1^\omega(F_L, \partial \bar{T}),$$

and the isomorphism in the statement follows from Lemma 5.1. □

**Corollary 5.14** (of the proof)**.** *Given a proper loop $\alpha \subset F$, consider its push-off $\alpha^-$ and its "trace" $S^- \subset S^3$,* i.e., *a cylinder contained in a regular neighborhood of $\alpha$ and such that $S^- \cap F = \alpha$ and $\partial S^- = \alpha - \alpha^-$. Then, the twisted boundary $\partial^\omega S^- + \alpha^-$ is equal to $\tau\big(R(\omega)(\alpha)\big) \in H_1^\omega(F_L)$, see* (5.5) *and Lemma 5.11.*

*Proof.* Clearly, $\partial^\omega S^- + \alpha^-$ is homologous to the image under $p$ in (5.12) of the cycle

$$\sum_{i=1}^{\mu} \operatorname{rel}_i^- \alpha + \sum_{1\leqslant i<j\leqslant\mu} \sum_{\mathfrak{c}\in\mathfrak{C}_{ij}} \langle \alpha, \mathfrak{c}_i \rangle \mathfrak{c},$$

(see Figure 6 for a simple example.) Then, by (5.13), for all $i < j$ and $\mathfrak{c} \in \mathfrak{C}_{ij}$, we have $\mathfrak{c} = \operatorname{sg}\mathfrak{c} \cdot \widetilde{\omega}_j^{-1} \mathfrak{c}_i$ in $H_1^\omega(F_L)$ and, using (5.2), we obtain

$$\begin{aligned} \sum_{i=1}^{\mu} \operatorname{rel}_i^- \alpha + \sum_{1\leqslant i<j\leqslant\mu} \widetilde{\omega}_j^{-1} \sum_{\mathfrak{c}\in\mathfrak{C}_{ij}} \operatorname{sg}\mathfrak{c} \langle \alpha, \mathfrak{c}_i \rangle \mathfrak{c}_i &\overset{(5.2)}{=\!=} \sum_{i=1}^{\mu} \operatorname{rel}_i^- \alpha + \sum_{1\leqslant i<j\leqslant\mu} \widetilde{\omega}_j^{-1} \operatorname{rel}_{ij} \alpha \\ &= \sum_{i=1}^{\mu} \widetilde{\omega}_i \underbrace{\left( \widetilde{\omega}_i^{-1} \operatorname{rel}_i^- \alpha + \sum_{j=i+1}^{\mu} \widetilde{\omega}_i^{-1} \widetilde{\omega}_j^{-1} \operatorname{rel}_{ij} \alpha \right)}_{R_i}. \end{aligned}$$

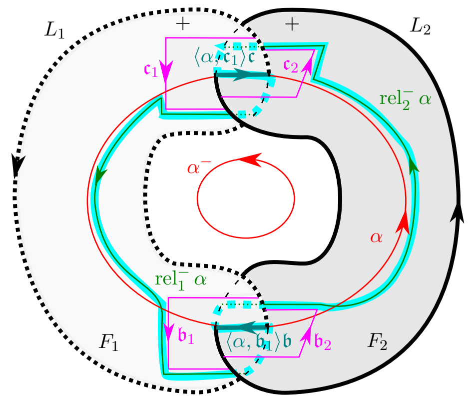


FIGURE 6. The push off $\alpha^-$ is to be thought of as located "behind" the surface $F_1 \cup F_2$. With the orientations depicted, together $\alpha$ and $-\alpha^-$ are the obvious boundary of the cylinder $S^-$ (not in the picture). The different elements of the cycle described at the beginning of the proof of Corollary 5.14, $\operatorname{rel}_i^- \alpha$ and $\langle \alpha, \mathfrak{c}_i \rangle \mathfrak{c}$, are highlighted.

Now, by (5.5), each $R_i$ is the $i$-th component of (a representative of) $R(\omega)(\alpha)$, and the statement follows from the definition of $\tau$ in Lemma 5.11. □

We proceed with the computation of the twisted homology of $S_L$ and $S_L \smallsetminus K$. We have fixed isomorphisms

$$H_*^\omega(S_L, F_L) = H_*(S, F), \qquad H_*^\omega(S_L \smallsetminus K, F_L) = H_*(S \smallsetminus K, F),$$

see (5.6). In particular,

$$H_1^\omega(S_L, F_L) = H_1^\omega(S_L \smallsetminus K, F_L) = 0$$

(recall that we assume $F$ connected and $\kappa \neq 0$) and, by the respective exact sequences of pairs $(S, F)$ and $(S \smallsetminus K, F)$,

$$H_2^\omega(S_L, F_L) = H_1(F), \qquad H_2^\omega(S_L \smallsetminus K, F_L) = \operatorname{Ker} \kappa \subset H_1(F).$$

Now, from the corresponding twisted exact sequences, and with the isomorphism $\tau$ given by Lemma 5.11 taken into account, we arrive at

$$H_1^\omega(S_L) = H^1(F)/\operatorname{Im} d, \qquad H_1^\omega(S_L \smallsetminus K) = H^1(F)/d(\operatorname{Ker} \kappa), \tag{5.15}$$

where $d$ is the composed map

$$d\colon H_1(F) \xrightarrow{\partial^{-1}} H_2(S, F) = H_2^\omega(S_L, F_L) \xrightarrow{\partial^\omega} H_1^\omega(F_L) \xrightarrow{\tau^{-1}} H^1(F). \tag{5.16}$$

5.4. **The twisted homomorphisms.** We still assume that $F$ is connected and $\kappa \neq 0$. By (5.15), for $X := S_L$ or $X := S_L \smallsetminus K$, we have natural epimorphisms

$$\pi_X\colon H^1(F) \twoheadrightarrow H_1^\omega(X). \tag{5.17}$$

Composing the inclusion with Alexander duality, we obtain a homomorphism

$$\mathrm{D}\colon H_1^\omega(X \smallsetminus F_L) = H_1(X \smallsetminus F_L) \to H_1(S^3 \smallsetminus F) \xrightarrow{\simeq} H^1(F).$$

Consider also the "orthogonal projection"

$$\begin{aligned} \mathrm{pr}_X\colon H_1^\omega(X \smallsetminus F_L) &\longrightarrow H_1^\omega(X \smallsetminus F_L),\\ \alpha &\longmapsto \alpha && \text{if } X = S_L,\\ \alpha &\longmapsto \alpha - \ell k(\alpha, K)m && \text{if } X = S_L \smallsetminus K. \end{aligned}$$

**Lemma 5.18.** *For $X = S_L$ or $S_L \smallsetminus K$ and any class $\alpha \in H_1^\omega(X \smallsetminus F_L)$, the image of $\mathrm{pr}_X(\alpha)$ under the inclusion homomorphism $H_1^\omega(X \smallsetminus F_L) \to H_1^\omega(X)$ is $\pi_X(\mathrm{D}(\alpha))$.*

*Proof.* The statement is a geometric version of Lemma 5.11. The class $\alpha' := \mathrm{pr}_X(\alpha)$ is represented by a cycle in $X \smallsetminus F_L$, which bounds a Seifert surface $G \subset S^3 \smallsetminus K$. (This is why we subtract $\ell k(\alpha, K)m$ in the case $X = S_L \smallsetminus K$: we want a Seifert surface disjoint from $K$.) Set $G_L := G \cap S_L$. We can choose the surface $G_L$ so that it cuts on $F$ a collection of arcs $\alpha_i \subset F_i^\circ$ with $\partial\alpha_i \subset \partial_L F_i^\circ$. Then, $\mathrm{D}(\alpha')$ is represented by

$$\sum_{i=1}^{\mu} \alpha_i \in \bigoplus_{i=1}^{\mu} H_1(F_i^\circ, \partial_L F_i^\circ) \longrightarrow H^1(F),$$

see Lemma 5.1, whereas the twisted boundary is

$$\partial^\omega G_L - \alpha' = -\sum_{i=1}^{\mu} \widetilde{\omega}_i \alpha_i = -\tau(\mathrm{D}(\alpha)), \tag{5.19}$$

*cf.* Lemma 5.11, implying that $\alpha' = \tau(\mathrm{D}(\alpha))$ in $H_1^\omega(X)$. □

**Corollary 5.20.** *For $X = S_L$ or $S_L \smallsetminus K$, let $\alpha \in H_1^\omega(X \smallsetminus F_L)$ be the class of $[K]$ or $\ell$, respectively. Then, the image of $\alpha$ in $H_1^\omega(X)$ is $\pi_X(\kappa)$.* ◁

**Lemma 5.21.** *The homomorphism $d$ in (5.16) equals $-E(\omega)$.*

**Lemma 5.22.** *For each $\alpha \in H_1(F)$, one has*

$$\pi_{S_L \smallsetminus K}\bigl(E(\omega)(\alpha)\bigr) = -\langle \alpha, \kappa \rangle m$$

*in $H_1^\omega(S_L \smallsetminus K)$, see (5.17).*

*Proof of Lemmas 5.21 and 5.22.* Let $\alpha \subset F$ be a proper loop and consider its push-off $\alpha^- \subset S^3 \smallsetminus (K \cup F)$. Let $S^-$ be the trace cylinder as in Corollary 5.14, and let $G$ be a Seifert surface bounded by $\alpha^-$. (For Lemma 5.22, we replace $\alpha^-$ with its projection $\mathrm{pr}(\alpha^-) = \alpha^- - \langle \alpha, \kappa \rangle m$ in order to keep $S$ in $S^3 \smallsetminus K$; details are left to the reader.)

Defining $G_L := G \cap S_L$ and letting $\bar{S} := G_L \cup S^-$, we have $\partial\bar{S} = \alpha$. On the other hand, the twisted boundary

$$\partial^\omega \bar{S} = (\partial^\omega S^- + \alpha^-) + (\partial^\omega G_L - \alpha^-) = \tau\bigl(R(\omega)(\alpha)\bigr) - \tau\bigl(\Theta^-(\alpha)\bigr)$$

is given by Corollary 5.14 and (5.19), and the statements follow from (5.5). □

**Corollary 5.23** (of Lemma 5.21 and (5.15))**.** *There are canonical, up to multiplication by integral powers of $\omega_i$'s, isomorphisms*

$$H_1^\omega(S_L) = H^1(F)/\operatorname{Im} E(\omega), \qquad H_1^\omega(S_L \smallsetminus K) = H^1(F)/E(\omega)(\operatorname{Ker}\kappa). \qquad ◁$$

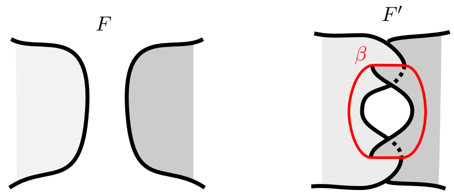


FIGURE 7. To the left a local picture of a disconnected C-complex $F$. To the right, the complex $F'$, obtained by adding a pair of close clasps to $F$. We have $H_1(F';\mathbb{Z}) = H_1(F;\mathbb{Z}) \oplus \mathbb{Z}\beta$.

*Proof of* Theorem 4.3. If $\kappa = 0$, then $K$ bounds a Seifert surface disjoint from $F$ and, hence, $K/L \equiv 0$, which agrees with the statement of the theorem.

Therefore, till the rest of the proof we assume that $\kappa \neq 0$. Assume also that $F$ is connected, so that we can use the results of §5.3 and §5.4. Abbreviate $E := E(\omega)$, so that $E^* = E(\omega^{-1})$ and $\operatorname{Ker} E^{\perp} = \operatorname{Im} E^*$. Then, in view of Corollary 5.23, the last two cases in the statement, as well as the finiteness of the slope in the first case, are given by Proposition 2.6. To compute this finite slope in the first case, we compare Corollary 5.20 and Lemma 5.22: if $\kappa = E(\alpha)$, then $\ell = -\langle \alpha, \kappa \rangle m$ in $H_1^{\omega}(S_L \smallsetminus K)$.

Finally, if $F$ is not connected, we can reduce inductively the number of components by introducing pairs of close clasps as in Figure 7. If $F'$ is obtained from $F$ by introducing one such pair, connecting two distinct components, then $H_1(F';\mathbb{Z}) = H_1(F;\mathbb{Z}) \oplus \mathbb{Z}\beta$, where $\beta$ is a small proper loop running through the two clasps, and, extending the existing pair of dual bases by $\beta \in H_1(F)$ and $\beta^* \in H^1(F)$, the other data are

$$\Theta'^{\varepsilon} = \Theta^{\varepsilon} \oplus [0], \qquad \kappa' = \kappa \oplus [0].$$

Obviously, this modification does not affect the result of the computation. □

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Department of Mathematics, Bilkent University, 06800 Ankara, Turkey
*Email address*: `degt@fen.bilkent.edu.tr`

Laboratoire de Mathématiques et leurs applications, UMR CNRS 5142, Université de Pau et des Pays de l'Adour, Avenue de l'Université, BP 1155 64013 Pau Cedex, France
*Email address*: `vincent.florens@univ-pau.fr`

School of Mathematics and Statistics, University of Glasgow, Glasgow, UK.
*Email address*: `ana.lecuona@glasgow.ac.uk`

## Marginal notes

1/ 4: **A:** We have never introduced $\Delta_L$. We decide it's "classic enough"? Who are we to explain $\Delta$ to knot theorists? :)

2/ 4: Is it a divisor by any chance? good idea to explain the relation to $K$; extended a little

3/ 4: notation fixed throughout