ON DEFORMATIONS OF SINGULAR PLANE SEXTICS

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Abstract. We study complex plane projective sextic curves with simple singularities up to equisingular deformations. It is shown that two such curves are deformation equivalent if and only if the corresponding pairs are diffeomorphic. A way to enumerate all deformation classes is outlined, and a few examples are considered, including classical Zariski pairs; in particular, promising candidates for homeomorphic but not diffeomorphic pairs are found.

1. Introduction

1.1. Motivation and principal results. Following the real algebraic geometry tradition, an equisingular deformation of complex plane projective algebraic curves is called a *rigid isotopy*. Whenever two curves $C_1, C_2 \subset \mathbb{P}^2$ are rigidly isotopic, the pairs $(\mathbb{P}^2, C_i)$, $i = 1, 2$, are homeomorphic and, in the case of simple singularities only, also diffeomorphic. In his celebrated paper [40], O. Zariski constructed a pair of irreducible curves C_1, C_2 of degree six that have the same set of singularities (six cusps) but are not rigidly isotopic; in fact, the complements $\mathbb{P}^2 \smallsetminus C_i$, $i = 1, 2$, are not homeomorphic. E. Artal [1] suggested to call such curves Zariski pairs. More precisely, a *Zariski pair* is a pair of reduced plane curves C_1, C_2 having the same combinatorial type of singularities but non-homeomorphic pairs $(\mathbb{P}^2, C_i)$; see Section 4.1 for details and various ramifications. The first degree where Zariski pairs exist is six, as the rigid isotopy class of a plane curve of degree up to five is determined by its combinatorial data, see [10].

In my thesis (see [8] and [11]), I generalized Zariski's example and found all pairs of irreducible sextics $C \subset \mathbb{P}^2$ that have the same singularities and, as in Zariski's original case, differ by their Alexander polynomial (see Section 4.4 for more details); to avoid confusion with Artal's definition above, we call such curves *classical Zariski pairs*. I also conjectured that, up to equisingular deformation, an irreducible sextic is determined by its set of singularities and its Alexander polynomial. (The conjecture was based on the calculation for a few special cases and the fact that the assertion does hold if the curves have at least one non-simple singular point, see [9].) The conjecture was soon disproved by H. Tokunaga [34], who constructed a pair of irreducible sextics C_1, C_2 with the same sets of singularities and Alexander polynomials. Still, Tokunaga's curves differ by the fundamental group $\pi_1(\mathbb{P}^2 \smallsetminus C_i)$. In a recent series of papers [2]–[4] Artal *et al.* constructed a number of new examples of not rigidly isotopic pairs (C_1, C_2) of sextics; for many pairs the fundamental

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groups $\pi_1(\mathbb{P}^2 \smallsetminus C_i)$ are calculated and shown to coincide. Thus, the question arises whether the curves constitute Zariski pairs, *i.e.*, whether $(\mathbb{P}^2, C_1)$ and $(\mathbb{P}^2, C_2)$ are homeomorphic. We show that they are not diffeomorphic. More precisely, the following theorem holds.

1.1.1. Theorem. *Two sextic curves $C_1, C_2 \subset \mathbb{P}^2$ with simple singularities only are rigidly isotopic if and only if there exists a diffeomorphism $f\colon (\mathbb{P}^2, C_1) \to (\mathbb{P}^2, C_2)$ that is* regular *in the sense that each singular point of C_1 has a neighborhood U such that the restriction $f|_U$ is complex analytic.*

This theorem is proved in Section 3.5.

1.1.2. Remark. The requirement that f should be a diffeomorphism is not a mere technical assumption; it is used essentially in the proof as a means of comparing the orientations of the homological types of C_1 and C_2 (see Section 3.2). Since pairs of sextics that differ solely by the orientation of their homological types do exist (*e.g.*, Proposition 5.4.4), one may anticipate that they would provide examples of homeomorphic but not diffeomorphic pairs.

As Theorem 1.1.1 settles the relative Dif = Def problem for plane sextics, it simplifies the process of finding Zariski pairs. For example, according to J.-G. Yang [39] there is a five page long list of sextics with maximal total Milnor number $\mu = 19$. The rigid isotopy classes of such curves are described by definite lattices, which tend to have very few isometries; hence, there should be a great deal of not rigidly isotopic pairs sharing the same sets of singularities.

The proof of Theorem 1.1.1 is based on an explicit description of the moduli space of sextics, see Theorems 3.4.1 and 3.4.2, which, in turn, is a rather standard application of the global Torelli theorem for $K3$-surfaces and the surjectivity of the period map. As another application, Theorem 3.4.2 reduces the rigid isotopy classification of plane sextics to an arithmetic question about lattices. We outline the principal steps of enumerating abstract homological types, see Section 5.1, and apply the scheme to two polar cases, those of curves with few singularities and curves with many singularities. In the former case, we prove Corollary 5.2.2 and Theorem 5.2.1, which give simple sufficient conditions for a set of singularities/configuration to be realized by a single rigid isotopy class. As a further application, we enumerate all curves constituting classical Zariski pairs without nodes, see Theorem 5.3.2. For Zariski's original example the theorem states that *plane sextics with six cusps form exactly two deformation families.* To my knowledge, this fact is new: contrary to the common belief, Zariski himself has only asserted the existence of *at least* two families.

In the latter case (maximal total Milnor number $\mu = 19$), the problem reduces to enumerating certain positive definite lattices of rank 2 and their isometries. The algorithm can easily be implemented (in fact, I do have it implemented in `Maple`), and, when combined with Yang's algorithm [39] for enumerating the configurations, it should produce a complete list of rigid isotopy classes. However, instead of compiling a long computer aided table, I illustrate the approach by studying a few examples (see Propositions 5.4.1–5.4.8) that were first considered in [2]–[4].

Undoubtedly, the most remarkable example is that given by Proposition 5.4.4, where two curves differ by the orientation of their homological types. It is worth mentioning that found in the literature are a great number of various deformation classification problems related to the global Torelli theorem for $K3$-surfaces (in the real case, see recent papers [33] and [14] and the survey [15] for further references; in the complex case, see, *e.g.*, V. Nikulin [30], A. Degtyarev *et al.* [14], Sh. Mukai [28], Sh. Kondō [22] and [23], and G. Xiao [38]). To my knowledge, the study of singular plane sextics is the only case so far where the orientation of maximal positive definite subspaces is involved in an essential way!

1.2. Contents of the paper. In §2, we outline the principal notions and results of Nikulin's theory of discriminant forms of even integral lattices. It is largely based on Nikulin's original paper [29]. A preliminary calculation involving certain definite lattices is also made here. In §3, the relation between plane sextics and $K3$-surfaces is explained, the moduli space is described, and Theorem 1.1.1 is proved. In §4, we discuss a few results relating the geometry of a sextic and the arithmetic properties of its homological type. Finally, §5 deals with the classification of oriented abstract homological types, which enumerate the rigid isotopy classes of sextics. We outline the general scheme and apply it to a few particular examples.

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2. Integral lattices

2.1. Finite quadratic forms. A *finite quadratic form* is a finite abelian group $\mathcal{L}$ equipped with a nonsingular quadratic form, *i.e.*, a map $q\colon \mathcal{L} \to \mathbb{Q}/2\mathbb{Z}$ satisfying $q(x+y) = q(x) + q(y) + 2b(x,y)$ for all $x, y \in \mathcal{L}$ and some nonsingular symmetric bilinear form $b\colon \mathcal{L} \otimes \mathcal{L} \to \mathbb{Q}/\mathbb{Z}$. If q is understood, we write x^2 and $x \cdot y$ for $q(x)$ and $b(x,y)$, respectively.

The bilinear form b is determined by q; it is called the bilinear form *associated* with q, and q is called a *quadratic extension* of b.

The group of automorphisms of $\mathcal{L}$ preserving q is denoted by $\operatorname{Aut}\mathcal{L}$.

The *Brown invariant* of a finite quadratic form $\mathcal{L}$ is the residue $\operatorname{Br}\mathcal{L} \in \mathbb{Z}/8\mathbb{Z}$ defined *via* the Gauss sum

$$\exp\bigl(\tfrac{1}{4}i\pi \operatorname{Br}\mathcal{L}\bigr) = |\mathcal{L}|^{-\frac{1}{2}} \sum_{x \in \mathcal{L}} \exp\bigl(i\pi x^2\bigr).$$

The Brown invariant is additive: $\operatorname{Br}(\mathcal{L}_1 \oplus \mathcal{L}_2) = \operatorname{Br}\mathcal{L}_1 + \operatorname{Br}\mathcal{L}_2$.

Clearly, each finite quadratic form $\mathcal{L}$ splits canonically into orthogonal sum of its primary components: $\mathcal{L} = \bigoplus \mathcal{L} \otimes \mathbb{Z}_p$, summation over all primes p. For a prime p, let $\mathcal{L}_p = \mathcal{L} \otimes \mathbb{Z}_p$ be the p-primary part of $\mathcal{L}$. Denote by $\ell(\mathcal{L})$ the minimal number of generators of $\mathcal{L}$, and let $\ell_p(\mathcal{L}) = \ell(\mathcal{L}_p)$. Obviously, $\ell(\mathcal{L}) = \max_p \ell_p(\mathcal{L})$.

For a fraction $\frac{m}{n} \in \mathbb{Q}/2\mathbb{Z}$ with $(m,n) = 1$ and $mn = 0 \bmod 2$, let $\langle \frac{m}{n} \rangle$ be the nondegenerate quadratic form on $\mathbb{Z}/n\mathbb{Z}$ sending the generator to $\frac{m}{n}$. For an integer

$k \geqslant 1$, let $\mathcal{U}_{2^k}$ and $\mathcal{V}_{2^k}$ be the quadratic forms on the group $(\mathbb{Z}/2^k\mathbb{Z})^2$ defined by the matrices

$$\mathcal{U}_{2^k} = \begin{bmatrix} 0 & \alpha_k \\ \alpha_k & 0 \end{bmatrix}, \qquad \mathcal{V}_{2^k} = \begin{bmatrix} \alpha_{k-1} & \alpha_k \\ \alpha_k & \alpha_{k-1} \end{bmatrix}, \quad \text{where } \alpha_k = \frac{1}{2^k}.$$

(When speaking about the matrix of a finite quadratic form, we assume that the diagonal elements are defined modulo $2\mathbb{Z}$ whereas all other elements are defined modulo $\mathbb{Z}$.) According to Nikulin [29], each finite quadratic form is an orthogonal sum of cyclic summands $\langle \frac{m}{n} \rangle$ and summands of the form $\mathcal{U}_{2^k}$, $\mathcal{V}_{2^k}$. The Brown invariants of these elementary blocks are as follows: if p is an odd prime, then

$$\operatorname{Br}\Bigl\langle \frac{2a}{p^{2s-1}} \Bigr\rangle = 2\Bigl(\frac{a}{p}\Bigr) - \Bigl(\frac{-1}{p}\Bigr) - 1, \quad \operatorname{Br}\Bigl\langle \frac{2a}{p^{2s}} \Bigr\rangle = 0 \quad (\text{for } s \geqslant 1 \text{ and } (a,p) = 1).$$

If $p = 2$, then

$$\operatorname{Br}\Bigl\langle \frac{a}{2^k} \Bigr\rangle = a + \frac{1}{2}k(a^2 - 1) \bmod 8 \quad (\text{for } k \geqslant 1 \text{ and odd } a \in \mathbb{Z}),$$
$$\operatorname{Br}\mathcal{U}_{2^k} = 0, \quad \operatorname{Br}\mathcal{V}_{2^k} = 4k \bmod 8 \quad (\text{for all } k \geqslant 1).$$

Quite a number of relations, *i.e.*, isomorphisms between various combinations of the aforementioned forms, is also listed in [29]. These observations make the classification of finite quadratic forms rather straightforward, although tedious. Two simple known results used in the sequel are listed below. More details on quadratic forms on 2-primary groups can be found in [13] and [32].

2.1.1. Proposition. *Let $p \neq 2$ be an odd prime. Then a quadratic form on a group $\mathcal{L}$ of exponent p is determined by its rank $\ell(\mathcal{L}) = \ell_p(\mathcal{L})$ and Brown invariant* $\operatorname{Br}\mathcal{L}$.

A finite quadratic form is called *even* if x^2 is an integer for each element $x \in \mathcal{L}$ of order 2; otherwise, it is called *odd*. Clearly, a form is odd if and only if it contains $\langle \pm\frac{1}{2} \rangle$ as an orthogonal summand.

2.1.2. Proposition (see [37] or [19]). *A quadratic form on a group $\mathcal{L}$ of exponent 2 is determined by its rank $\ell(\mathcal{L}) = \ell_2(\mathcal{L})$, parity (even or odd), and Brown invariant* $\operatorname{Br}\mathcal{L}$.

2.2. Even integral lattices and discriminant forms. An (*integral*) *lattice* is a finitely generated free abelian group L equipped with a symmetric bilinear form $\varphi\colon L \otimes L \to \mathbb{Z}$. When the form is understood, we will freely use the multiplicative notation $u \cdot v = \varphi(u,v)$ and $u^2 = \varphi(u,u)$. A lattice L is called *even* if $u^2 = 0 \bmod 2$ for each $u \in L$; otherwise, it is called *odd*.

Since the transition matrix from one integral basis to another one has determinant ± 1, the *determinant* $\det L = \det\varphi \in \mathbb{Z}$ is well defined. The lattice L is called *non-degenerate* if $\det L \neq 0$; it is called *unimodular* if $\det L = \pm 1$. The *signature* of a non-degenerate lattice L is the pair $(\sigma_+ L, \sigma_- L)$ of its inertia indices. Recall that $\sigma_+ L$ is the dimension of any maximal positive definite subspace of the vector space $L \otimes \mathbb{R}$. Recall, further, that all maximal positive definite subspaces of $L \otimes \mathbb{R}$ can be oriented in a coherent way. For example, the orientations of two such subspaces ω_1,

ω_2 can be compared using the orthogonal projection $\omega_2 \to \omega_1$, which is necessarily injective and hence bijective.

Given a lattice L, we denote by $O(L)$ the group of isometries of L, and by $O^+(L) \subset O(L)$ its subgroup consisting of the isometries preserving the orientation of maximal positive definite subspaces. Clearly, either $O^+(L) = O(L)$ or $O^+(L) \subset O(L)$ is a subgroup of index 2. In the latter case, each element of $O(L) \smallsetminus O^+(L)$ is called a $+$-*disorienting* isometry. (The awkward terminology is chosen to avoid confusion with isometries reversing the orientation of L itself.)

If L is a non-degenerate lattice, the dual group $L^* = \operatorname{Hom}(L, \mathbb{Z})$ can be identified with the subgroup $\bigl\{x \in L\otimes\mathbb{Q} \bigm| x\cdot y \in \mathbb{Z} \text{ for all } y \in L\bigr\}$. The quotient L^*/L is called the *discriminant group* of L and is denoted by $\mathcal{L}$ or $\operatorname{discr} L$. One has $|\mathcal{L}| = |\det L|$ and $\ell(\mathcal{L}) \leqslant \operatorname{rk} L$. The discriminant group inherits from $L \otimes \mathbb{Q}$ a non-degenerate symmetric bilinear form $b\colon \mathcal{L} \otimes \mathcal{L} \to \mathbb{Q}/\mathbb{Z}$ and, if L is even, its quadratic extension $q\colon \mathcal{L} \to \mathbb{Q}/2\mathbb{Z}$. Thus, the discriminant of an even lattice is a finite quadratic form.

Two integral lattices L_1, L_2 are said to have the same *genus* if all their localizations $L_i \otimes \mathbb{R}$ and $L_i \otimes \mathbb{Q}_p$ are isomorphic (over $\mathbb{R}$ and $\mathbb{Q}_p$, respectively). Each genus is known to contain finitely many isomorphism classes. The relation between the genus of a lattice and its discriminant form is given by the following two statements (see also Section 2.5 below).

2.2.1. Theorem (see [29]). *The genus of an even integral lattice L is determined by its signature (σ_+L, σ_-L) and discriminant form* $\operatorname{discr} L$.

In what follows, the genus of even integral lattices determined by a signature (σ_+, σ_-) and a discriminant form $\mathcal{L}$ is referred to as the *genus* $(\sigma_+, \sigma_-; \mathcal{L})$.

2.2.2. Theorem (van der Blij formula, see [6]). *For any nondegenerate even integral lattice L one has* $\operatorname{Br} \mathcal{L} = \sigma_+L - \sigma_-L \bmod 8$.

Since the construction of the discriminant form $\mathcal{L}$ is natural, there is a canonical homomorphism $O(L) \to \operatorname{Aut} \mathcal{L}$. Its image is denoted by $\operatorname{Aut}_L \mathcal{L}$. Of special importance are so called reflections of L: given a vector $a \in L$, the *reflection* against the hyperplane orthogonal to a (for short, reflection defined by a) is the automorphism

$$t_a\colon L \to L, \quad x \mapsto x - 2\frac{a\cdot x}{a^2}a.$$

It is easy to see that t_a is an involution, *i.e.*, $t_a^2 = \mathrm{id}$. The reflection t_a is well defined whenever $a \in (a^2/2)L^*$. In particular, t_a is well defined if $a^2 = \pm1$ or ±2; in this case the induced automorphism of the discriminant group $\mathcal{L}$ is the identity and t_a extends to any lattice containing L.

2.3. Special lattices and notation. Given a lattice L and an integer n, we denote by $L(n)$ the lattice obtained by multiplying all values by n (*i.e.*, the quadratic form $x \mapsto nx^2$ defined on the same group L). For finite quadratic forms the multiplication operation is meaningful only for $n = -1$, and we abbreviate $-\mathcal{L} = \mathcal{L}(-1)$.

The notation nL, $n \geqslant 1$, stands for the direct sum of n copies of L.

The *hyperbolic plane* is the lattice $\mathbf{U}$ spanned by two vectors u, v so that $u^2 = v^2 = 0$, $u \cdot v = 1$. Any pair (u, v) as above is called a *standard basis* for $\mathbf{U}$. In fact, it is unique up to transposing u and v and multiplying one or both of them by (-1). The hyperbolic plane is an even unimodular lattice of signature $(1, 1)$.

A *root* in a lattice L is an element $v \in L$ of square -2. Given L, we denote by $\mathfrak{r}L \subset L$ the sublattice generated by all roots of L. A *root system* is a negative definite lattice generated by its roots. Every root system admits a unique decomposition into an orthogonal sum of irreducible root systems, the latter being either $\mathbf{A}_p$, $p \geqslant 1$, or $\mathbf{D}_q$, $q \geqslant 4$, or $\mathbf{E}_6$, $\mathbf{E}_7$, $\mathbf{E}_8$. The discriminant forms are as follows:

$$\operatorname{discr}\mathbf{A}_p = \langle -\tfrac{p}{p+1}\rangle, \quad \operatorname{discr}\mathbf{D}_{2k+1} = \langle -\tfrac{2k+1}{4}\rangle,$$
$$\operatorname{discr}\mathbf{D}_{8k\pm2} = 2\langle \mp\tfrac{1}{2}\rangle, \quad \operatorname{discr}\mathbf{D}_{8k} = \mathcal{U}_2, \quad \operatorname{discr}\mathbf{D}_{8k+4} = \mathcal{V}_2,$$
$$\operatorname{discr}\mathbf{E}_6 = \langle \tfrac{2}{3}\rangle, \quad \operatorname{discr}\mathbf{E}_7 = \langle \tfrac{1}{2}\rangle, \quad \operatorname{discr}\mathbf{E}_8 = 0.$$

The orthogonal group of a root system L is a semi-direct product of the group generated by reflections (defined by the roots of L), which acts simply transitively on the set of Weyl chambers of L, and the group of symmetries of any fixed Weyl chamber (or Dynkin graph) of L. As a consequence, the following statement holds:

2.3.1. Proposition. *For a root system L, the subgroup $\operatorname{Aut}_L\mathcal{L}$ coincides with the image in $\operatorname{Aut}\mathcal{L}$ of the group of symmetries of any fixed Weyl chamber.*

If L is an irreducible root systems other than $\mathbf{A}_p$ or $\mathbf{D}_q$ with $q = 8k+4 \geqslant 12$, one has $\operatorname{Aut}_L\mathcal{L} = \operatorname{Aut}\mathcal{L}$. If $L = \mathbf{A}_p$, the image $\operatorname{Aut}_L\mathcal{L}$ is the subgroup $\{\pm\operatorname{id}\}$. In the case $L = \mathbf{D}_{8k+4}$, $k \geqslant 1$, the full orthogonal group $\operatorname{Aut}\mathcal{L}$ is the group S_3 of permutations of the three elements of square 1 mod $2\mathbb{Z}$, whereas the image $\operatorname{Aut}_L\mathcal{L}$ is generated by one of the three transpositions.

Further details on irreducible root systems are found in N. Bourbaki [7].

2.4. Definite lattices of rank 2. Each positive definite even lattice N of rank 2 has a unique representation by a matrix of the form

$$\begin{bmatrix} 2a & b \\ b & 2c \end{bmatrix}, \quad 0 < a \leqslant c, \quad 0 \leqslant b \leqslant a. \tag{2.4.1}$$

Denote the lattice represented by (2.4.1) by $\mathbf{M}(a,b,c)$. Let (u,v) be a basis in which the quadratic form is given by (2.4.1). Then, depending on a, b, and c, the orthogonal group $O(N)$ is one of the groups described below:

- $0 < b < a < c$: the group $O(N) \cong \mathbb{Z}/2\mathbb{Z}$ is generated by $-\operatorname{id}$;
- $0 < b < a = c$: the group $O(N) \cong \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ is generated by $-\operatorname{id}$ and the transposition $(u,v) \mapsto (v,u)$;
- $b = 0$, $a < c$: the group $O(N) \cong \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ is generated by t_u and t_v;
- $b = 0$, $a = c$: then $N = 2\mathbf{A}_1(-a)$ and $O(N) \cong \mathbb{D}_4$ is the group of symmetries of a square; it is generated by t_u and the transposition $(u,v) \mapsto (v,u)$;
- $b = a < c$: the group $O(N) \cong \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ is generated by $-\operatorname{id}$ and t_u;
- $b = a = c$: then $N = \mathbf{A}_2(-a)$ and $O(N) \cong \mathbb{D}_6$ is the group of symmetries of a regular hexagon; it is generated by t_u and the transposition $(u,v) \mapsto (v,u)$.

All results above are classical and well known. The inequalities $a \leqslant c$ and $|b| \leqslant a$ can be achieved by a sequence of transpositions $(u,v) \mapsto (v,u)$ and transformations $(u,v) \mapsto (u, v \pm u)$. Then, assuming that the matrix has the form (2.4.1), for a vector $xu + yv \in N$ one has

$$(xu+yv)^2 = 2ax^2 + 2bxy + 2cy^2 \geqslant 2a(x^2+y^2) - 2a|xy| \geqslant a(x^2+y^2).$$

Since x and y are integers, it immediately follows that u is a shortest vector and, unless $a = c$, the only shortest vectors are $\pm u$. If $a = c$, there are two more shortest vectors $\pm v$, and if also $b = a$, there are yet two more, $\pm(u - v)$. From here, one can easily deduce the uniqueness of representation (2.4.1). The description of the orthogonal group is also straightforward: one observes that u should be taken to a shortest vector and then, assuming u fixed, the only nontrivial isometry of the Euclidean plane $N \otimes \mathbb{R}$ is the reflection against the line spanned by u; it remains to enumerate the few cases when this reflection is defined over $\mathbb{Z}$.

2.5. Nikulin's existence and uniqueness results. Let p be a prime. The notion of lattice and its discriminant form extends to the case of finitely generated free $\mathbb{Z}_p$-modules. (In the case $p = 2$, to define the quadratic form on the discriminant group one still needs to require that the lattice should be even.) The discriminant of a p-adic lattice L_p is a finite $\mathbb{Z}_p$-module $\mathcal{L}_p$ (in other words, $p^k \mathcal{L}_p = 0$ for some k large enough), and one has $|\mathcal{L}_p| = |\det L_p| \bmod \mathbb{Z}_p^*$. For an integral lattice L one has $\operatorname{discr}(L \otimes \mathbb{Z}_p) = (\operatorname{discr} L) \otimes \mathbb{Z}_p = \mathcal{L}_p$.

According to Nikulin [29], given a prime p and a $\mathbb{Q}/2\mathbb{Z}$-valued quadratic form on a finite $\mathbb{Z}_p$-module $\mathcal{L}$, there is a p-adic lattice L such that $\operatorname{rk} L = \ell_p(\mathcal{L})$ and $\operatorname{discr} L = \mathcal{L}$. Unless $p = 2$ and $\mathcal{L}$ is odd, such a lattice L is determined by $\mathcal{L}$ uniquely up to isomorphism; in particular, the ratio $\det L/|\mathcal{L}|$ is a well defined element of the group $\mathbb{Z}_p^*/(\mathbb{Z}_p^*)^2$. We will denote it by $\det_p \mathcal{L}$. In the exceptional case $p = 2$, $\mathcal{L}$ odd there are two lattices L as above, the ratio of their determinants being $5 \in \mathbb{Z}_2^*/(\mathbb{Z}_2^*)^2$.

2.5.1. Theorem (see Theorem 1.10.1 in [29]). *Let $\mathcal{L}$ be a finite quadratic form and let $\sigma_\pm$ be a pair of integers. Then, the following four conditions are necessary and sufficient for the existence of an even integral lattice L whose signature is (σ_+, σ_-) and whose discriminant form is $\mathcal{L}$*:

(1) $\sigma_\pm \geqslant 0$ *and* $\sigma_+ + \sigma_- \geqslant \ell(\mathcal{L})$;
(2) $\sigma_+ - \sigma_- = \operatorname{Br} \mathcal{L} \bmod 8$;
(3) *for each* $p \neq 2$, *either* $\sigma_+ + \sigma_- > \ell_p(\mathcal{L})$ *or* $\det_p \mathcal{L}_p = (-1)^{\sigma_-} \bmod (\mathbb{Z}_p^*)^2$;
(4) *either* $\sigma_+ + \sigma_- > \ell_2(\mathcal{L})$, *or* $\mathcal{L}_2$ *is odd, or* $\det_2 \mathcal{L}_2 = \pm 1 \bmod (\mathbb{Z}_2^*)^2$.

2.5.2. Theorem (see Theorem 1.13.2 in [29]). *Let L be an indefinite even integral lattice,* $\operatorname{rk} L \geqslant 3$. *The following two conditions are sufficient for L to be unique in its genus*:

(1) *for each* $p \neq 2$, *either* $\operatorname{rk} L \geqslant \ell_p(\mathcal{L}) + 2$ *or* $\mathcal{L}_p$ *contains a subform isomorphic to* $\langle a/p^k \rangle \oplus \langle b/p^k \rangle$, $k \geqslant 1$, *as an orthogonal summand*;
(2) *either* $\operatorname{rk} L \geqslant \ell_2(\mathcal{L}) + 2$ *or* $\mathcal{L}_2$ *contains a subform isomorphic to* $\mathcal{U}_{2^k}$, $\mathcal{V}_{2^k}$, *or* $\langle a/2^k \rangle \oplus \langle b/2^{k+1} \rangle$, $k \geqslant 1$, *as an orthogonal summand.*

2.5.3. Theorem (see Theorem 1.14.2 in [29]). *Let L be an indefinite even integral lattice,* $\operatorname{rk} L \geqslant 3$. *The following two conditions are sufficient for L to be unique in its genus and for the canonical homomorphism* $O(L) \to \operatorname{Aut} \mathcal{L}$ *to be onto*:

(1) *for each* $p \neq 2$, $\operatorname{rk} L \geqslant \ell_p(\mathcal{L}) + 2$;
(2) *either* $\operatorname{rk} L \geqslant \ell_2(\mathcal{L}) + 2$ *or* $\mathcal{L}_2$ *contains a subform isomorphic to* $\mathcal{U}_2$ *or* $\mathcal{V}_2$ *as an orthogonal summand.*

2.6. Extensions. From now on we confine ourselves to even lattices. An *extension* of an even lattice S is an even lattice L containing S. Two extensions $L_1 \supset S$ and $L_2 \supset S$ are called *isomorphic* (*strictly isomorphic*) if there is an isomorphism $L_1 \to L_2$ preserving S (respectively, identical on S). More generally, one can fix a subgroup $A \subset O(S)$ and speak about *A-isomorphisms* and *A-automorphisms* of extension, *i.e.*, isometries whose restriction to S belongs to A.

Any extension $L \supset S$ of finite index admits a unique embedding $L \subset S \otimes \mathbb{Q}$. If S is nondegenerate, then L belongs to S^* and thus defines a subgroup $\mathcal{K} = L/S \subset \mathcal{S}$, called the *kernel* of the extension. Since L itself is an integral lattice, the kernel $\mathcal{K}$ is isotropic, *i.e.*, the restriction to $\mathcal{K}$ of the discriminant quadratic form is identically zero. Conversely, given an isotropic subgroup $\mathcal{K} \subset \mathcal{S}$, the subgroup

$$L = \bigl\{x \in S^* \mid (x \bmod S) \in \mathcal{K}\bigr\} \subset S^*$$

is an extension of S. Thus, the following statement holds:

2.6.1. Proposition (see [29]). *Let S be a nondegenerate even lattice, and fix a subgroup $A \subset O(S)$. The map $L \mapsto \mathcal{K} = L/S \subset \mathcal{S}$ establishes a one-to-one correspondence between the set of A-isomorphism classes of finite index extensions $L \supset S$ and the set of A-orbits of isotropic subgroups $\mathcal{K} \subset \mathcal{S}$. Under this correspondence, one has* $\operatorname{discr} L = \mathcal{K}^\perp/\mathcal{K}$.

An isometry $f\colon S \to S$ extends to a finite index extension $L \supset S$ defined by an isotropic subgroup $\mathcal{K} \subset \mathcal{S}$ if and only if the automorphism $\mathcal{S} \to \mathcal{S}$ induced by f preserves $\mathcal{K}$ (as a set).

2.6.2. Remark. Since a finite index extension $L \supset S$ has the same signature as S, Proposition 2.6.1 implies, in particular, that $\operatorname{Br}(\mathcal{K}^\perp/\mathcal{K}) = \operatorname{Br}\mathcal{S}$ for any isotropic subgroup $\mathcal{K} \subset \mathcal{S}$. This observation facilitates the calculation of the Brown invariant; for example, it can be used to reduce the list of values of Br given in Section 2.1 to a few special cases.

2.6.3. Corollary. *Any imprimitive extension of a root system $S = 3\mathbf{A}_2$, $\mathbf{A}_5 \oplus \mathbf{A}_2$, $\mathbf{A}_8$, $\mathbf{E}_6 \oplus \mathbf{A}_2$, $2\mathbf{A}_4$, $\mathbf{A}_5 \oplus \mathbf{A}_1$, $\mathbf{A}_7$, $\mathbf{D}_8$, $\mathbf{E}_7 \oplus \mathbf{A}_1$, $4\mathbf{A}_1$, $\mathbf{A}_3 \oplus 2\mathbf{A}_1$, or $\mathbf{D}_q \oplus 2\mathbf{A}_1$ with $q < 12$ or $q \neq 0 \bmod 4$ contains a finite index extension $R \supset S$, where R is a root system strictly larger than S.*

Proof. The extensions are easily enumerated using Proposition 2.6.1. (In fact, in all cases except $S = \mathbf{D}_8 \oplus 2\mathbf{A}_1$, a nontrivial finite order extension is unique up to isometry.) The statement follows then from a direct calculation, using the fact that each lattice $\mathbf{E}_6$, $\mathbf{E}_7$, $\mathbf{E}_8$ is unique in its genus and the known embedding $2\mathbf{A}_1 \subset \mathbf{D}_q$ with $(2\mathbf{A}_1)^\perp_{\mathbf{D}_q} \cong \mathbf{D}_{q-2}$. $\square$

Another extreme case is when $S \subset L$ is a primitive nondegenerate sublattice and L is a unimodular lattice. Then L is a finite index extension of the orthogonal sum $S \oplus S^\perp$, both S and $S^\perp$ being primitive in L. Since $\operatorname{discr} L = 0$, the kernel $\mathcal{K} \subset \mathcal{S} \oplus \operatorname{discr} S^\perp$ is the graph of an anti-isometry $\mathcal{S} \to \operatorname{discr} S^\perp$. Conversely, given a lattice N and an anti-isometry $\kappa\colon \mathcal{S} \to \mathcal{N}$, the graph of κ is an isotropic subgroup $\mathcal{K} \subset \mathcal{S} \oplus \mathcal{N}$ and the resulting extension $L \supset S \oplus N \supset S$ is a unimodular primitive extension of S with $S^\perp \cong N$.

Let N and $\kappa\colon \mathcal{S} \to \mathcal{N}$ be as above, and let $s\colon S \to S$ and $t\colon N \to N$ be a pair of isometries. Then the direct sum $s \oplus t\colon S \oplus N \to S \oplus N$ preserves the graph of κ (and, thus, extends to L) if and only if $\kappa \circ s = t \circ \kappa$. (We use the same notation s and t for the induced homomorphisms on $\mathcal{S}$ and $\mathcal{N}$, respectively.) Summarizing, one obtains the following statement:

2.6.4. Proposition (see [29]). *Let S be a nondegenerate even lattice, and let s_+, s_- be nonnegative integers. Fix a subgroup $A \subset O(S)$. Then the A-isomorphism class of a primitive extension $L \supset S$ of S to a unimodular lattice L of signature (s_+, s_-) is determined by*

(1) *a choice of a lattice N in the genus $(s_+ - \sigma_+ S, s_- - \sigma_- S; -\mathcal{S})$, and*
(2) *a choice of a bi-coset of the canonical left-right action of $A \times \operatorname{Aut}_N \mathcal{N}$ on the set of anti-isometries $\mathcal{S} \to \mathcal{N}$.*

If a lattice N and an anti-isometry $\kappa\colon \mathcal{S} \to \mathcal{N}$ as above are chosen (and thus an extension L is fixed), an isometry $t\colon N \to N$ extends to an A-automorphism of L if and only if the composition $\kappa^{-1} \circ t \circ \kappa \in \operatorname{Aut} \mathcal{S}$ is in the image of A.

2.6.5. Remark. Proposition 2.6.4 can be regarded as the algebraic counterpart of the Meyer-Vietoris exact sequence of the gluing of two 4-manifolds *via* a diffeomorphism of their boundaries. The lattices in question are the intersection index forms on the 2-homology of the manifolds, and the discriminant forms are the linking coefficient forms on the 1-homology of the boundary. The anti-isometry κ as above is the homomorphism induced by the identification of the boundaries (which is orientation reversing). For more details, see, *e.g.*, O. Ivanov and N. Netsvetaev [20] and [21].

3. The moduli space

3.1. Plane sextics and $K3$-surfaces. A *rigid isotopy* of plane projective algebraic curves is an equisingular deformation or, equivalently, an isotopy in the class of algebraic curves. Since, in this paper, we deal with simple singularities only, the choice of a category (topological, smooth, piecewise linear) for this definition is irrelevant. Indeed, recall that one of the fifteen definitions of simple singularities, see [18], is that they are 0-modal, *i.e.*, their differential type is determined by their topological type.

Let $C \subset \mathbb{P}^2$ be a reduced sextic with simple singular points. Consider the following diagram:

$$\begin{array}{ccc} X & \longleftarrow & \bar{X} \\ {\scriptstyle p}\big\downarrow & & {\scriptstyle \bar{p}}\big\downarrow \\ \mathbb{P}^2 & \xleftarrow{\ \pi\ } & \bar{Y}, \end{array}$$

where X is the double covering of $\mathbb{P}^2$ branched at C, $\bar{X}$ is the minimal resolution of singularities of X, and $\bar{Y}$ is the minimal embedded resolution of singularities of C such that all odd order components of the divisorial pull-back π^*C of C are nonsingular and disjoint. It is well known that X is a singular $K3$-surface and that $\bar{X}$ is a double covering of $\bar{Y}$ ramified at the union of the odd order components of π^*C.

Let $L_X = H_2(\bar{X})$; it is a lattice isomorphic to $2\mathbf{E}_8 \oplus 3\mathbf{U}$. (In what follows we identify the homology and cohomology of $\bar{X}$ *via* the Poincaré duality isomorphism.) Introduce the following vectors and sublattices:

- $\sigma_X \subset L_X$, the set of the classes of the exceptional divisors appearing in the blow-up $\bar{X} \to X$;
- $\Sigma_X \subset L_X$, the sublattice generated by σ_X;
- $h_X \in L_X$, the pull-back of the hyperplane section class $[\mathbb{P}^1] \in H_2(\mathbb{P}^2)$;

- $S_X = \Sigma_X \oplus \langle h_X \rangle \subset L_X$;
- $\tilde{\Sigma}_X \subset \tilde{S}_X \subset L_X$, the primitive hulls of Σ_X and S_X, respectively;
- $\omega_X \subset L_X \otimes \mathbb{R}$, the oriented 2-subspace spanned by the real and imaginary parts of the class of a holomorphic 2-form on $\bar{X}$ (the 'period' of $\bar{X}$).

Clearly, the isomorphism class of the collection (L_X, h_X, σ_X) is both a deformation invariant of curve C and a topological invariant of pair $(\bar{Y}, \pi^*C)$; it is called the *homological type* of C. By an isomorphism between two collections (L', h', σ') and (L'', h'', σ'') we mean an isometry $L' \to L''$ taking h' and σ' onto h'' and σ'', respectively.

Recall that ω_X is a positive definite subspace and that the Picard group $\operatorname{Pic}\bar{X}$ can be identified with the lattice $\omega_X^\perp \cap L_X$. In particular, $\omega_X \in \tilde{S}_X^\perp \otimes \mathbb{R}$. Recall also that the Kähler cone V_X^+ of $\bar{X}$ can be given by

$$V_X^+ = \bigl\{ x \in \omega_X^\perp \bigm| x^2 > 0 \text{ and } x \cdot [E] > 0 \text{ for any } (-2)\text{-curve } E \subset \bar{X} \bigr\}.$$

The projectivization $\mathbb{P}(V_X^+)$ is one of the (open) fundamental polyhedra of the group of motions of the hyperbolic space $\mathbb{P}(\{x \in \omega_X^\perp \,|\, x^2 > 0\})$ generated by the reflections defined by the roots of $\operatorname{Pic}\bar{X}$. The walls bounding V_X^+ are precisely those defined by the classes of the irreducible (-2)-curves in $\bar{X}$, and the integral classes in the closure $\overline{V}_X^+$ are the numerically effective divisors on $\bar{X}$.

In particular, σ_X is a 'standard' basis of the root system Σ_X, so that the cone

$$W_X = \bigl\{ x \in \Sigma_X \otimes \mathbb{R} \bigm| x \cdot r > 0 \text{ for each } r \in \sigma_X \bigr\}$$

is a Weyl chamber of Σ_X. Clearly, W_X and σ_X determine each other.

3.1.1. Remark. Instead of the oriented real subspace ω_X one often considers the Hodge subspace $H^{2,0}(X) \subset L_X \otimes \mathbb{C}$ or, equivalently, the class $\omega_{\mathbb{C}} \in L_X \otimes \mathbb{C}$ of a holomorphic 2-form on $\bar{X}$, the latter being defined up to a nonzero factor and satisfying the conditions $\omega_{\mathbb{C}}^2 = \bar{\omega}_{\mathbb{C}}^2 = 0$, $\omega_{\mathbb{C}} \cdot \bar{\omega}_{\mathbb{C}} > 0$. Then ω_X is the real part of the space $H^{2,0} \oplus \overline{H^{2,0}}$, or, equivalently, ω_X is spanned by the real part $\operatorname{Re}\omega_{\mathbb{C}}$ and imaginary part $\operatorname{Im}\omega_{\mathbb{C}}$. Conversely, $\omega_{\mathbb{C}}$ can be recovered as $x + iy$, where x, y is any positively oriented orthonormal basis for ω_X.

3.2. Homological types. The set of simple singularities of a plane sextic $C \subset \mathbb{P}^2$ can be viewed as a root system Σ with a distinguished 'standard' basis σ, or, equivalently, a distinguished Weyl chamber

$$W = \bigl\{ x \in \Sigma \otimes \mathbb{R} \bigm| x \cdot r > 0 \text{ for each } r \in \sigma \bigr\}.$$

Similar to Section 3.1, let $S = \Sigma \oplus \langle h \rangle$, $h^2 = 2$. One has $\mathcal{S} = \operatorname{discr}\Sigma \oplus \langle \frac{1}{2} \rangle$. An isometry of S is called *admissible* if it preserves both h and σ (as a set). The group $O_h(S) \subset O(S)$ of admissible isometries is the group of symmetries of the distinguished Weyl chamber W. Hence, its image $\operatorname{Aut}_h \mathcal{S} \subset \operatorname{Aut}\mathcal{S}$ coincides with the subgroup $\operatorname{Aut}_\Sigma \operatorname{discr}\Sigma$, see Proposition 2.3.1. In particular, $\operatorname{Aut}_h \mathcal{S}$ is independent of the choice of σ.

3.2.1. Definition. Let Σ and h be as above. A *configuration* is a finite index extension $\tilde{S} \supset S = \Sigma \oplus \langle h \rangle$ satisfying the following conditions:

1. $\mathfrak{r}\tilde{\Sigma} = \Sigma$, where $\tilde{\Sigma} = h_{\tilde{S}}^\perp$ is the primitive hull of Σ in $\tilde{S}$ and $\mathfrak{r}\tilde{\Sigma} \subset \tilde{\Sigma}$ is the sublattice generated by the roots of $\tilde{\Sigma}$, see 2.3;
2. there is no root $r \in \Sigma$ such that $\frac{1}{2}(r + h) \in \tilde{S}$.

An isometry of a configuration $\tilde{S}$ is called *admissible* if it preserves S and induces an admissible isometry of S.

The group of admissible isometries of $\tilde{S}$ and its image in $\operatorname{Aut}\tilde{\mathcal{S}}$ are denoted by $O_h(\tilde{S})$ and $\operatorname{Aut}_h\tilde{\mathcal{S}}$, respectively. Since $\Sigma = \mathfrak{r}\tilde{\Sigma}$ is a characteristic sublattice of $\tilde{\Sigma} = h^\perp_{\tilde{S}}$, any isometry of $\tilde{S}$ preserving h preserves Σ. Hence, one has $\operatorname{Aut}_h\tilde{\mathcal{S}} = \bigl\{s \in \operatorname{Aut}_h\mathcal{S} \mid s(\mathcal{K}) \subset \mathcal{K}\bigr\}$, where $\mathcal{K}$ is the kernel of the extension $\tilde{S} \supset S$.

3.2.2. Definition. An *abstract homological type* (extending a fixed set of simple singularities (Σ, σ)) is an extension of the orthogonal sum $S = \Sigma \oplus \langle h\rangle$, $h^2 = 2$, to a lattice L isomorphic to $2\mathbf{E}_8 \oplus 3\mathbf{U}$ so that the primitive hull $\tilde{S}$ of S in L is a configuration. An *isomorphism* between two abstract homological types $L_i \supset S_i \supset \sigma_i \cup \{h_i\}$, $i = 1, 2$, is an $O_h(S)$-isomorphism of extensions, see Section 2.6; in other words, it is an isometry $L_1 \to L_2$ taking h_1 to h_2 and σ_1 to σ_2 (as a set).

An abstract homological type is uniquely determined by the collection (L, h, σ); then Σ is the sublattice spanned by σ, and $S = \Sigma \oplus \langle h\rangle$. The lattices $\tilde{\Sigma}$ and $\tilde{S}$ are defined as the respective primitive hulls.

3.2.3. Definition. An *orientation* of an abstract homological type $\mathcal{H} = (L, h, \sigma)$ is a choice of one of the two orientations of positive definite 2-subspaces of the space $\tilde{S}^\perp \otimes \mathbb{R}$. (Recall that $\sigma_+\tilde{S}^\perp = 2$ and, hence, all positive definite 2-subspaces of $\tilde{S}^\perp \otimes \mathbb{R}$ can be oriented in a coherent way.) The type $\mathcal{H}$ is called *symmetric* if $(\mathcal{H}, \theta)$ is isomorphic to $(\mathcal{H}, -\theta)$ (for some orientation θ of $\mathcal{H}$). In other words, $\mathcal{H}$ is symmetric if it has an automorphism whose restriction to $\tilde{S}^\perp$ is $+$-disorienting.

3.3. Marked sextics. Let (Σ, σ) and $S = \Sigma \oplus \langle h\rangle$ be as in Section 3.2, and fix an extension $L \supset S$ with $L \cong 2\mathbf{E}_8 \oplus 3\mathbf{U}$. A *marking* (more precisely, (L, h, σ)*-marking*) of a singular plane sextic $C \subset \mathbb{P}^2$ is an isometry $\varphi\colon L_X \to L$ taking h_X and σ_X onto h and σ, respectively (see Section 3.1 for the notation). A *marked* sextic is a sextic supplied with a distinguished marking.

The following statement, based on the surjectivity of the period map for $K3$-surfaces, is essentially contained in T. Urabe [35].

3.3.1. Proposition. *Let (L, h, σ) be a collection as above, and let ω be an oriented positive definite 2-subspace in $S_L^\perp \otimes \mathbb{R}$. Then there exists a singular plane sextic $C \subset \mathbb{P}^2$ and an (L, h, σ)-marking $\varphi\colon L_X \to L$ taking ω_X to ω if and only if the following conditions are satisfied*:

(1) *(L, h, σ) is an abstract homological type*;
(2) *every root $r \in L$ orthogonal to both h and ω belongs to Σ.*

We precede the proof with a lemma. Denote by Γ the group generated by the reflections defined by the roots of the lattice $\omega^\perp \cap L$.

3.3.2. Lemma. *Let (L, h, σ) and ω be as in Proposition* 3.3.1, *and assume that conditions* 3.3.1(1), (2) *are satisfied. Then there is a unique open convex cone $V^+ = V^+(\omega) \subset \omega^\perp$ such that*

- *the projectivization $\mathbb{P}(V_X^+)$ is one of the fundamental polyhedra of the action of Γ on the hyperbolic space $\mathbb{P}(\{x \in \omega^\perp \mid x^2 > 0\})$;*
- *the closure $\overline{V}^+$ contains h;*
- *the intersection $V^+ \cap (\Sigma \otimes \mathbb{R})$ is the Weyl chamber W defined by σ.*

Proof. Condition 3.3.1(2) implies that W extends to a Weyl chamber W' in the negative definite space $h^{\perp}_{\omega^{\perp}}$; it is characterized by the requirement that $W' \cdot r > 0$ for each $r \in \sigma$. Then, Vinberg's algorithm [36] applied to h extends $\mathbb{P}(W')$ to a unique fundamental polyhedron P of Γ whose closure $\overline{P}$ contains the class $h/\mathbb{R}^*$. The connected component of the cone $\{x \in \omega^{\perp} \mid x/\mathbb{R}^* \in P\}$ containing W is the desired cone V^+. $\square$

Proof of Proposition 3.3.1. In the presence of (2), condition (1) is equivalent to the requirement that

(3) there is no element $u \in \omega^{\perp} \cap L$ with $u^2 = 0$ and $u \cdot h = 1$.

In this form, it is obvious that the conditions are necessary: (3) is necessary for the linear system h to define a degree 2 map $\bar{X} \to \mathbb{P}^2$, see [35], and (2) means that the curves contracted by this map are exactly those defined by the elements of σ, *i.e.*, the sextic does have the prescribed set of singularities.

Prove the sufficiency. Due to the surjectivity of the period map, there is a $K3$-surface $\bar{X}$ and an isomorphism $\varphi\colon H_2(\bar{X}) \to L$ taking ω_X to ω. The image $\varphi(V^+_X)$ of the the Kähler cone V^+_X of $\bar{X}$ is a fundamental domain of the action of Γ on one of the two halves of the positive cone $\{x \in \omega^{\perp} \mid x^2 > 0\}$. Hence, composing φ with an element of Γ and, if necessary, multiplication by -1, one can assume that φ takes V^+_X to the cone $V^+(\omega)$ given by Lemma 3.3.2. Then the pull-back $h_X = \varphi^{-1}(h)$ belongs to the closure $\overline{V}_X$; hence, it is numerically effective and, due to condition (3), it defines a degree 2 map $p\colon \bar{X} \to \mathbb{P}^2$, see [35]. The elements of the pull-back $\sigma_X = \varphi^{-1}(\sigma)$ define (some of) the walls of the Kähler cone and, hence, are realized by irreducible (-2)-curves in $\bar{X}$; due to condition (2), they are all the (-2)-curves contracted by p. Thus, φ is the desired marking. $\square$

3.4. Moduli. In view of Proposition 3.3.1, when speaking about (L, h, σ)-marked sextics, one can assume that $\mathcal{H} = (L, h, \sigma)$ is an abstract homological type. Since the period ω_X changes continuously within a family, the orientation of the image $\varphi(\omega_X)$ is an additional discrete invariant of deformations in the class of marked plane sextics.

3.4.1. Theorem. *For each abstract homological type $\mathcal{H} = (L, h, \sigma)$ there are exactly two rigid isotopy classes of $\mathcal{H}$-marked plane sextics. They differ by the orientation of the positive definite* 2*-subspace $\varphi(\omega_X) \subset \tilde{S}^{\perp} \otimes \mathbb{R}$.*

Proof. The existence of at least two rigid isotopy classes that differ by the orientation of $\varphi(\omega_X)$ is given by Proposition 3.3.1. Thus, it suffices to show that any two $\mathcal{H}$-marked $K3$-surfaces $(\bar{X}_0, \varphi_0)$, $(\bar{X}_1, \varphi_1)$ satisfying 3.3.1(2) and such that the images $\varphi_t(\omega_X)$, $t = 0, 1$, have coherent orientations can be connected by a family $(\bar{X}_t, \varphi_t)$, $t \in [0, 1]$ of $\mathcal{H}$-marked $K3$-surfaces still satisfying 3.3.1(2). Then the linear systems $h_t = \varphi_t^{-1}(h)$ would define a family of degree 2 maps $\bar{X}_t \to \mathbb{P}^2$ and, since 3.3.1(2) holds for each t, the resulting family $C_t \in \mathbb{P}^2$ of the branch curves would be equisingular.

Consider the space $\widetilde{\Omega}$ of pairs (ω, ρ), where $\omega \subset L \otimes \mathbb{R}$ is an oriented positive definite 2-subspace and $\rho \in L \otimes \mathbb{R}$ is a positive vector ($\rho^2 > 0$) orthogonal to ω. Let

$$\widetilde{\Omega}_0 = \widetilde{\Omega} \smallsetminus \bigcup_{r \in L,\, r^2 = -2} \bigl\{(\omega, \rho) \in \widetilde{\Omega} \bigm| \omega \cdot r = \rho \cdot r = 0\bigr\}.$$

According to A. Beauville [5], $\widetilde{\Omega}$ is a fine period space of marked quasi-polarized $K3$-surfaces, a *quasi-polarization* being a class of a Kähler metric.

Let $\Omega(\mathcal{H}) \cong O(2,d)/SO(2) \times O(d)$ be the space of oriented positive definite 2-subspaces $\omega \subset S^\perp \otimes \mathbb{R}$ (here $d = 19 - \operatorname{rk} S$), and let $\Omega_0(\mathcal{H}) \subset \Omega(\mathcal{H})$ be the set of subspaces ω satisfying 3.3.1(2). Since $\mathcal{H}$ is an abstract homological type, $\Omega_0(\mathcal{H})$ is obtained from $\Omega(\mathcal{H})$ by removing a countable number of codimension 2 subspaces $H_r = \{\omega \mid \omega \cdot r = 0\}$, $r \in h^\perp \smallsetminus \Sigma$, $r^2 = -2$. Condition 3.2.1(1) implies that none of H_r coincides with $\Omega(\mathcal{H})$ and, hence, $\Omega_0(\mathcal{H})$ is nonempty. Since $\Omega(\mathcal{H})$ has two connected components, so does $\Omega_0(\mathcal{H})$. The components differ by the orientation of the subspaces.

Now, let $\widetilde{\Omega}_0(\mathcal{H}) \subset \widetilde{\Omega}_0$ be the subspace $\{(\omega,\rho) \mid \omega \in \Omega_0(\mathcal{H}),\ \rho \in V^+(\omega)\}$, where $V^+(\omega)$ is the cone given by Lemma 3.3.2. In view of Proposition 3.3.1 and Lemma 3.3.2, Beauville's result cited above implies that $\widetilde{\Omega}_0(\mathcal{H})$ is a fine period space of $\mathcal{H}$-marked quasi-polarized plane sextics. On the other hand, the natural projection $\widetilde{\Omega}_0(\mathcal{H}) \to \Omega_0(\mathcal{H})$, $(\omega,\rho) \mapsto \omega$, has contractible fibers (the cones $V^+(\omega)$) and, outside a countable union of codimension 2 subsets H_r, $r \in L \smallsetminus \Sigma$, $r^2 = -2$, it is a locally trivial fibration. Hence, the period space $\widetilde{\Omega}_0(\mathcal{H})$ has two connected components, and the statement follows. □

3.4.2. Theorem. *The map sending a plane sextic $C \subset \mathbb{P}^2$ to the pair consisting of its homological type (L_X, h_X, σ_X) and the orientation of the space ω_X establishes a one-to-one correspondence between the set of rigid isotopy classes of sextics with a given set of singularities (Σ, σ) and the set of isomorphism classes of oriented abstract homological types extending (Σ, σ).*

Proof. The statement is an immediate consequence of Theorem 3.4.1 and the obvious fact that any two $\mathcal{H}$-markings of a given sextic differ by an isometry of the abstract homological type $\mathcal{H}$. □

3.4.3. Remark. Marked sextics satisfying conditions 3.3.1(1) and (2) can be regarded as ample marked $\tilde{S}$-polarized $K3$-surfaces in the sense of Nikulin [30]. (The ampleness of the polarization follows from condition 3.3.1(2).) Their period space is constructed in Dolgachev [16], based directly on the global Torelli theorem and the surjectivity of the period map. It is shown that the period space has two connected components interchanged by the complex conjugation.

3.5. Proof of Theorem 1.1.1. The 'only if' part of the statement is obvious. We will prove the 'if' part under the assumption that C_1 has at least one singular point. (Otherwise the two sextics are nonsingular and, hence, rigidly isotopic.)

The regularity condition implies that f preserves the complex orientations of both $\mathbb{P}^2$ and C_1; in particular, the induced map $f_*\colon H_2(\mathbb{P}^2) \to H_2(\mathbb{P}^2)$ takes $[\mathbb{P}^1]$ to $[\mathbb{P}^1]$. Furthermore, f lifts to a diffeomorphism $\bar{Y}_1 \to \bar{Y}_2$ and, hence, to a diffeomorphism $\bar{f}\colon \bar{X}_1 \to \bar{X}_2$ of the corresponding $K3$-surfaces (see Section 3.1 for the notation). The induced homomorphism $\bar{f}_*\colon L_{X_1} \to L_{X_2}$ takes h_{X_1} and σ_{X_1} to h_{X_2} and σ_{X_2}, respectively. Hence, for each marking $\varphi\colon L_{X_2} \to L$ of C_2 the composition $\varphi \circ \bar{f}_*$ is a marking of C_1. The crucial observation is the fact that, according to S. K. Donaldson [17], the map $\bar{f}_*$ induced by a *diffeomorphism* of $K3$-surfaces preserves the orientation of the (positive definite) 3-subspace spanned by the period ω_{X_1} and a Kähler class ρ_{X_1}. Since Kähler classes ρ_{X_1} and ρ_{X_2} can be chosen arbitrary close to h_{X_1} and h_{X_2}, respectively (recall that h_{X_1} and h_{X_2} belong to the closures of the respective Kähler cones), the latter assertion means that the orientations of $\varphi(\omega_{X_2})$ and $\varphi \circ \bar{f}_*(\omega_{X_2})$ agree, and Theorem 3.4.1 implies that C_1 and C_2 are rigidly isotopic. $\square$

4. Geometry of plain sextics

In this section, we discuss the relation between the geometry of a plane sextic and its homological type. We start with introducing several versions of the notion of Zariski pair (Section 4.1) and outlining Yang's algorithm recovering the combinatorial data of a curve from its configuration (Section 4.2). Sections 4.3 and 4.4 give a simple characterization of, respectively, reducible and abundant sextics.

4.1. Zariski pairs. Two reduced curves $C_1, C_2 \subset \mathbb{P}^2$ are said to have the same *combinatorial data* if there exist irreducible decompositions $C_i = C_{i,1} + \ldots + C_{i,k_i}$, $i = 1, 2$, such that:

(1) $k_1 = k_2$ and $\deg C_{1,j} = \deg C_{2,j}$ for all $j = 1, \ldots, k_1$;
(2) there is a one-to-one correspondence between the singular points of C_1 and those of C_2 preserving the topological types of the points;
(3) two singular points $P_i \in C_i$, $i = 1, 2$, corresponding to each other are related by a local homeomorphism such that if a branch at P_1 belongs to a component $B_{1,j}$ then its image belongs to $B_{2,j}$.

For an irreducible curve C, its combinatorial data are determined by the degree $\deg C$ and the set of topological types of the singularities of C.

One of the principal question in topology of plane curves is the extent to which the combinatorial data of a curve determine its global behavior. In order to formalize this question, Artal [1] suggested the notion of Zariski pair.

4.1.1. Definition. Two reduced curves $C_1, C_2 \subset \mathbb{P}^2$ are said to form a *Zariski pair* if

(1) C_1 and C_2 have the same combinatorial data, and
(2) the pairs $(\mathbb{P}^1, C_1)$ and $(\mathbb{P}^2, C_2)$ are not homeomorphic.

4.1.2. Remark. Cited above is the more suitable definition used in the subsequent papers. The original definition suggested in [1] requires, instead of 4.1.1(1), that the pairs (T_1, C_1) and (T_2, C_2) should be diffeomorphic, where T_i is a regular neighborhood of C_i, $i = 1, 2$. If the singularities involved are simple, the two definitions are equivalent as, on the one hand, simple singularities are 0-modal and, on the other hand, simple curve singularities are distinguished by their links (which is a straightforward consequence of their classification).

Condition (2) in Definition 4.1.1 varies from paper to paper: one can replace it with the negation of any reasonable 'global' equivalence relation. For example, instead of (2) it is sometimes required that the complements $\mathbb{P}^2 \smallsetminus C_1$ and $\mathbb{P}^2 \smallsetminus C_2$ should not be homeomorphic. Relevant for the present paper are the following notions:

– *regular Zariski pair*, with 4.1.1(2) replaced by 'the pairs $(\mathbb{P}^1, C_1)$ and $(\mathbb{P}^2, C_2)$ are not regularly diffeomorphic in the sense of Theorem 1.1.1;'
– *classical Zariski pair*, with 4.1.1(2) replaced by 'the Alexander polynomials $\Delta_{C_1}(t)$ and $\Delta_{C_2}(t)$ differ,' see Section 4.4 for details.

Theorems 1.1.1 and 3.4.2 state that, in order to construct examples of regular Zariski pairs, it suffices to find curves with the same combinatorial data but not isomorphic oriented homological types. The notion of classical Zariski pair is of a historical interest, as it was the Alexander polynomial that was used to distinguish the curves in the first examples. In Section 5.3 below we enumerate the deformation families of unnodal curves whose Alexander polynomial is not determined by the combinatorial data.

4.2. Configurations and combinatorial data. Let $C_1, C_2 \subset \mathbb{P}^2$ be a pair of reduced plane sextics with simple singularities. Consider the corresponding oriented homological types $(\mathcal{H}_i, \theta_i) = (L_i, h_i, \sigma_i, \theta_i)$, $i = 1, 2$, and related lattices Σ_i, $\tilde{S}_i$, *etc.*, see Section 3.1. (To simplify the notation we use index i instead of X_i.) Recall that the finite index extension $\tilde{S} \supset \Sigma \oplus \langle h \rangle$ is called a configuration, see Definition 3.2.1.

$$
\begin{array}{ccl}
\Sigma_1 \cong \Sigma_2 & \Longleftrightarrow & \text{C_1 and C_2 have the same set of singularities} \\
\Uparrow & & \qquad\qquad\Uparrow \\
(\tilde{S}_1, h_1, \sigma_1) \cong (\tilde{S}_2, h_2, \sigma_2) & \Longrightarrow & \text{C_1 and C_2 have the same combinatorial data;} \\
 & & \Delta_{C_1}(t) = \Delta_{C_2}(t) \\
\Uparrow & & \qquad\qquad\Uparrow \\
\mathcal{H}_1 \cong \mathcal{H}_2 & \Longleftrightarrow & \text{C_1 is rigidly isotopic to either C_2 or its conjugate $\bar{C}_2$} \\
\Uparrow & & \qquad\qquad\Uparrow \\
(\mathcal{H}_1, \theta_1) \cong (\mathcal{H}_2, \theta_2) & \Longleftrightarrow & \text{C_1 is rigidly isotopic to C_2;} \\
 & & \text{$(\mathbb{P}^2; C_1)$ and $(\mathbb{P}^2; C_2)$ are regularly diffeomorphic}
\end{array}
$$

Diagram 1

Diagram 1 represents various relations between the geometric properties of a curve and arithmetic properties of its homological type. The equivalence in the first line of the diagram is obvious; the equivalences in the last two lines are the statement of Theorem 3.4.2 and the fact that the two components of the period space are interchanged by the complex conjugation.

Informally, the implication in the second line of the diagram states that the configuration $\tilde{S}_X$ encodes the existence of various auxiliary curves passing in a prescribed way through the singular points of *each* curve deformation equivalent to C. In short, this assertion follows from the fact that $\tilde{S}_X$ is the Picard group of a generic curve of the deformation family. The precise algorithm recovering the combinatorial data of a sextic C from its configuration $\tilde{S}_X$ is outlined at the end of this section. The relation between the configuration and the Alexander polynomial in the case of irreducible curves is discussed in Section 4.4.

Note that the implication in the second line of Diagram 1 is not invertible. There are pairs of curves with the same combinatorial data and/or Alexander polynomial but not isomorphic configurations, see, *e.g.*, Theorem 5.3.2 and Proposition 5.4.6.

Yang's algorithm. Assume that a sextic C splits into irreducible components $C_1, \ldots, C_k$. Consider the fundamental classes $[C], [C_i] \in H^2(\bar{X})$ in the homology of the covering $K3$-surface. They realize certain elements c, c_i of the group $\tilde{S}_X \subset S_X^*$, so that $c = \sum_i c_i$. The classes c, c_i are recovered from the combinatorial data of C: one has $c \cdot h_X = 6$, $c_i \cdot h_X = \deg C_i$, and the intersections $c \cdot \sigma_j$, $c_i \cdot \sigma_j$ with the classes of the exceptional divisors are determined by the incidence of the curves in the minimal resolution of singularities. In fact, to each local branch b at a simple singular point P one can assign an element $\bar{\alpha}(b)$ of the group $(\Sigma_P)^*$ dual to the lattice Σ_P spanned by the exceptional divisors at P. Then, for a component C_i of C, one has

$$c_i = \frac{1}{2}(\deg C_i)h_X + \sum_{b \in C_i} \bar{\alpha}(b). \tag{4.2.1}$$

Explicit expressions for the elements $\bar{\alpha}(b)$ are found in [39]. Next lemma is an immediate consequence of these formulas.

4.2.2. Lemma. *If a simple singular point P has more than one branch $b_1, \ldots, b_k$, then each residue $\bar{\alpha}(b_i) \bmod \Sigma_P \in \operatorname{discr} \Sigma_P$ is an element of order 2; these residues are subject to the only relation $\sum_{i=1}^k \bar{\alpha}(b_i) = 0 \bmod \Sigma_P$.*

Now, one can ignore the geometric setting and consider a *virtual decomposition*, *i.e.*, a decomposition $c = \sum_i c_i$ determined by a hypothetical set of combinatorial data of C. Certainly, *a priori* one can only assert that $c_i \in S_X^*$. The set of all virtual decompositions of c is partially ordered by degeneration, so that $c = c$ is the minimal element. The following statement is based on the Riemann-Roch theorem for $K3$-surfaces.

4.2.3. Theorem (see Yang [39]). *The actual combinatorial data of an irreducible plane sextic C with simple singularities is the one corresponding to the only maximal element in the set of the virtual decompositions $c = \sum_i c_i$ with all $c_i \in \tilde{S}_X$.*

4.3. Reducible curves. In this section, C is a reduced plane curve of arbitrary degree $d = 4m+2$. We still assume that all singularities of C are simple. As in the case of sextics, consider the double covering X branched over C and its minimal resolution $\bar{X}$. Certainly, $\bar{X}$ is not a $K3$-surface; however, since $\bar{X}$ is diffeomorphic to the double covering branched over a nonsingular curve, one still has $\pi_1(\bar{X}) = 0$ and $L_X = H_2(\bar{X})$ is an even lattice.

All lattices $\Sigma_X \subset S_X = \Sigma_X \oplus \langle h_X \rangle \subset \tilde{S}_X \subset L_X$ introduced in Section 3.1 for sextics still make sense in the general case.

4.3.1. Theorem. *Let C be a reduced plane curve of degree $4m+2$ and with simple singularities only. Then C is reducible if and only if the kernel $\mathcal{K}$ of the extension $\tilde{S}_X \supset S_X$ has elements of order 2.*

Proof. If C_i is a proper component of C, the residue $c_i \bmod S_X \in \mathcal{K}$ given by (4.2.1) is an element of order 2. It is nontrivial since C_i must pass through a singular point of C that is not entirely contained in C_i. (Clearly, the calculation of c_i, including Lemma 4.2.2, still applies to the general case of curves of degree $4m+2$.)

Now, assume that C is irreducible. Denote by $\bar{C} \subset \bar{X}$ the set theoretical pull-back of C; it is the union of the exceptional divisors and the proper pull-back, which can be identified with C itself. Consider the fundamental group $\pi = \pi_1(\mathbb{P}^2 \smallsetminus C)$ and the homomorphism $\kappa\colon \pi \to \mathbb{Z}/2\mathbb{Z}$ defining the double covering $\bar{X} \smallsetminus \bar{C} \to \mathbb{P}^2 \smallsetminus C$. One has $\operatorname{Ker} \kappa = \pi_1(\bar{X} \smallsetminus \bar{C})$.

The abelinization $\pi/[\pi,\pi] = H_1(\mathbb{P}^2 \smallsetminus C)$ is the cyclic group $\mathbb{Z}/(4m+2)\mathbb{Z}$; its 2-primary part is $\mathbb{Z}/2\mathbb{Z}$, and from the exact sequence

$$\{1\} \to \operatorname{Ker}\kappa/(\operatorname{Ker}\kappa)^2 \to \pi/(\operatorname{Ker}\kappa)^2 \to \mathbb{Z}/2\mathbb{Z} \to \{1\}$$

and properties of 2-groups one concludes that the group $\operatorname{Ker}\kappa/(\operatorname{Ker}\kappa)^2 = H_1(\bar{X} \smallsetminus \bar{C}; \mathbb{F}_2)$ is trivial. Then, from the Poincaré duality and the fact that $H^3(\bar{X}; \mathbb{F}_2) = 0$ it follows that the inclusion homomorphism $H^2(\bar{X}; \mathbb{F}_2) \to H^2(\bar{C}; \mathbb{F}_2)$ is onto and its dual $H_2(\bar{C}; \mathbb{F}_2) \to H_2(\bar{X}; \mathbb{F}_2)$ is a monomorphism. On the other hand, since C is irreducible and $[C] = (2m+1)h_X \bmod \Sigma_X$, the inclusion induces an isomorphism $H_2(\bar{C}; \mathbb{F}_2) = S_X \otimes \mathbb{F}_2$. Thus, the (mod2)-reduction $S_X \otimes \mathbb{F}_2 \to \tilde{S}_X \otimes \mathbb{F}_2$ is a monomorphism. This fact implies that $\mathcal{K}$ has no elements of order 2. $\square$

4.3.2. Remark. In the case of sextics, Theorem 4.3.1 can as well be deduced from Theorem 4.2.3. However, this would require a thorough analysis of a number of exceptional cases and eliminating them using conditions 3.2.1(1) and (2) and an extended version of Proposition 2.6.3.

4.4. Classical Zariski pairs. The *Alexander polynomial* of a degree m plane curve $C \subset \mathbb{P}^2$ can be defined as the characteristic polynomial of the deck translation action on $H_1(\bar{X}_m; \mathbb{C})$, where $\bar{X}_m$ is the desingularization of the m-fold cyclic covering of $\mathbb{P}^2$ ramified at C (see A. Libgober [24]–[27] for details). By a *classical Zariski pair* we mean a pair of curves that have the same combinatorial data and differ by their Alexander polynomial. (The truly classical Zariski pair, due to Zariski himself [40], is a pair of irreducible sextics with six cusps each, one of them having all cusps on a conic, and the other one not.)

The Alexander polynomials $\Delta_C(t)$ of all irreducible sextics C are found in [8] (see also [11]), where it is shown that $\Delta_C(t) = (t^2 - t + 1)^d$ and the exponent d is determined by the set of singularities of C unless the latter has the form

$$\Sigma = e\mathbf{E}_6 \oplus \bigoplus_{i=1}^{6} a_i\mathbf{A}_{3i-1} \oplus n\mathbf{A}_1, \quad 2e + \sum ia_i = 6. \tag{4.4.1}$$

If the set of singularities is as in (4.4.1), then d may *a priori* take values 0 or 1; in the latter case the curve is called *abundant*. The following statement is proved in [11].

4.4.2. Theorem. *For an irreducible plane sextic C with a set of singularities Σ as in* (4.4.1), *the following three conditions are equivalent*:

1. *C is abundant*;
2. *C is* tame, i.e., *it is given by an equation of the form $f_2^3 + f_3^2 = 0$, where f_2 and f_3 are some polynomials of degree* 2 *and* 3, *respectively*;
3. *there is a conic Q whose local intersection index with C at each singular point of C of type $\mathbf{A}_{3i-1}$* (*respectively,* $\mathbf{E}_6$) *is* $2i$ (*respectively,* 4).

Observe that the discriminant group of each lattice $\mathbf{A}_{3i-1}$ or $\mathbf{E}_6$ has a unique subgroup isomorphic to $\mathbb{Z}/3\mathbb{Z}$. Its nontrivial elements are the residues of $\bar{\beta}^{(1,2)}$, where $\bar{\beta}^{(1)}$ is the element given in some standard basis $e_1, e_2, \ldots$ by

$$\bar{\beta}^{(1)} = \frac{1}{3}(2e_1 + 4e_2 + \ldots + 2ie_i + (2i-1)e_{i+1} + \ldots + e_{2i-1}) \in (\mathbf{A}_{2i-1})^*,$$

$$\bar{\beta}^{(1)} = \frac{1}{3}(4e_1 + 5e_2 + 6e_3 + 4e_4 + 2e_5 + 3e_6) \in (\mathbf{E}_6)^*,$$

and $\bar\beta^{(2)}$ is obtained from $\bar\beta^{(1)}$ by the only nontrivial symmetry of the Dynkin graph. (In the case $\mathbf{E}_6$, the basis elements are numbered so that e_6 is attached to the short edge of the graph.) The following theorem characterizes abundant curves in terms of configurations.

4.4.3. Theorem. *Let C be a plane sextic with a set of singularities Σ as in* (4.4.1). *Then a reduced conic Q as in* 4.4.2(3) *exists if and only if the kernel $\mathcal{K}$ of the extension $\tilde S_X \supset S_X$ has* 3*-torsion. If this is the case, the* 3*-primary part of $\mathcal{K}$ is a cyclic group of order* 3 *generated by a residue of the form $\sum \bar\beta_i^{(1,2)} \bmod S_X$, where $\bar\beta_i^{(1,2)}$ are the elements defined above and the sum contains exactly one element for each singular point of C other than $\mathbf{A}_1$.*

Proof. Assume that a conic Q exists. Resolving the singularities, one can see that the proper pull-back of Q in $\bar Y$ does not intersect the branch locus; hence, Q lifts to a pair of rational curves (possibly, reducible) in $\bar X$. In the homology of $\bar X$, each of the lifts realizes a class of the form $q = h + \sum \bar\beta_i^{(1,2)}$, the summation involving exactly one element for each singular point other than $\mathbf{A}_1$. Hence, $q \in \tilde S_X$ and the residue $q \bmod S_X$ is a 3-torsion element of $\mathcal{K}$. Conversely, if a class q as above belongs to $\tilde S_X$, the Riemann-Roch theorem implies that q is realized by a rational curve. Its projection to $\mathbb{P}^2$ is a conic Q as in 4.4.2(3).

Show that any element $q \in \mathcal{K}$ of order 3 must be as in the statement (and hence give rise to a conic Q). Clearly, q is a linear combination of the residues $\bar\beta_i^{(\cdot)} \bmod S_X$. One has $(\bar\beta_i^{(\cdot)})^2 = -2i/3 \bmod 2\mathbb{Z}$ for $\bar\beta_i^{(\cdot)} \in (\mathbf{A}_{3i-1})^*$ and $(\bar\beta_i^{(\cdot)})^2 = 2/3 \bmod 2\mathbb{Z}$ for $\bar\beta_i^{(\cdot)} \in (\mathbf{E}_6)^*$. Since q is isotropic, it must either involve all singular points of C other than $\mathbf{A}_1$ or else belong to an orthogonal summand of S_X of the form $3\mathbf{A}_2$, $\mathbf{A}_5 \oplus \mathbf{A}_2$, $\mathbf{A}_8$, or $\mathbf{E}_6 \oplus \mathbf{A}_2$. The latter possibility is ruled out by condition 3.2.1(1) and Proposition 2.6.3.

If $q' \in \mathcal{K}$ were another element of order 3, $q' \neq \pm q$, then the sum $q + q'$ would be an order 3 element not involving all singular points. Hence, $\mathcal{K}$ contains at most two (opposite) elements of order 3.

Finally, if $q \in \mathcal{K}$ is an element of order 9, then either $3q$ is an element of order 3 not involving all singularities, or q is the sum of two generators of $\operatorname{discr}(2\mathbf{A}_8)$, or q is twice a generator of $\operatorname{discr}\mathbf{A}_{17}$. In the last two cases q cannot be isotropic. □

4.4.4. Corollary. *Each set of singularities Σ as in* (4.4.1) *extends to two isomorphism classes of configurations $\tilde S \supset S = \Sigma \oplus \langle h\rangle$ that may correspond to irreducible sextics, one abundant* ($\mathcal{K} = \mathbb{Z}/3\mathbb{Z}$) *and one not* ($\mathcal{K} = 0$).

Proof. The 2-primary part of the kernel $\mathcal{K}$ is trivial due to Theorem 4.3.1; the 3-primary part is given by Theorem 4.4.3. All extensions with $\mathcal{K} = \mathbb{Z}/3\mathbb{Z}$ are isomorphic to each other as the two elements $\bar\beta^{(1,2)}$ corresponding to each singularity are interchangeable by an admissible automorphism. Finally, $\mathcal{S}$ cannot have an isotropic subgroup of prime order other than 2 or 3. In fact, the only nontrivial p-primary component, $p \neq 2$ or 3, is $\mathcal{S} \otimes \mathbb{Z}_5 \cong \langle -\frac{2}{5}\rangle$ in the case $\Sigma = \mathbf{A}_{14} \oplus \mathbf{A}_2$. It has no isotropic elements. □

4.4.5. Remark. Since elements $\bar\beta^{(1,2)}$ are not symmetric, Theorem 4.4.3 implies that the singular points of an abundant curve admit a natural coherent 'orientation' (order of the exceptional divisors in $\bar X$). Geometrically, this order is selected by a choice of one of the two components of the pull-back of Q in $\bar X$. If the order of the exceptional divisors were fixed, instead of Corollary 4.4.4 one would have 2^{m-1} non-isomorphic abundant configurations, where m is the number of the singular points other than $\mathbf{A}_1$.

5. Examples

Theorem 3.4.2 reduces rigid isotopy classification of plane sextics to the enumeration of oriented abstract homological types. In this concluding section, we outline the principal steps of the classification and illustrate them on a few examples: sextics with few singularities, where Nikulin's theorems apply to give a unique rigid isotopy class (Section 5.2), unnodal classical Zariski pairs (Section 5.3), and a few recent examples of sextics with maximal total Milnor number (Section 5.4). Section 5.5 contains a few concluding remarks, speculations, and open problems.

5.1. Enumerating abstract homological types. Recall that an isomorphism of abstract homological types is defined as an isometry preserving the distinguished class h and distinguished basis σ (as a set). Next proposition states that σ can be ignored.

5.1.1. Proposition. *Let $\mathcal{H}_i = (L_i, h_i, \sigma_i)$, $i = 1, 2$, be two abstract homological types, and let S_i be the corresponding sublattices spanned by h_i and σ_i. Then $\mathcal{H}_1$, $\mathcal{H}_2$ are isomorphic if and only if there is an isometry $t\colon L_1 \to L_2$ taking S_1 to S_2 and h_1 to h_2.*

Proof. The extensions of the restriction $t_{\tilde{S}_1^\perp}$ to the whole lattice L_1 depend only on the induced map $\tilde{\mathcal{S}}_1 \to \tilde{\mathcal{S}}_2$. In view of Proposition 2.3.1, the image in $\operatorname{Aut} \tilde{\mathcal{S}_i}$ of the group of admissible isometries of $\tilde{S}_i$ coincides with the image of the group of isometries preserving h_i. $\square$

Fix a set of singularities Σ. The classification of oriented abstract homological types extending Σ is done in four steps.

Step 1: *enumerating the configurations $\tilde{S}$ extending* Σ. Due to Proposition 2.6.1, a configuration is determined by a choice of an isotropic subgroup $\mathcal{K} \subset \mathcal{S}$. Note that, given 3.2.1(1), condition 3.2.1(2) should only be checked for the direct summands of Σ isomorphic to $\mathbf{A}_1$, as for any other root $r \in \Sigma$ there is another root $r' \in \Sigma$ such that $r \cdot r' = 1$ and, hence, $r + h$ is primitive in $\tilde{S}$. We combine this observation and Corollary 2.6.3 to the following statement.

5.1.2. Proposition. *Let $\tilde{S}$ be a configuration extending a set of singularities Σ. Then each direct summand of Σ isomorphic to one of the root systems listed in Corollary 2.6.3 is primitive in $\tilde{S}$, and each sublattice $\mathbf{A}_1 \oplus \langle h\rangle$, where $\mathbf{A}_1$ is a direct summand of Σ, is primitive in $\tilde{S}$.*

Step 2: *enumerating the isomorphism classes of $\tilde{S}^\perp$.* The orthogonal complement $N = \tilde{S}^\perp_L$ has genus $(2, 19 - \operatorname{rk}\Sigma; -\tilde{\mathcal{S}})$. The existence of a lattice in this genus, whenever it holds, is given by Theorem 2.5.1. If N is indefinite, one would hope that a theorem similar to 2.5.2 would imply uniqueness. The case of definite lattices ($\operatorname{rk}\Sigma = 19$) is treated in Section 2.4. There are examples (see, *e.g.*, Proposition 5.4.4 below) when the genus does contain more than one isomorphism class.

Step 3: *enumerating the bi-cosets of* $\operatorname{Aut}_h \tilde{\mathcal{S}} \times \operatorname{Aut}_N \mathcal{N}$. Once the lattice $N = \tilde{S}^\perp$ is chosen, one can fix an anti-isometry $\tilde{\mathcal{S}} \to \mathcal{N}$ and, hence, an isomorphism $\operatorname{Aut}\mathcal{N} =$

$\operatorname{Aut}\tilde{\mathcal{S}}$; then, the extensions are classified by the quotient set $\operatorname{Aut}_h\tilde{\mathcal{S}}\backslash\operatorname{Aut}\tilde{\mathcal{S}}/\operatorname{Aut}_N\mathcal{N}$. Important special cases are those with $\operatorname{Aut}_h\tilde{\mathcal{S}} = \operatorname{Aut}\tilde{\mathcal{S}}$ (*cf.* Section 2.3) or $\operatorname{Aut}_N\mathcal{N} = \operatorname{Aut}\mathcal{N}$ (this would normally be given by Theorem 2.5.3). If none of the above applies, the isometries are to be described manually. There are examples (see Propositions 5.4.2, 5.4.6, and 5.4.8 below) when the quotient consists of more than one coset, thus giving rise to more than one abstract homological type.

Step 4: detecting whether the abstract homological types are symmetric. An abstract homological type is symmetric if and only if $\tilde{S}^\perp$ has a $+$-disorienting isometry t whose image in $\operatorname{Aut}\operatorname{discr}\tilde{S}^\perp = \operatorname{Aut}\tilde{\mathcal{S}}$ belongs to the product of the subgroup $O_h(\tilde{S})$ and the image of $O^+(\tilde{S}^\perp)$. Asymmetric abstract homological types do exist, see Proposition 5.4.4. Below is a sufficient condition for an abstract homological type to be symmetric.

5.1.3. Proposition. *Let $\mathcal{H} = (L, h, \sigma)$ be an abstract homological type. If the lattice $\tilde{S}^\perp$ contains a vector v of square 2, then $\mathcal{H}$ is symmetric.*

Proof. The reflection t_v reverses the orientation of one and, hence, any maximal positive definite subspace. On the other hand, it is obviously an automorphism of $\mathcal{H}$, as it acts identically on $\tilde{S}$. □

If a lattice N is unique in its genus, the existence of a vector $v \in N$ of square 2 can easily be expressed in terms of discriminant forms. Indeed, either one has $\langle v\rangle \oplus \langle v\rangle^\perp = N$ or $\langle v\rangle \oplus \langle v\rangle^\perp \subset N$ is a sublattice of index 2. In both cases, the discriminant $\operatorname{discr}\langle v\rangle^\perp$ is determined by that of N, and the question reduces to the existence of a lattice $\langle v\rangle^\perp$ within a prescribed genus, see Theorem 2.5.1. If it does exist, Proposition 2.6.4 implies that $\langle v\rangle \oplus \langle v\rangle^\perp$ is a sublattice of a lattice isomorphic to N. Next statement is a simple special case of the above observation.

5.1.4. Proposition. *Any indefinite even lattice N with $\operatorname{rk} N \geqslant \ell(\mathcal{N}) + 2$ has a vector of square 2.*

Proof. First, let $\operatorname{rk} N \geqslant 3$. Then Theorem 2.5.2 implies that N is unique in its genus. On the other hand, from Theorem 2.5.1 it follows that there exists a lattice N' of signature $(\sigma_+N - 1, \sigma_-N)$ whose discriminant form is $\mathcal{N} \oplus \langle -\frac{1}{2}\rangle$. (Since both signature and Brown invariant are additive, condition 2.5.1(2) holds automatically.) Then the sum $\langle v\rangle \oplus N'$, $v^2 = 2$, is an index 2 sublattice in a lattice isomorphic to N.

In the exceptional case $\operatorname{rk} N = 2$ one has $\ell(\mathcal{N}) = 0$, *i.e.*, N is unimodular. Then $N \cong \mathbf{U}$ and the statement is obvious. □

5.2. Sextics with few singularities. Let $\mu = \operatorname{rk}\Sigma = \operatorname{rk}\tilde{S} - 1$ be the total Milnor number of the singularities. One has $\mu \leqslant 19$, and the orthogonal complement $\tilde{S}^\perp$ has rank $21 - \mu$ and signature $(2, 19 - \mu)$.

5.2.1. Theorem. *Each configuration $\tilde{S} \supset S = \Sigma \oplus \langle h\rangle$ satisfying the inequality $\ell(\tilde{\mathcal{S}}) + \operatorname{rk}\Sigma \leqslant 19$ is realized by a unique rigid isotopy class of plane sextics.*

Proof. The inequality $\ell(\tilde{\mathcal{S}}) + \operatorname{rk}\Sigma \leqslant 19$ implies that $\mu = \operatorname{rk}\Sigma \leqslant 18$, as otherwise $\tilde{S}^\perp$ would be a unimodular even lattice of signature $(2, 0)$. Thus, $\tilde{S}^\perp$ is indefinite, $\operatorname{rk}\tilde{S}^\perp \geqslant 3$, and $\operatorname{rk}\tilde{S}^\perp \geqslant \ell(\operatorname{discr}\tilde{S}^\perp) + 2$. Hence, Theorem 2.5.3 applies to $\tilde{S}^\perp$ and $\tilde{S}$ extends to a unique abstract homological type, which is symmetric due to Propositions 5.1.3 and 5.1.4. (The existence of a lattice $\tilde{S}^\perp$ realizing the given genus follows from Theorem 2.5.1, as condition 2.5.1(2) holds automatically.) □

5.2.2. Corollary. *Each configuration extending a set of singularities Σ satisfying the inequality $\ell(\operatorname{discr}\Sigma) + \operatorname{rk}\Sigma \leqslant 19$ is realized by a unique rigid isotopy class of plane sextics.*

Proof. Let $\tilde{S}$ be a configuration extending Σ. Then, for each prime $p \neq 2$, one has $\ell_p(\tilde{\mathcal{S}}) \leqslant \ell_p(\operatorname{discr}\Sigma) \leqslant 19 - \mu$. For $p = 2$ one has $\ell_2(\tilde{\mathcal{S}}) \leqslant \ell_2(\operatorname{discr}\Sigma) + 1 \leqslant 20 - \mu$. However, since $\ell_2(\mathcal{N}) = \operatorname{rk} N \bmod 2$ for each lattice N, the latter inequality still implies $\ell_2(\tilde{\mathcal{S}}) \leqslant 19 - \mu$, and Theorem 5.2.1 applies. $\square$

5.3. Classical Zariski pairs. Consider a set of singularities of a classical Zariski pair of irreducible curves, *i.e.*, a set Σ as in (4.4.1). Let

$$g(\Sigma) = 10 - 3e - \sum_{i=1}^{6} a_i \left[\frac{3i}{2}\right] - n$$

be its virtual genus. In [11] it is conjectured that, if $g(\Sigma) > 0$ (respectively, $g(\Sigma) = 0$), then Σ is realized by exactly two (respectively, one) rigid isotopy classes of irreducible sextics, one abundant and one not (respectively, one abundant). We prove the conjecture in the case $n = 0$.

5.3.1. Remark. Now, it seems clear that the nonexistence part of the conjecture (the case $g(\Sigma) = 0$) is wrong: a simple estimate and Theorem 2.5.1 show that *most* sets of singularities are realized by both abundant and non-abundant curves.

The uniqueness part seems to follow more or less directly from Theorem 2.5.3 for all abundant curves, as well as for all curves with $e \leqslant 1$. However, as there still are quite a number of details to be double checked (and the non-abundant case with $e > 1$ requires tedious manual calculations, *cf.* the proof of Theorem 5.3.2), I will consider the general case in a separate paper.

5.3.2. Theorem. *Any set of singularities Σ of the form*

$$\Sigma = e\mathbf{E}_6 \oplus \bigoplus_{i=1}^{6} a_i \mathbf{A}_{3i-1}, \quad 2e + \sum i a_i = 6,$$

is realized by exactly two rigid isotopy classes of irreducible plane sextics, one abundant and one not.

Proof. We need to show that, assuming that the number of nodes $n = 0$, each of the two configurations given by Corollary 4.4.4 extends to a unique homological type, which is symmetric. One has $\mu = \operatorname{rk}\Sigma = 18 - (m - e)$, where m is the total number of points in Σ. Since each singular point considered contributes exactly one to both $\ell(\operatorname{discr}\Sigma)$ and $\ell_3(\operatorname{discr}\Sigma)$, one has $\ell(\operatorname{discr}\Sigma) + \mu = 18 + e$. Thus, if $e = 0$ or 1, the statement of the theorem follows directly from Corollary 5.2.2. It remains to consider the three sets $\Sigma = 2\mathbf{E}_6 \oplus \mathbf{A}_5$, $2\mathbf{E}_6 \oplus 2\mathbf{A}_2$, or $3\mathbf{E}_6$ corresponding to $e = 2, 3$.

If $\tilde{S}$ is an abundant configuration without points of type $\mathbf{A}_7$ or $\mathbf{A}_{17}$, then $\ell_3(\tilde{S}) = \ell_3(\operatorname{discr}\Sigma) - 2$. If also $e \geqslant 2$, then still $\ell(\tilde{\mathcal{S}}) + \mu \leqslant 19$ and Theorem 5.2.1 applies.

The remaining three cases, the non-abundant configurations $\tilde{S} = S$ with $e = 2$ or 3, are considered below.

In the first two cases ($e = 2$), the uniqueness of the lattice $\tilde{S}^\perp$ in its genus (Step 2 in Section 5.1) follows from Theorem 2.5.2. We will show that the homomorphism

$O(\tilde S^\perp) \to \operatorname{Aut}\tilde{\mathcal S}$ is onto and that $\tilde S^\perp$ has a $+$-disorienting isometry whose image in $\operatorname{Aut}\tilde{\mathcal S}$ belongs to the product of $\operatorname{Aut}_h\tilde{\mathcal S}$ and the image of $\operatorname{Aut}^+\tilde S^\perp$ (see Steps 3 and 4 in Section 5.1).

The case $\Sigma = 2\mathbf{E}_6 \oplus \mathbf{A}_5$. One has $\operatorname{discr}\tilde S^\perp \cong -\tilde{\mathcal S} \cong 2\langle -\frac12\rangle \oplus 3\langle -\frac23\rangle$, so that one can take $\tilde S^\perp = \langle -2\rangle \oplus \langle 6\rangle \oplus \mathbf{U}(3)$. Let a and b be generators of the $\langle -2\rangle$- and $\langle 6\rangle$-summands, respectively, and let u, v be a standard basis for the $\mathbf{U}(3)$-summand.

Consider the 3-primary part $\operatorname{discr}\tilde S^\perp \otimes \mathbb{Z}_3 \cong 3\langle -\frac23\rangle$. Its automorphisms are permutations of the three generators and multiplication of some of them by (-1). One can take for the generators the classes $\beta^\pm = [b^\pm/3]$ and $\gamma = [c/3]$, where $b^\pm = b \pm (u-2v)$ and $c = u - v$ are vectors of square (-6). Then the reflections $t_{b^\pm}$ and t_c, which are well defined elements of $O^+(\tilde S)^\perp$, act on $\operatorname{discr}\tilde S^\perp$ by multiplying the corresponding generators by (-1). The reflection t_{u+v} transposes β^+ and β^-, and the reflection t_{b+u} transposes β^- and γ. Since $(u+v)^2 = (b+u)^2 = 6$, the latter two reflections are $+$-disorienting. All isometries mentioned act identically on $\operatorname{discr}\tilde S^\perp \otimes \mathbb{Z}_2$, and together they generate the group $\operatorname{Aut}(\operatorname{discr}\tilde S^\perp \otimes \mathbb{Z}_3)$.

The 2-primary part is $\operatorname{discr}\tilde S^\perp \otimes \mathbb{Z}_2 \cong 2\langle -\frac12\rangle$. Its only nontrivial automorphism is realized by the reflection t_{a+b}, which acts identically on $\operatorname{discr}\tilde S^\perp \otimes \mathbb{Z}_3$.

Since the homomorphism $O(\tilde S^\perp) \to \operatorname{Aut}\operatorname{discr}\tilde{\mathcal S}^\perp$ is onto, one can assume that $-\operatorname{discr}\tilde{\mathcal S}^\perp$ and $\tilde{\mathcal S}$ are identified so that β^+ and β^- are generators of the two copies of $\operatorname{discr}\mathbf{E}_6$ in $\tilde{\mathcal S}$. Then they can be transposed by an admissible isometry of $\tilde S$. On the other hand, the transposition $\beta^+ \leftrightarrow \beta^-$ is realized by a $+$-disorienting isometry of $\tilde S^\perp$. Hence, the abstract homological type is symmetric.

The case $\Sigma = 2\mathbf{E}_6 \oplus 2\mathbf{A}_2$. One has $\operatorname{discr}\tilde S^\perp \cong -\tilde{\mathcal S} \cong \langle -\frac12\rangle \oplus 4\langle -\frac23\rangle$, so that $\tilde S^\perp = \langle -2\rangle \oplus 2\mathbf{U}(3)$. Let c be a generator of the $\langle -2\rangle$-summand, and let (u_1, v_1) and (u_2, v_2) be some standard bases for the two $\mathbf{U}(3)$-summands.

It suffices to consider the 3-primary part $\operatorname{discr}\tilde S^\perp \otimes \mathbb{Z}_3$. One can take for a basis the classes $\alpha_i = [a_i/3]$ and $\beta^\pm = [b^\pm/3]$, where $a_i = u_i - v_i$, $i = 1, 2$, and $b^\pm = (u_1 - 2v_1) \pm (u_2 + v_2)$ are vectors of square (-6). The reflections t_{a_i} and $t_{b^\pm}$ multiply the corresponding generators by (-1), and modulo these automorphisms each vector of square $(-2/3)$ in $\operatorname{discr}\tilde S^\perp$ is either one of the four generators or their sum $\alpha_1 + \alpha_2 + \beta^+ + \beta^- = [(v_1 + u_2 - v_2)/3]$. Thus, each element $\chi \in \operatorname{discr}\tilde S^\perp$ of square $(-2/3)$ can be realized by a vector $x \in \tilde S^\perp$ of square (-6). The orthogonal complement $\langle x\rangle^\perp_{\tilde S^\perp}$ has the genus of the lattice $\tilde S^\perp$ considered in the previous case; due to the results obtained there, any such element χ can be taken to α_1, and then any automorphism of the complement $\langle \alpha_1\rangle^\perp_{\operatorname{discr}\tilde S^\perp}$ can be realized by an isometry of $\langle a_1\rangle^\perp_{\tilde S^\perp}$.

A consideration similar to the previous case shows that $\tilde S^\perp$ has a $+$-disorienting isometry that extends to an admissible isometry of L.

The case $\Sigma = 3\mathbf{E}_6$. One has $\tilde{\mathcal S} \cong \langle \frac12\rangle \oplus 3\langle \frac23\rangle$. The automorphisms of $\tilde{\mathcal S}$ are permutations of the three generators of order 3 or multiplication of some of them by (-1). Each such automorphism is realized by an admissible isometry of $\tilde S$ (respectively, permutation of the $\mathbf{E}_6$-components and the nontrivial symmetries of the Dynkin graphs of some of them), and it remains to show that $\tilde S^\perp$ is unique in its genus and has a $+$-disorienting isometry. The uniqueness in the genus follows from Theorem 2.5.2; one can take $\tilde S^\perp = \langle 6\rangle \oplus \mathbf{U}(3)$, and the generator of the $\langle 6\rangle$-summand defines a $+$-disorienting reflection. $\square$

5.3.3. Remark. The case of abundant unnodal curves is treated in [11] geometrically. Since unnodal curves of the form $f_2^3 + f_3^2 = 0$ are generic (for given polynomials f_2, f_3), this case reduces to the classification of certain reducible quintics. This is done in [10]. Thus, essentially new in Theorem 5.3.2 is the case of non-abundant curves.

5.4. Maximal sextics. The maximal value of the total Milnor number $\mu = \operatorname{rk}\Sigma$ is 19. When $\mu = 19$, the orthogonal complement $\tilde{S}^\perp$ is a positive definite lattice of rank 2. This case, although not covered by general Theorems 2.5.2 and 2.5.3, can easily be handled directly, see Section 2.4. It is rather straightforward to extend Yang's algorithm [39] listing the maximal sets of singularities and corresponding configurations in order to produce a listing of all maximal (in the sense $\mu = 19$) rigid isotopy classes. I am planning to publish the results in a forthcoming paper.

Below, we treat manually a few examples. I have chosen the sets of singularities that were studied in details in a recent series of papers by Artal *et al.*, see [2]–[4], so that the classification obtained arithmetically could be compared with the known geometric properties of the curves. The principal purpose of this section is to illustrate the phenomena that take place and the number of details that should be taken into account in an attempt to realize the algorithm programmatically.

In the proofs below, the isomorphism classes within a given genus for $\tilde{S}^\perp$ can be found *via* `Maple`, using Section 2.4. Indeed, any lattice $N = \mathbf{M}(a, b, c)$ with a given discriminant form $\mathcal{N}$ must satisfy $|\mathcal{N}|/4 \leqslant ac \leqslant |\mathcal{N}|/3$, and it remains to enumerate the triples (a, b, c), calculate the discriminant forms, and compare them to $\mathcal{N}$. We merely indicate the result by a sentence like 'the only possibility for $\tilde{S}^\perp$ is . . . .'

5.4.1. Proposition. *The set of singularities $\Sigma = \mathbf{D}_{19}$ extends to a unique abstract homological type, which is symmetric.*

Proof. One has $\mathcal{S} \cong \langle -\frac{3}{4}\rangle \oplus \langle \frac{1}{2}\rangle$. This form has no isotropic subgroups; hence, always $\tilde{S} = S$, and the only possibility for $\tilde{S}^\perp$ is $\mathbf{M}(1, 0, 2)$. The only nontrivial automorphism of $\operatorname{discr}\tilde{S}^\perp$ is the multiplication of an order 4 element by (-1); it is realized by a reflection in $\tilde{S}^\perp$. Hence, there is a unique abstract homological type, and it is symmetric due to Proposition 5.1.3. $\square$

5.4.2. Proposition. *The set of singularities $\Sigma = \mathbf{A}_{19}$ admits a unique configuration $\tilde{S}$. It extends to two abstract homological types, which are both symmetric. The two lattices $\tilde{S}^\perp$ are isomorphic.*

5.4.3. Remark. Sextics with this set of singularities were studied in Artal *et al.* [3], where the two rigid isotopy classes were discovered. The only difference between the two homological types is the anti-isometry identifying the discriminant groups of $\tilde{S}$ and $\tilde{S}^\perp$. (One also observes a similar phenomenon in Propositions 5.4.6 and 5.4.8 below, where the curves are reducible.) Thus, it *looks* like the two pairs in question are obtained by gluing diffeomorphic pieces *via* different diffeomorphisms of their boundaries, *cf.* Remark 2.6.5. At present, I do not know whether this claim is true, as of course the global Torelli theorem only applies to whole $K3$-surfaces.

Up to projective equivalence, each rigid isotopy class consists of a single curve C_i, $i = 1, 2$, and the two curves are indeed very similar to each other. For example, disregarding the hyperplane section class h in the calculation above, one can easily show that the two covering $K3$-surfaces $\bar{X}_i$ are deformation equivalent. The fundamental groups $\pi_1(\mathbb{P}^2 \smallsetminus C_i)$ were calculated in [3], and it was shown that they are isomorphic to each other. Whether the two complements $\mathbb{P}^2 \smallsetminus C_i$ themselves are diffeo-/homeomorphic still remains an open question.

Proof. One has $\mathcal{S} \cong \langle -\frac{19}{20}\rangle \oplus \langle \frac{1}{2}\rangle \cong \langle \frac{4}{5}\rangle \oplus \langle \frac{1}{4}\rangle \oplus \langle \frac{1}{2}\rangle$, the first two summands being generated by 4α and 5α, where α is a canonical generator of the group $\operatorname{discr} \mathbf{A}_{19}$. Since $\mathcal{S}$ has no isotropic subgroups, one has $\tilde{S} = S$, and the only possibility for the orthogonal complement $\tilde{S}^\perp$ is $\mathbf{M}(1, 0, 10)$. The automorphism group $\operatorname{Aut} \mathcal{S} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ consists of the automorphisms

$$(4\alpha, 5\alpha) \mapsto (\epsilon_1 \cdot 4\alpha, \epsilon_2 \cdot 5\alpha), \quad \epsilon_1, \epsilon_2 = \pm 1,$$

whereas the images of both $O(\Sigma)$ and $O(\tilde{S}^\perp)$ are generated by $-\operatorname{id}$, corresponding to $\epsilon_1 = \epsilon_2 = -1$. Hence, $\tilde{S}$ extends to two distinct abstract homological types. They are both symmetric due to Proposition 5.1.3. $\square$

5.4.4. Proposition. *The set of singularities $\Sigma = \mathbf{A}_{18} \oplus \mathbf{A}_1$ admits a unique configuration $\tilde{S}$. It extends to two abstract homological types, which differ by the lattice $\tilde{S}^\perp$. One of the homological types is symmetric, the other one is not, so that there are three rigid isotopy classes of sextics with this set of singularities.*

5.4.5. Remark. This set of singularities was first studied in Artal *et al.* [3]. The most remarkable fact is the existence of two rigid isotopy classes that differ by the orientation of their homological types. This example may be a first candidate for a pair of sextic curves C_1, C_2 with homeomorphic but not diffeomorphic pairs $(\mathbb{P}^2, C_i)$, see Remark 1.1.2. According to the description of the orthogonal groups given in Section 2.4, this situation should be rather typical for the maximal Milnor number: any abstract homological type with $\tilde{S}^\perp \cong \mathbf{M}(a, b, c)$, $0 < b < a < c$, would be asymmetric.

All three curves are given by Galois conjugate equations defined over $\mathbb{Q}(\beta)$, where β is a root of $19s^3 + 50s^2 + 36s + 8$. In [3], there are more examples of curves given by complex conjugate equations; the corresponding sets of singularities are $\mathbf{A}_{16} \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$ and $\mathbf{A}_{15} \oplus \mathbf{A}_4$.

Proof. One has $\mathcal{S} \cong \langle -\frac{18}{19}\rangle \oplus \langle -\frac{1}{2}\rangle \oplus \langle \frac{1}{2}\rangle$. The only imprimitive extension of S would contradict Proposition 5.1.2; hence, there is a unique configuration $\tilde{S} = S$. The genus of $\tilde{S}^\perp$ contains two isomorphism classes: $\mathbf{M}(1, 0, 19)$ and $\mathbf{M}(4, 2, 5)$. The only nontrivial automorphism of $\tilde{\mathcal{S}}$ is realized by the isometry $-\operatorname{id} \in O(\tilde{S}^\perp)$. Hence, each of the isomorphism classes gives rise to a unique abstract homological type. The abstract homological type with $\tilde{S}^\perp = \mathbf{M}(1, 0, 19)$ is symmetric due to Proposition 5.1.3; the one with $\tilde{S}^\perp = \mathbf{M}(4, 2, 5)$ is not symmetric as $\tilde{S}^\perp$ has no $+$-disorienting isometries. $\square$

5.4.6. Proposition. *The set of singularities $\Sigma = 2\mathbf{A}_9 \oplus \mathbf{A}_1$ admits two distinct configuration $\tilde{S}$, with $[\tilde{S} : S] = 2$ or 10. The former configuration extends to two abstract homological types (with isomorphic lattices $\tilde{S}^\perp$), the latter extends to one. All extensions are symmetric, so that altogether there are three rigid isotopy classes of sextics with this set of singularities.*

5.4.7. Remark. Although the configurations differ, all three curves have the same combinatorial data. The case $[\tilde{S}:S]=10$ can be told apart by the existence of an extra line in a special position with respect to the curve, see Artal *et al.* [4].

Proof. One has $\mathcal{S}\cong 2\langle -\frac{9}{10}\rangle\oplus\langle -\frac{1}{2}\rangle\oplus\langle\frac{1}{2}\rangle\cong 2\langle\frac{2}{5}\rangle\oplus 3\langle -\frac{1}{2}\rangle\oplus\langle\frac{1}{2}\rangle$. Let $\alpha_{1,2}$, β, and γ be generators of the summands $\operatorname{discr}\mathbf{A}_9$, $\operatorname{discr}\mathbf{A}_1$, and $\operatorname{discr}\langle 2\rangle$, respectively.

Since $\ell_2(\mathcal{S})=4$, the kernel $\mathcal{K}=\tilde{S}/S$ must contain elements of order 2. The isotropic elements of order 2 are $\beta+\gamma$ and $5\alpha_{1,2}+\gamma$. The former cannot belong to $\mathcal{K}$ due to Proposition 5.1.2; the two latter are interchangeable by an admissible isometry. Hence, one can assume that $\mathcal{K}\otimes\mathbb{Z}_2\cong\mathbb{Z}/2\mathbb{Z}$ is generated by $5\alpha_1+\gamma$.

The 5-primary part $\mathcal{K}\otimes\mathbb{Z}_5$ may be either trivial or one of the two order 5 subgroups generated by $2\alpha_1\pm 4\alpha_2$. In the latter case the two subgroups are conjugate by an admissible isometry preserving $5\alpha_1+\gamma$. Hence, up to admissible isometry, there are two configurations, which differ by the index $[\tilde{S}:S]$.

If $[\tilde{S}:S]=10$, then $\tilde{\mathcal{S}}\cong 2\langle -\frac{1}{2}\rangle$ and $\tilde{S}^\perp\cong\mathbf{M}(1,0,1)$. The homomorphism $O(\tilde{S}^\perp)\to\operatorname{Aut}\operatorname{discr}\tilde{S}^\perp$ is onto, and the only resulting abstract homological type is symmetric due to Proposition 5.1.3.

If $[\tilde{S}:S]=10$, then $\tilde{\mathcal{S}}\cong 2\langle\frac{2}{5}\rangle\oplus 2\langle -\frac{1}{2}\rangle$ and $\tilde{S}^\perp\cong\mathbf{M}(5,0,5)$. The group $\operatorname{Aut}\tilde{\mathcal{S}}$ is generated by the multiplications $(-1)_i\colon\alpha_i\mapsto -\alpha_i$, $i=1,2$, the transposition tr_5 of α_1 and α_2, and the transposition tr_2 of the generators of the two summands of order 2. The subgroup $\operatorname{Aut}_h\tilde{\mathcal{S}}$ is generated by $(-1)_1$ and $(-1)_2$, and the image of $O(\tilde{S}^\perp)$ in $\operatorname{Aut}\tilde{\mathcal{S}}$ is generated by $(-1)_1$, $(-1)_2$, and the composition $\mathrm{tr}_5\circ\mathrm{tr}_2$. Hence, $\tilde{S}$ extends to two distinct abstract homological types, and they are both symmetric (as $\tilde{S}^\perp$ has a $+$-disorienting isometry that extends to L by an admissible isometry of $\tilde{S}$). □

5.4.8. Proposition. *The set of singularities* $\Sigma=\mathbf{E}_6\oplus\mathbf{A}_7\oplus\mathbf{A}_3\oplus\mathbf{A}_2\oplus\mathbf{A}_1$ *admits a unique configuration* $\tilde{S}$*. It extends to two abstract homological types, which are both symmetric. The two lattices* $\tilde{S}^\perp$ *are isomorphic.*

5.4.9. Remark. Sextics with this set of singularities are all reducible, splitting to a singular quintic and a line. They were studied in Artal *et al.* [2].

Proof. One has $\mathcal{S}\cong\langle\frac{2}{3}\rangle\oplus\langle -\frac{7}{8}\rangle\oplus\langle -\frac{3}{4}\rangle\oplus\langle -\frac{2}{3}\rangle\oplus\langle -\frac{1}{2}\rangle\oplus\langle\frac{1}{2}\rangle$. Since $\ell_2(\mathcal{S})=4$, the kernel $\mathcal{K}$ must contain elements of order 2. The only isotropic element of order 2 not contradicting Proposition 5.1.2 is $4\beta_7+\beta_1+\gamma$, where β_i is a generator of $\operatorname{discr}\mathbf{A}_i$, $i=1,2,3,7$, and γ is the generator of $\operatorname{discr}\langle 2\rangle$. Thus, there is a unique configuration $\tilde{S}$, and the group $\tilde{\mathcal{S}}\cong\langle\frac{2}{3}\rangle\oplus\langle -\frac{3}{8}\rangle\oplus\langle -\frac{3}{4}\rangle\oplus\langle -\frac{2}{3}\rangle$ is generated by α (a generator of $\operatorname{discr}\mathbf{E}_6$), $\beta_7'=\beta_7+\gamma$, β_3, and β_2. The group $\operatorname{Aut}\tilde{\mathcal{S}}$ is generated by the multiplications of generators by (-1), which lift to admissible isometries of S, and the involution $\varphi\colon(\beta_7',\beta_3)\mapsto(3\beta_7'+2\beta_3,\beta_3+4\beta_7')$, which is not in the image of $O_h(S)$.

The only possibility for $\tilde{S}^\perp$ is $\mathbf{M}(6,0,12)$. Since the involution φ above does not lift to $\tilde{S}^\perp$, there are two abstract homological types extending $\tilde{S}$. Each of them is symmetric, as the $+$-disorienting reflection defined by the vector of square 24 extends to L by an admissible isometry of $\tilde{S}$. □

5.5. Concluding remarks.

5.5.1. Examples with $\mu<19$. One of the by-products of the calculation in Section 5.4 is the fact that each of the four steps outlined in Section 5.1 does matter, in the sense that there are pairs of sextics that diverge at that particular step. However, in all these examples one has $\mu = 19$, *i.e.*, the moduli space is discrete, and I do not know a single examples of a pair of not rigidly isotopic sextics with $\mu < 19$ that share the same configuration. (A number of families with $\mu = 18$ is considered in [3], where it is proved that the curves are determined by their combinatorial data. The members of classical Zariski pairs considered in Section 5.3 differ by their configurations.)

At present, I do not know how general this phenomenon is and how it could be proved/disproved without essentially enumerating all rigid isotopy classes.

5.5.2. Asymmetric homological types. The existence of two opposite orientations of periods of marked $K3$-surfaces is a well known fact. The two orientations are interchanged by the canonical real structure on the moduli space, sending a $K3$-surface to its conjugate. Thus, asymmetric abstract homological types give rise to moduli spaces without real points. It would be interesting to find similar examples with $\mu < 19$, when the modular space has positive dimension, or to prove that such examples do not exist, *cf.* Section 5.5.1. (Note that, *a priori*, the moduli space may have no real points even if the homological type is symmetric.)

For plane curves, the canonical real structure on the moduli space is induced by the standard complex conjugation on $\mathbb{P}^2$. In particular, the pairs $(\mathbb{P}^2, C_i)$ corresponding to two conjugate curves *are* diffeomorphic, but the diffeomorphism is not regular and, most importantly, it reverses the orientation of complex curves, inducing (-1) in $H_2(\mathbb{P}^2)$, *cf.* Section 5.5.4.

Apparently, the existence of asymmetric homological types is due to the fact that we consider $K3$-surfaces with a fixed polarization (which prohibits certain obvious changes of the marking), whereas no involution that would interchange the components is assumed *a priori* (as in the case of real curves and surfaces). Probably, one can anticipate a similar phenomenon in the case of quartic surfaces in $\mathbb{P}^3$, see Section 5.5.6.

5.5.3. Conjugacy over number fields. Comparing the results obtained in Section 5.4 with the geometric properties of the curves discovered in [2]–[4], one can observe that, in the case of maximal total Milnor number $\mu = 19$, *all sextics with a given configuration $\tilde{S}$ are given by equations Galois conjugate over a certain finite extension of* $\mathbb{Q}$. (Remind that curves with $\mu = 19$ are rigid, *i.e.*, their moduli spaces are discrete.) At present, I do not know how general this statement is and how it can be obtained arithmetically. Another piece of substantiating evidence is given by the material of §4, where various geometric properties of curves that should remain invariant under all, not necessarily continuous, Galois transformations are expressed in terms of the configuration only.

5.5.4. The regularity condition. The regularity condition in Theorem 1.1.1 seems to be a purely technical assumption; it is needed to assure that the diffeomorphism f lifts to the minimal resolutions of singularities and then, further, to the nonsingular double coverings. I believe that any diffeomorphism preserving the complex orientations gives rise to a regular one. To prove/disprove this statement one would need to know the diffeotopy classification of auto-diffeomorphisms of the link (*i.e.*, boundary of a regular neighborhood) of each simple singularity of surfaces. Unfortunately, I do not know any results in this direction.

Certainly, if it turns out that the regularity assumption can be dropped, one would still have to require that f preserve the complex orientation of both $\mathbb{P}^2$

and the curves, *i.e.*, that the induced homomorphism $f_*\colon H_*(\mathbb{P}^2) \to H_*(\mathbb{P}^2)$ is the identity.

5.5.5. Other equivalence relations. In the definition of admissible isometry, it is required that the distinguished basis σ of Σ should only be fixed as a set. Geometrically, this means that neither the order of the singular points nor their 'orientation' are assumed fixed. If the order matters, one should fix the basis and consider isomorphism classes of lattice polarized $K3$-surfaces in the sense of Nikulin [30]; they are classified by oriented abstract homological types up to isometries identical on $\Sigma \oplus \langle h \rangle$.

An interesting example of multiple equivalence classes is described in Remark 4.4.5. A more straightforward example is given by Proposition 5.4.6, where the two $\mathbf{A}_9$ points cannot be transposed by a rigid isotopy: they are distinguished by the combinatorial data of the curves.

Alternatively, one can compare curves using various relaxed equivalence relations: homeomorphism of the pairs, diffeo-/homeomorphism of the complement spaces, *etc.* For most curves, this problem still remains open.

5.5.6. Quartic surfaces. The classification of singular quartic surfaces in $\mathbb{P}^3$ should be very similar to the classification of plane sextics. The proof of the corresponding counterpart of Theorem 1.1.1 would repeat literally the contents of §3, with $h^2 = 2$ replaced with $h^2 = 4$. It is worth mentioning that, as in the case of plane sextics, the rigid isotopy class of a quartic surface with at least one non-simple singular point is determined by its combinatorial data, see [12].

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