QUINTICS IN $\mathbb{C}p^2$ WITH NONABELIAN FUNDAMENTAL GROUP

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This paper is dedicated to *V. A. Rokhlin*

Abstract. The fundamental groups of all the complex plane projective quintics are listed; some new examples of curves with abelian and nonabelian fundamental group are constructed.

1. Introduction

Given an algebraic curve $C \in \mathbb{C}p^2$, its fundamental group Π_C is defined to be $\pi_1(\mathbb{C}p^2 \smallsetminus C)$. The problem of studying this group was first posed by, probably, O. Zariski [Z1], and since then just a few results in this direction have been obtained: on one hand, it is known that Π_C is abelian provided that the singularities of the curve are simple enough (see Deligne [De] and Nori [N]), and, on the other hand, there are a few examples of curves with nonabelian fundamental group (see, e.g., [A1], [D3], [M], [O1], [O2], [S1], [S2], [Z1], [Z2]). Though, what is known is quite enough to show that the fundamental group is an interesting invariant of algebraic curves; e.g., to my knowledge it is Π_C (more precisely, the Alexander polynomial, which is a purely algebraic invariant of the group) that distinguishes nonisotopic equisingular irreducible curves in all known examples.

It is clear that the abelinization of Π_C depends only on the components of C: if $C = \sum r_i C_i$ with C_i irreducible and reduced and $\deg C_i = d_i$, then the abelinization is $\prod \langle a_i \rangle / (\sum d_i a_i)$. Thus, the problem is only interesting when Π_C is nonabelian. The main result of the paper is the complete list of all the quintics with nonabelian fundamental group (see 3.3). The most interesting examples are certainly the two irreducible quintics; for one of them Π_C is finite, for the other it is infinite. (The fundamental group of a quintic with the singular set $A_6 \sqcup 3A_2$ was independently calculated by B. Artal in his recent paper [A2].) As a by-product of the techniques used we also obtain two new series of examples of curves with controllable fundamental groups:

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one series consists of curves with 'deep' singularities whose group is abelian (see 3.2), the other one produces new curves with nonabelian (and sometimes finite) group (see 3.1).

The principal tool used in the paper is a slight modification of well-known van Kampen's method (see §4), which allows one to overcome the usual difficulty with the 'global' braid monodromy when the curve has deep singularities. It is used to prove Theorems 3.1 and 3.2 (see §5) and to find the groups of all the irreducible curves (see §6). The calculation for reducible curves (which is absolutely similar and even easier, but involves too many curves to consider) can be found in [D1]; details will appear elsewhere.

I would like to express my gratitude to O. Viro, who inspired this work, and V. Kharlamov for his helpful remarks. My deepest gratitude is to V. A. Rokhlin, whose influence to Leningrad topology school in general and to myself in particular cannot be overestimated; in this work I used quite a few of his ideas, remarks, and observations. I am also thankful to Max-Planck-Institut für Mathematik: but for its hospitality, these results would never have been completed and published.

2. Notation

2.1 Group notation.

2.1.1. Given a group G, denote by KG and $K'G$ its first and second commutants, respectively: $KG = [G,G]$ and $K'G = K(KG)$.

2.1.2. Given $a, b \in G$, let $[a,b] = a^{-1}b^{-1}ab$.

2.1.3. Some standard groups:

- F_p is the free group of rank p;
- $T_{p,q}$ is the fundamental group of a toric link of type (p,q): if $p=2$, then $T_{2,2r} = \langle a, b \mid (ab)^r = (ba)^r \rangle$ and $T_{2,2r+1} = \langle a, b \mid (ab)^r a = b(ab)^r \rangle$; if g.c.d.$(p,q)=1$, then $T_{p,q} = \langle a, b \mid a^p = b^q \rangle$;
- B_p is the braid group on p strings:

  $$B_p = \langle \sigma_1 \ldots \sigma_{p-1} \mid [\sigma_i, \sigma_j] = 1 \text{ for } |i-j|>1,\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \rangle;$$

  in particular, $B_3 = T_{2,3}$;
- $G(T)$ and $G_p(T)$, where $T \in \mathbb{Z}[t]$ is an integral polynomial, are the extensions

  $$\{1\} \longrightarrow \mathbb{Z}[t]/T \longrightarrow G(T) \longrightarrow \mathbb{Z} \longrightarrow \{1\} \quad \text{and}$$
  $$\{1\} \longrightarrow \mathbb{Z}_p[t]/T \longrightarrow G_p(T) \longrightarrow \mathbb{Z} \longrightarrow \{1\},$$

  respectively, where the conjugation action of the generator of the quotient $\mathbb{Z}$ on the kernel is the multiplication by t;

- we use some notations from [CM], attaching Gr to avoid confusion. Thus,

$$\begin{aligned}
\mathrm{Gr}(p,q,r) &= \bigl\langle \alpha,\beta,\gamma \bigm| \alpha^p=\beta^q=\gamma^r=\alpha\beta\gamma=1\bigr\rangle,\\
\mathrm{Gr}\langle p,q,r\rangle &= \bigl\langle \alpha,\beta,\gamma \bigm| \alpha^p=\beta^q=\gamma^r=\alpha\beta\gamma\bigr\rangle,\\
\mathrm{Gr}\langle\!\langle p,q\,|\,r\rangle\!\rangle &= \bigl\langle \alpha,\beta \bigm| \alpha^p=\beta^q=1,\ (\alpha\beta)^r=(\beta\alpha)^r\bigr\rangle, \quad (r\in\mathbb{Z}).
\end{aligned}$$

The last notation is also used for $r=(2k+1)/2\in\frac{1}{2}\mathbb{Z}$; in this case the last relation reads $(\alpha\beta)^k\alpha=\beta(\alpha\beta)^k$, and it is shown in [CM] that $\mathrm{Gr}\langle\!\langle p,p\,|\,(2k+1)/2\rangle\!\rangle=\mathrm{Gr}\langle\!\langle 2,p\,|\,2k+1\rangle\!\rangle$.

2.2. Other notation.

2.1.1. We use Arnol′d's notation for the types of singular points (see [AVG]). In particular, A_p denotes a singularity given locally by $x^2+y^{p+1}=0$. A set of singularities is denoted like this: $5A_1\sqcup 2A_2\sqcup\ldots$

2.2.2. A curve is said to be *of type* $aC_p\sqcup bC_q\sqcup\ldots$ if it has a irreducible components of degree p, b irreducible components of degree q, etc.

2.2.3. $C_d(\Sigma)$, where Σ is a list of singularities, denotes an irreducible curve of degree d whose set of singular points is Σ. (If $d\leqslant 5$, such a curve is unique up to rigid isotopy.)

2.2.4. The mutual position of an irreducible curve C and a line L is denoted by a list $\{\ldots\}$ whose elements correspond to the intersection points of L and C:

$\times d$ – L meets C with multiplicity d at a nonsingular point of C;
A_p – L intersects C transversally at a singular point of C of type A_p;
A_p^* – L is tangent to C at a singular point of C of type A_p.

Remark. The notion of transversal intersection and tangency for the types A_p is obvious; the tangency is always assumed to have the smallest possible multiplicity.

3. Main results

3.1. Proposition. *Given four integers $p,r\geqslant 0$ and $a,b>0$ such that $ap<b(2r+1)$, there exists an irreducible curve C of degree $2b(2r+1)-ap$ with the fundamental group*

$$\bigl\langle \alpha_1,\alpha_2 \bigm| \alpha_1^p=\alpha_2^p,(\alpha_1\alpha_2)^r\alpha_1=\alpha_2(\alpha_1\alpha_2)^r,\alpha_1^{ap}=(\alpha_1\alpha_2)^{b(2r+1)}\bigr\rangle.$$

This group is abelian only if $r=0$ or $p=1$; otherwise, it is finite only if $p=2$ or $(p,r)=(3,1)$, $(3,2)$, $(4,1)$, or $(5,1)$.

3.2. Proposition. *If C is an irreducible curve of an odd degree $2k+1$ with a singular point adjacent to the semiquasihomogeneous singularity of type $(k,4k)$, then $\pi_1(\mathbb{C}\mathrm{p}^2 \smallsetminus C)$ is abelian.*

3.3. Quintics with nonabelian fundamental group. *The following is the complete list of complex plane projective quintics whose fundamental group Π is nonabelian:*

3.3.1 Irreducible quintics.

$C_5(3A_4)$: $\Pi = \langle a, b \mid b = ab^4a,\ a^2 = b^2a^3b^2 \rangle$:
- $\Pi/K\Pi = \mathbb{Z}_5$;
- $K\Pi/K'\Pi = \mathbb{Z}_2[t]/(t^4+t^3+t^2+t+1)$;
- $K'\Pi = \mathbb{Z}_2 \times \mathbb{Z}_2$ is a central subgroup of Π;
- $\operatorname{ord}\Pi = 320$.

$C_5(A_6 \sqcup 3A_2)$: $\Pi = \langle u, v \mid u = v^2u^2v^2,\ v^2 = uv^5u \rangle$
$= \langle u, v \mid u^3 = v^7 = (uv^2)^2 \rangle$:
$\Pi/u^3 = \mathrm{Gr}(2,3,7)$ is infinite.

3.3.2 The quintics of type $C_4 \sqcup C_1$ (see 2.2.4).

$C_4(3A_2)$	$\sqcup\{\times 2, \times 2\}$:	Π is given below
	$\sqcup\{\times 2, \times 1, \times 1\}$ or $\{A_2^*, \times 1\}$:	$\Pi = B_3$
	$\sqcup$ otherwise:	$\Pi = G_3(t+1)$
$C_4(2A_2 \sqcup A_1)$	$\sqcup\{\times 4\}$:	$\Pi = B_4$
	$\sqcup\{\times 2, \times 2\}$:	$\Pi = B_3$
$C_4(2A_2)$	$\sqcup\{\times 4\}$ or $\{\times 2, \times 2\}$:	$\Pi = B_3$
$C_4(A_4 \sqcup A_2)$	$\sqcup\{\times 3, \times 1\}$:	$\Pi = \mathbb{Z} \times \mathrm{Gr}\langle 2,3,5\rangle$
	$\sqcup\{A_4^*\}$:	$\Pi = B_3$
	$\sqcup\{A_2, \times 2\}$:	$\Pi = G_5(t+1)$
$C_4(A_3 \sqcup A_2)$	$\sqcup\{A_2, \times 2\}$:	$\Pi = B_3$
$C_4(A_6)$	$\sqcup\{A_6, \times 2\}$:	$\Pi = B_3$
$C_4(A_5)$	$\sqcup\{\times 4\}$ or $\{\times 2, \times 2\}$:	$\Pi = B_3$
$C_4(E_6)$	$\sqcup\{\times 4\}$:	$\Pi = T_{3,4}$
	$\sqcup\{\times 2, \times 2\}$:	$\Pi = B_3$

The fundamental group of a curve of type $C_4(3A_2) \sqcup \{\times 2, \times 2\}$ is

$$\Pi = \langle a, b, c \mid aba = bab,\ bcb = cbc,\ abcb^{-1}a = bcb^{-1}abcb^{-1} \rangle.$$

3.3.3. The quintics of type $C_3 \sqcup C_2$. The only such quintic with nonabelian fundamental group has the cubic component of type $C_3(A_2)$ which intersects the other component (the quadric) at two points with multiplicity 3 at each. The group is $\Pi = \langle a, b \mid [a^3, b] = 1,\ ab^2 = ba^2 \rangle$, and one has:

- $\Pi/K\Pi = \mathbb{Z}$;
- $K\Pi$ is the quaternion group $\langle i, j \mid i^2 = j^2 = (ij)^2 \rangle$, and the conjugation by the generator of $\Pi/K\Pi$ acts via $i \mapsto j$, $j \mapsto ij$;
- $K\Pi/K'\Pi = \mathbb{Z}_2[t]/(t^2+t+1)$;
- $K'\Pi = \mathbb{Z}_2$ is a central subgroup of Π.

3.3.4. The quintics of type $C_3 \sqcup 2C_1$. The position of each of the linear components in respect to the cubic is denoted using 2.2.4. If the two linear components intersect each other at a point in the cubic, the corresponding elements in their lists are underlined.

$C_3(A_2) \sqcup \{\times 3\} \sqcup \{\times 2, \times 1\}$:	Π is given below
$C_3(A_2) \sqcup \{A_2^*\} \sqcup \{\times 3\}$:	$\Pi = T_{2,6}$
$C_3(A_2) \sqcup \{\underline{\times 3}\} \sqcup \{A_2, \underline{\times 1}\}$:	$\Pi = T_{2,4}$
$C_3(A_2) \sqcup \{\underline{\times 3}\} \sqcup \{\underline{\times 1}, \times 1, \times 1\}$:	$\Pi = \mathbb{Z} \times B_3$
$C_3(A_2) \sqcup \{\times 3\} \sqcup \{A_2, \times 1\}$:	$\Pi = \mathbb{Z} \times B_3$
$C_3(A_2) \sqcup \{\times 3\} \sqcup \{\times 1, \times 1, \times 1\}$:	$\Pi = \mathbb{Z} \times B_3$
$C_3(A_2) \sqcup \{\underline{\times 2}, \times 1\} \sqcup \{\times 2, \underline{\times 1}\}$:	$\Pi = \mathbb{Z} \times B_3$
$C_3(A_2) \sqcup \{A_2, \underline{\times 1}\} \sqcup \{\times 2, \underline{\times 1}\}$:	$\Pi = G(t^2-1)$
$C_3(A_1) \sqcup \{\times 3\} \sqcup \{\times 3\}$:	$\Pi = G(t^3-1)$
$C_3(A_1) \sqcup \{\underline{\times 3}\} \sqcup \{\times 2, \underline{\times 1}\}$:	$\Pi = G(t^2-1)$
$C_3(A_1) \sqcup \{\times 2, \underline{\times 1}\} \sqcup \{\times 2, \underline{\times 1}\}$:	$\Pi = G(t^2-1)$

The fundamental group of a curve of type $C_3(A_2) \sqcup \{\times 3\} \sqcup \{\times 2, \times 1\}$ is

$$\Pi = \langle a, b, c \mid aca = cac, [b, c] = 1, (ab)^2 = (ba)^2 \rangle$$

3.3.5. The quintics of type $2C_2 \sqcup C_1$.

- the two quadrics have an intersection point of multiplicity 4. If the linear component is their common tangent at this point, then $\Pi = F_2$; otherwise, $\Pi = T_{2,4}$;
- the two quadrics touch each other at two points. If the linear component passes through these two points, then $\Pi = F_2$; otherwise, $\Pi = T_{2,4}$;
- the two quadrics have a common point of multiplicity 3, and the linear component is their common tangent at this point. $\Pi = \mathbb{Z} \times B_3$.

3.3.6. The quintics of type $C_2 \sqcup 3C_1$.
The three linear components have a common point.

- if two of them are tangent to the quadric, then
  $\Pi = \langle a, b, c \mid [a, b] = [a, c^{-1}bc] = 1,\ (bc)^2 = (cb)^2 \rangle$;

- otherwise, $\Pi = \mathbb{Z} \times F_2$.

The three linear components do not have a common point, and two of them are tangent to the quadric.

- if the third line is also a tangent, then $\Pi = \langle a, b, c \mid (ab)^2 = (ba)^2,\ (ac)^2 = (ca)^2,\ [b, c] = 1\rangle$;
- if the third line passes through the tangency points of the first two, then $\Pi = \mathbb{Z} \times F_2$;
- otherwise, $\Pi = \mathbb{Z} \times T_{2,4}$.

3.3.7. The quintics of type $5C_1$. The fundamental group depends on the singular points of multiplicity greater than 2:

- if there is a 5-ple point, then $\Pi = F_4$;
- if there is a quadruple point, then $\Pi = \mathbb{Z} \times F_3$;
- if there are two triple points, then $\Pi = F_2 \times F_2$;
- if there is only one triple point, then $\Pi = \mathbb{Z} \times \mathbb{Z} \times F_2$.

4. Van Kampen's method

Below we give a description of a slight modification of well-known van Kampen's method (see [vK]). The principal difference from the classical version is that we do not assume the projection generic; its center may belong to the curve and even be one of its singular points.

4.1. General idea. Let $C \in \mathbb{C}\mathrm{p}^2$ be an algebraic curve. Pick a point $O \in \mathbb{C}\mathrm{p}^2$ and a line L_0 through O. Consider the canonical projection $\mathrm{pr}\colon \mathbb{C}\mathrm{p}^2 \smallsetminus O \to \mathbb{C}\mathrm{p}^1$ and pick a generic fiber L of pr. Then van Kampen's method gives a representation of the fundamental group of $\mathbb{C}\mathrm{p}^2 \smallsetminus C$ which consists of:

(4.1.1) one generator α_i for each intersection point $S_i \in C \cap L$ other than O;
(4.1.2) one generator γ_j for each singular fiber L_j of pr (see 4.2) other than L_0;
(4.1.3) relations $\gamma_j^{-1}\alpha_i\gamma_j = m_j\alpha_j$, where $m_j\colon \langle\alpha_1, \dots\rangle \to \langle\alpha_1, \dots\rangle$ is the braid monodromy along γ_j (see 4.3);
(4.1.4) one relation $\overline{\gamma}_j = 1$ for each singular fiber L_j, $j \geqslant 1$, which is not a component of C; here $\overline{\gamma}_j = \gamma_j w_j$ for a certain word w_j in $\alpha_1, \dots$ (see 4.5);
(4.1.5) relation $\alpha_1 \dots \gamma_1 \dots = 1$, present if L_0 is not a component of C.

4.2. Singular fibers and generators. A fiber L of pr is called *singular* (in respect to C) if $\#(L \cap C) \neq \deg \mathrm{pr}_C$. Thus, L is singular if it either is a component of C, or is tangent to C, or intersects C at a singular point other than O, or is tangent to a branch of C at O (i.e., the proper transforms of L and C in the blow-up of $\mathbb{C}\mathrm{p}^2$ at O meet at a point of the exceptional divisor). Let $L_1, \dots, L_q$ be all the singular fibers other than L_0. Pick some small disjoint closed disks $d_j \subset \mathbb{C}\mathrm{p}^1$ about $\operatorname{pr} L_j$ and let $\widetilde{d}_j = \operatorname{pr}^{-1} d_j \cup O$. Fix another line $M \not\ni O$ close to L_0. (More precisely, we let $M = M^{(1)}$, where $M^{(t)}$ is a perturbation of $L_0 = M^{(0)}$ so small that for each $t \in (0,1]$ the line $M^{(t)}$ meets $C \cup L_0$ transversally and does not intersect C in $\bigcup \partial \widetilde{d}_j$.) Let $S = L \cap M$. Choose a system of simple disjoint (except S) paths σ_j connecting S and $\partial \widetilde{d}_j \cap M$ and let γ_j be the loop which goes along σ_j, then along the circle $\partial \widetilde{d}_j \cap M$ in the positive direction, and then comes back to S along σ_j^{-1}. We assume that σ_j are chosen so that $\gamma_1 \dots \gamma_q$ is homotopic to a large circle in M surrounding all the $M \cap L_j$, $j \geqslant 1$. Then $\gamma_1, \dots, \gamma_q$ form a standard simple basis of $\pi_1(M \setminus \bigcup_{j \geqslant 0} L_j, S)$.

Remark. Note that, unlike the classical construction, γ_j surrounds not only L_j, but also the branches of C at O tangent to L_j. Hence, in general γ_j may not be contractible in $\mathbb{C}\mathrm{p}^2 \setminus C$.

The generators $\alpha_1, \dots \alpha_p$ are constructed in a similar manner, as a standard simple basis of $\pi_1(M \setminus O \cup \bigcup S_i, S)$, where $S_1, \dots, S_p$ are all the intersection points $L \cap C$ other than O.

4.3. Braid monodromy. Let $s\colon I \to Y$ be a path in $Y = \mathbb{C}\mathrm{p}^2 \setminus C \cup \bigcup_{j \geqslant 0} L_j$, and let L' and L'' be the fibers of pr through $s(0)$ and $s(1)$ respectively. The *braid monodromy* along s (relative to C) is the homeomorphism $m_s\colon (L'; O, s(0), C \cap L') \to (L''; O, s(1), C \cap L'')$, defined up to relative homotopy, constructed as follows: Consider the fibration $s^*\mathrm{pr}\colon (s^*Y, s^*C) \to I$. It is trivial. Moreover, its restriction to s^*C is trivialized (as its fiber is discrete), and this trivialization extends to s^*Y, which gives a fiberwise homeomorphism of s^*Y to the cylinder $L' \times I$. By definition, m_s is the composition of the inclusion of the base over 0 and projection to the base over 1, which is L''.

Since m_s is defined up to relative homotopy, it induces a well defined isomorphism (also denoted by m_s) $\pi_1(L' \setminus O \cup C, s(0)) \to \pi_1(L'' \setminus O \cup C, s(1))$.

To simplify the notation, denote $m_j = m_{\gamma_j}$. From the Serre exact sequence of the fibration $\mathrm{pr}|_Y$ it immediately follows that $\pi_1(Y, S)$ is generated by $\alpha_1, \dots, \alpha_p, \gamma_1, \dots, \gamma_q$, and the defining relations are (4.1.3).

Consider now a small analytical branch B at a point of a singular fiber L_j different from O. Our next goal is to express the loop ∂B in terms of the standard generators. Assume that the disk d_j (see 4.2) is so small that the restriction of pr to $B \cap \widetilde{d}_j$ is proper and all the fibers over $d_j \setminus \operatorname{pr} L_j$ are transversal to B. Then, given a path in $\widetilde{d}_j$, one can obviously speak about the braid monodromy in respect to $C \cup B$. Assume for a moment that the base fiber L is in $\partial \widetilde{d}_j$. (Afterwards we can drag it back along σ_j^{-1} and translate everything via the braid monodromy.) Let $\{P_1, \dots, P_r\} = B \cap L$. Denote by β' the loop $B \cap \partial \widetilde{d}_j$ starting at P_1. Pick a path ω in L connecting S and P_1

and disjoint from C, and let $\beta = \omega \cdot \beta' \cdot \omega^{-1}$.

4.3.1. Proposition. *One has $\beta = \gamma_j^r w$, where $r = \#(B \cap L)$ and w is the word in $\alpha_1, \dots, \alpha_p$ corresponding to the loop $m_j^r \omega \cdot \omega^{-1}$.*

Proof. The statement is obvious if $r = 1$: the 'square' drawn by ω when it is dragged along γ_j gives a homotopy $\gamma_j \sim \omega \cdot \beta' \cdot m_j \omega^{-1}$. In the general case, assume that the points P_k are ordered so that m_j induces the cyclic permutation $(P_1, \dots, P_k)$, and denote by β_k, $k = 1, \dots, r$, the loop which goes from S to P_k along $m_j^{k-1}\omega$, then goes along $B \cap \partial\tilde{d}_j$ to P_{k+1}, and comes back to S along $m_j^k \omega^{-1}$. (We let $P_{r+1} = P_1$.) Then similar arguments show that $\gamma_j \sim \beta_k$ for all k. On the other hand, $\beta_1 \dots \beta_r = \beta \cdot \omega \cdot m_j^r \omega^{-1}$, and the result follows. □

4.4. Patching L_0 (relation (4.1.5)). It is clear that patching L_0 ads to the representation a relation $\gamma_0 = 1$, where γ_0 is a small loop in M around $L_0 \cap M$. On the other hand, in $\pi_1(M \smallsetminus \bigcup L_i)$ one has $\gamma_0^{-1} = \overline{\alpha}_1 \dots \overline{\alpha}_p \gamma_1 \dots \gamma_q$, where $\overline{\alpha}_i$ are some appropriate loops surrounding the intersection points $M \cap C$, and rotating M about S to L shows that $\overline{\alpha}_1 \dots \overline{\alpha}_p = \alpha_1 \dots \alpha_p$. This gives (4.1.5).

4.5. Patching the singular fibers (relations (4.1.4)). Patching a fiber L_j adds a relation $\overline{\gamma}_j = 1$, where $\overline{\gamma}_j$ is a small loop in Y about L_j. To construct such a loop, choose another line M', which intersects L_j 'far' from C (more precisely, we require that $M' \cap \tilde{d}_j$ should not intersect C), and let $\overline{\gamma}_j$ be the loop $M' \cap \partial\tilde{d}_j$, connected to a point in $M \cap \partial\tilde{d}_j$ along a fiber and then to S along σ_j. Proposition 4.3.1 gives $\overline{\gamma}_j = \gamma_j w_j$, where w_j is a word in $\alpha_1, \dots, \alpha_p$, which can be easily found using the local monodromy about L_j.

4.6. Birational transformations. Let $\mathbb{C}\mathrm{p}^2 \xleftarrow{\rho} X \xrightarrow{\overline{\rho}} \mathbb{C}\mathrm{p}^2$ be a birational transformation of $\mathbb{C}\mathrm{p}^2$. (Here ρ and $\overline{\rho}$ are two sequences of blow-ups.) Consider a curve C in the first copy of $\mathbb{C}\mathrm{p}^2$ and denote by $\overline{C}$ its proper transform in the second copy. Let E_k (resp. $\overline{E}_l$) be the projections to the first (resp. second) copy of $\mathbb{C}\mathrm{p}^2$ of the exceptional divisors of $\overline{\rho}$ (resp. ρ). Then it is clear that $\pi_1(\mathbb{C}\mathrm{p}^2 \smallsetminus C \cup \bigcup E_k) = \pi_1(\mathbb{C}\mathrm{p}^2 \smallsetminus \overline{C} \cup \bigcup \overline{E}_l)$, and, hence, $\pi_1(\mathbb{C}\mathrm{p}^2 \smallsetminus C)$ can be obtained from $\pi_1(\mathbb{C}\mathrm{p}^2 \smallsetminus \overline{C} \cup \bigcup \overline{E}_l)$ by adding the relations corresponding to gluing in all the E_k's. Such a relation can be chosen in the form $[\partial b_k] = 1$ or $[\partial \overline{b}_k] = 1$, where b_k is a small analytical branch transversal to E_k and disjoint from C, and $\overline{b}_k$ is its proper transform in the second copy of $\mathbb{C}\mathrm{p}^2$. Now, $[\partial \overline{b}_k]$ can be found using Proposition 4.3.1.

We will use the three well-known quadratic birational transformations. Each of them is determined by its three fundamental points (O_1, O_2, O_3) and is denoted by $T(O_1, O_2, O_3)$. (Some of the fundamental points may be infinitely near; the fact that O' is infinitely near to O is denoted by $O \leftarrow O'$.)

The fundamental points (of both T and T^{-1}), exceptional divisors E_k and $\overline{E}_l$, and branches $\overline{b}_k$ are shown in Fig. 1–3, which also represent the intermediate configuration appearing in X.

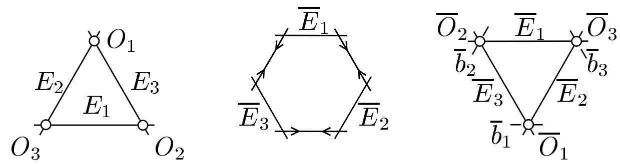


FIGURE 1. $T(O_1, O_2, O_3)$

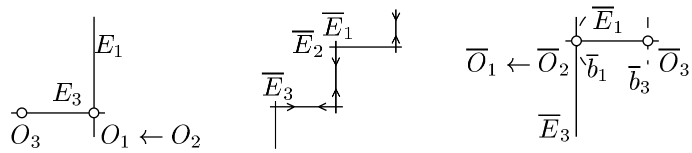


FIGURE 2. $T(O_1 \leftarrow O_2, O_3)$

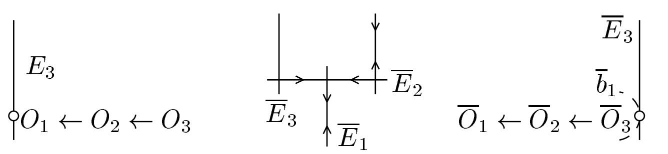


FIGURE 3. $T(O_1 \leftarrow O_2 \leftarrow O_3)$

4.7. First results. As an immediate consequence of the above machinery, one obtains the following:

4.6.1. Proposition. *Suppose that C has a singular point O of multiplicity m and does not have linear components through O. Then $\pi_1(\mathbb{C}\mathrm{p}^2 \smallsetminus C)$ admits a representation with at most $(\deg C - m)$ generators.*

4.6.2. Corollary. *If an irreducible curve C has a singular point of multiplicity $(\deg C - 1)$, then $\pi_1(\mathbb{C}\mathrm{p}^2 \smallsetminus C)$ is abelian.*

4.6.3. Proposition. *If C has a singular point of multiplicity $(\deg C - 1)$ and consists of r components, $r \geqslant 2$, then $\pi_1(\mathbb{C}\mathrm{p}^2 \smallsetminus C) = \mathbb{Z} \times F_{r-2}$.*

5. Curves with deep singularities

In this section we consider a curve C with a singular point of multiplicity $(\deg C-2)$. The main results are Theorem 5.2, which is proved in 5.4, and proof of Propositions 3.1 and 3.2.

5.1. Classification (see [D2]). Throughout this section we assume fixed a curve C and a singular point O of C of multiplicity $(\deg C-2)$. Let $C=\overline{C}\cup\overline{L}$, where $\overline{C}$ has no linear components through O and $\overline{L}$ is the union of all such components of C, and let $L_1,\dots,L_q$ be all the singular fibers (in respect to C) of the projection from O. Consider the blow-up of $\mathbb{C}\mathrm{p}^2$ at O and denote by E the exceptional divisor and by $\widetilde{C}$, $\widetilde{L}$, and $\widetilde{L}_j$ the proper transforms of $\overline{C}$, $\overline{L}$, and L_j respectively.

5.1.1. Definition. A pair (p,q) of nonnegative integers is called *admissible* if either $p=q$ or the smallest of p,q is even. The *reduced type* of a singular fiber L_j is the admissible pair (p_j,q_j) defined as follows: p_j is the local intersection index of $\widetilde{C}$ and E at $\widetilde{L}_j\cap E$, and $q_j=0$, 1, or $k\geqslant 2$ if, respectively, $\widetilde{C}$ intersects $\widetilde{L}_j$ transversally, is tangent to $\widetilde{L}_j$, or has a singular point of type A_{k-1} on $\widetilde{L}_j$ (see Fig. 4). Note that if $p_j>q_j$, then $\widetilde{C}$ has two branches intersecting $\widetilde{L}_j$, and one of them has greater local intersection index with E; this branch will be called the *principal branch* of C at L_j. The *formula* of C is the set $\{(p_j,q_j)\}$ of the reduced types of all the singular fibers enriched with the following two additional structures:

(1) if L_j is a component of C, its reduced type is marked;
(2) if all the q_j's are even, then the types (p_j,q_j) with $p_j>q_j$ split into two classes $\mathcal{B}_1$, $\mathcal{B}_2$ in the following way: under the hypotheses, $\overline{C}$ consists of two components $\overline{C}_1$, $\overline{C}_2$, and we say that $(p_j,q_j)\in\mathcal{B}_r$ iff the principal branch at L_j is in $\overline{C}_r$. (If there is q_j odd, we let $\mathcal{B}_2=\varnothing$.)

5.1.2. Proposition (see [D2]). *The pair (C,O) is determined by its formula up to rigid isotopy (i.e., isotopy through algebraic curves with distinguished singular point). Furthermore, any abstract formula (i.e., a finite set $\{(p_j,q_j)\}$ of admissible pairs enriched with the above additional structures) with $\sum q_j=2\sum p_j+2$ is realized by an algebraic curve of degree $\sum p_j+2+\{$number of marked pairs$\}$; this curve is irreducible iff there is no marked pairs and there is at least one pair (p_j,q_j) with q_j odd.*

5.2. Theorem. *Denote by σ the automorphism of $\langle\alpha_1,\alpha_2\rangle$ which takes α_1 to $\alpha_1\alpha_2\alpha_1^{-1}$ and α_2 to α_1. Given a curve C with a singular point O of multiplicity $(\deg C-2)$, the fundamental group $\pi_1(\mathbb{C}\mathrm{p}^2\smallsetminus C)$ admits the following*

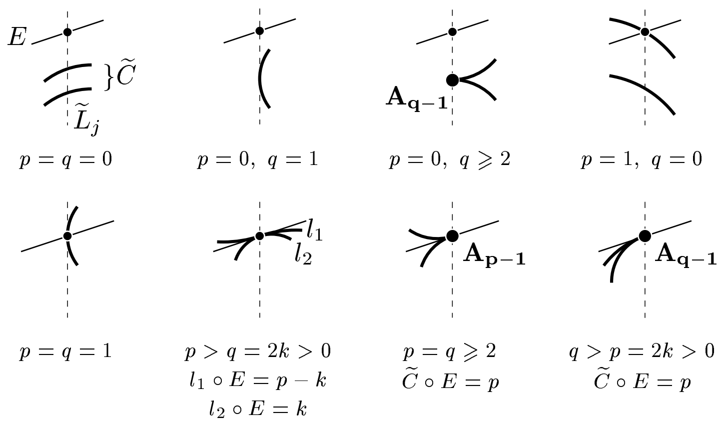


FIGURE 4

representation:

(1) *there are generators* γ_j, $j = 1, \ldots, q$, *corresponding to the singular fibers* L_j *(or pairs* (p_j, q_j)*), and two more generators* α_1, α_2*;*
(2) *there is relation* $\alpha_1\alpha_2\gamma_1 \ldots \gamma_q = 1$*;*
(3) *each marked pair* (p_j, q_j) *gives two relations* $\alpha_i = \sigma^{q_j - 2p_j}\alpha_i$, $i = 1, 2$*;*
(4) *each nonmarked pair* (p_j, q_j) *gives one of the following relations:*
 (a) $p_j = q_j = 2k$: $\quad \gamma_j = (\alpha_1\alpha_2)^k$;
 (b) $p_j = q_j = 2k + 1$: $\quad \alpha_1 = \alpha_2, \quad \gamma_j = \alpha_1^{p_j}$;
 (c) $q_j > p_j = 2k$: $\quad \alpha_i = \sigma^{q_j - p_j}\alpha_i,\ i = 1, 2, \quad \gamma_j = (\alpha_1\alpha_2)^k$;
 (d) $p_j > q_j = 2k$: $\quad [\alpha_r, \alpha_s^{p_j - q_j}] = 1, \quad \gamma_j = \alpha_s^{q_j - p_j}(\alpha_1\alpha_2)^{p_j - k}$,
 where $(p_j, q_j) \in \mathcal{B}_r$ *and* $s = 3 - r$*.*

This theorem is proved in 5.4.

5.3. The local monodromy. Fix a singular fiber L_j of type (p_j, q_j) and assume that the base fiber L is in $\partial\widetilde{d}_j$ (see the notation in §4). Let α_1, α_2 be the two generators of $\pi_1(L \smallsetminus C \cup O, S)$; if $p_j > q_j$, we assume that α_1 corresponds to the principal branch. Keeping in mind other applications, let us also consider several branches $B_1, \ldots, B_k$, which meet L_j transversally 'far' from C, M, and M' (i.e., $B_i \cap \widetilde{d}_j$ does not intersect these curves), and complete α_1, α_2 to a simple basis $\alpha_1, \alpha_2, \beta_1, \ldots, \beta_k$ of $\pi_1(L \smallsetminus (C \cup O \cup \bigcup B_i), S)$ (see Fig. 5, where $S' = M' \cap L$ and $P_i = B_i \cap L$). Let $\rho = \beta_1 \ldots \beta_k$.

Considering model examples, one can easily find the braid monodromy m_j along $\gamma_j = \partial\widetilde{d}_j$ and the word w_j (see 4.5). The results are listed below, where we give the monodromy operator m_j, the word $w_j = m_j\omega \cdot \omega^{-1}$, and

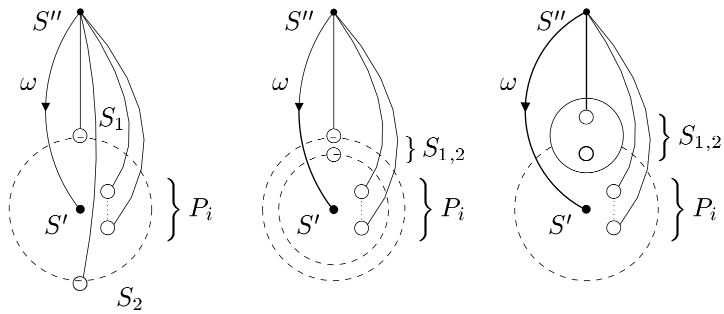


FIGURE 5

an equation of the model curve in an affine coordinate system (x, y) in which L_j, L, and M' have equations $x = 0$, $x = \epsilon$ and $y = 0$ respectively, $0 < \epsilon \ll 1$, and d_j is the disk $|x| < \epsilon$.

5.3.1. The case $p_j = q_j$. *The model curve is $x^{p_j} y^2 = 1$. The points S_1, S_2 are rotating through $-p_j\pi$ about S'. Thus, $m_j = \delta^{p_j}$, where δ is the operator corresponding to the rotation through $-\pi$:*

$$\begin{aligned} \delta\colon\quad \alpha_1 &\mapsto \rho^{-1}\alpha_2\rho, \\ \alpha_2 &\mapsto \rho^{-1}\alpha_2^{-1}\rho\alpha_1\rho^{-1}\alpha_2\rho, \\ \beta_i &\mapsto \rho^{-1}\alpha_2^{-1}\rho\beta_i\rho^{-1}\alpha_2\rho, \end{aligned}$$

and $w_j^{-1} = \alpha_1 \cdot \delta\alpha_1 \cdot \ldots \cdot \delta^{p_j-1}\alpha_1$.

5.3.2. The case $p_j > q_j = 2k$. *The model is $(x^k y - 1)(x^{p_j-k} y - 1) = 0$. The points S_1, S_2 are rotating about S' through $2(k - p_j)\pi$ and $-2k\pi$ respectively. Thus, $m_j = \delta_1^{p_j-k}\delta_2^k$, where δ_i is the operator corresponding to rotating S_i through -2π (obviously $[\delta_1, \delta_2] = 1$):*

$$\begin{aligned} \delta_1\colon\quad \alpha_1 &\mapsto \rho^{-1}\alpha_2^{-1}\alpha_1\alpha_2\rho, \\ \alpha_2 &\mapsto \rho^{-1}\alpha_2^{-1}\alpha_1^{-1}\alpha_2\rho\alpha_2\rho^{-1}\alpha_2^{-1}\alpha_1\alpha_2\rho, \\ \beta_i &\mapsto \rho^{-1}\alpha_2^{-1}\alpha_1^{-1}\alpha_2\rho\beta_i\rho^{-1}\alpha_2^{-1}\alpha_1\alpha_2\rho, \\ \delta_2\colon\quad \alpha_1 &\mapsto \alpha_1, \\ \alpha_2 &\mapsto \rho^{-1}\alpha_2\rho, \\ \beta_i &\mapsto \rho^{-1}\alpha_2^{-1}\rho\beta_i\rho^{-1}\alpha_2\rho, \end{aligned}$$

and $w_j^{-1} = \alpha_1 \cdot (\alpha_2 \cdot \delta_2\alpha_2 \cdot \ldots \cdot \delta_2^{k-1}\alpha_2) \cdot \alpha_1^{-1} \cdot (\alpha_1 \cdot \delta_1\alpha_1 \cdot \ldots \cdot \delta_1^{p_j-k-1}\alpha_1)$.

5.3.3. The case $q_j > p_j = 2k$. *The model curve is $(x^k y - 1)^2 = y^{q_j - p_j}$. The small disk containing S_1 and S_2 is translated along the large circle through $-2k\pi$ and rotates about its center through $(q_j - 2p_j)\pi$. If δ_1 and δ_2 correspond* *to the rotation through* $-\pi$ *and translating through* -2π *respectively, then* $[\delta_1, \delta_2] = 1$ *and* $m_j = \delta_1^{2p_j - q_j} \delta_2^k$:

$$\begin{aligned}
\delta_1\colon\ & \alpha_1 \mapsto \alpha_2, \\
& \alpha_2 \mapsto \alpha_2^{-1}\alpha_1\alpha_2, \\
& \beta_i \mapsto \beta_i, \\
\delta_2\colon\ & \alpha_1 \mapsto \rho^{-1}\alpha_1\rho, \\
& \alpha_2 \mapsto \rho^{-1}\alpha_2\rho, \\
& \beta_i \mapsto \rho^{-1}\alpha_2^{-1}\alpha_1^{-1}\rho\beta_i\rho^{-1}\alpha_1\alpha_2\rho,
\end{aligned}$$

and $w_j^{-1} = \alpha_1\alpha_2 \cdot \delta_2(\alpha_1\alpha_2) \cdot \ldots \cdot \delta_2^{k-1}(\alpha_1\alpha_2)$

5.4. Proof of Theorem 5.2. The group is found by van Kampen's method, using the results of 5.3, where we let $\beta_1 = \cdots = \beta_k = \rho = 1$. This gives the generators $\gamma_1, \ldots, \gamma_q$ and α_1, α_2 (the two latters generate the group of a fixed generic fiber L) and the braid monodromy $m_j = \sigma^{q_j - 2p_j}$, which provides for relations 5.2 (2). (Note that 5.3 gives the monodromy in some *local* generators α_1', α_2' of the group of a generic fiber L' close to L_j. However, α_1', α_2' differ from α_1, α_2 by the action of the braid group B_2, i.e., by a power of σ. Hence, the monodromy has the same form in α_1, α_2.)

Patching a nonmarked fiber L_j gives an additional relation $\gamma_j = w_j^{-1}$, which in some local generators α_1', α_2' has the form (cf. 5.2 (4)):

(a) $p_j = q_j = 2k$: $\gamma_j = (\alpha_1'\alpha_2')^k$;
(b) $p_j = q_j = 2k+1$: $\gamma_j = \alpha_1'(\alpha_1'\alpha_2')^k$;
(c) $q_j > p_j = 2k$: $\gamma_j = (\alpha_1'\alpha_2')^k$;
(d) $p_j > q_j = 2k$: $\gamma_j = \alpha_1'\alpha_2'^{\,q_j - p_j + 1}(\alpha_1'\alpha_2')^{p_j - k - 1}$.

Combining this with $\gamma_j^{-1}\alpha_i'\gamma_j = \sigma^{q_j - 2p_j}\alpha_i'$, $i = 1, 2$, after simplification one gets 5.2 (4) written in α_1', α_2'. Now it suffices to notice that the resulting normal subgroup is σ-invariant, and hence these relations can be written in α_1, α_2. The only exception is case (d), $p_j > q_j$, when the curve has two asymmetric branches at L_j; in this case the relations are only σ^2-invariant and, hence, one should take into account the permutation monodromy. □

5.5. Simplification of the group. Combining like relations provided by Theorem 5.2, one arrives to a representation which has one or several relations from the following list:

(5.5.1)	$\alpha_1^p = \alpha_2^p,$	$p \geqslant 0,$
(5.5.2)	$(\alpha_1\alpha_2)^r\alpha_1 = \alpha_2(\alpha_1\alpha_2)^r,$	$q \geqslant 0,$
(5.5.3)	$\alpha_1^u = (\alpha_1\alpha_2)^w,$	$2w - u = \deg C > 0,$
(5.5.4)	$[\alpha_1^p, \alpha_2] = [\alpha_1, \alpha_2^q] = 1,$	$p, q \geqslant 0,$
(5.5.5)	$(\alpha_1\alpha_2)^r = (\alpha_2\alpha_1)^r,$	$r \geqslant 0,$
(5.5.6)	$\alpha_1^u\alpha_2^v = (\alpha_1\alpha_2)^w,$	$2w - u - v = \deg C > 0,$
(5.5.7)	$\gamma_j^{-1}\alpha_i\gamma_j = \sigma^{s_j}\alpha_i,\ i = 1, 2.$	

More precisely, one has:

5.5.8. Proposition. *The fundamental group of a curve C with a singular point of multiplicity* $(\deg C - 2)$ *has one of the following representations:*

$$
\begin{aligned}
O\ &= \bigl\langle \alpha_1, \alpha_2 \bigm| (5.5.1)\text{–}(5.5.3) \textit{ with } p\,|u \textit{ and } (2r+1)|w \bigr\rangle,\\
O'\ &= \bigl\langle \alpha_1, \alpha_2 \bigm| (5.5.1), (5.5.2) \bigr\rangle,\\
O'_k &= \bigl\langle \alpha_1, \alpha_2, \gamma_1, \ldots, \gamma_k \bigm| (5.5.1), (5.5.2), (5.5.7) \bigr\rangle,\\
E\ &= \bigl\langle \alpha_1, \alpha_2 \bigm| (5.5.4)\text{–}(5.5.6) \textit{ with } p\,|u,\ q\,|v, \textit{ and } r\,|w \bigr\rangle,\\
E'\ &= \bigl\langle \alpha_1, \alpha_2 \bigm| (5.5.4), (5.5.5) \bigr\rangle,\\
E'_k &= \bigl\langle \alpha_1, \alpha_2, \gamma_1, \ldots, \gamma_k \bigm| (5.5.4), (5.5.5), (5.5.7) \bigr\rangle,
\end{aligned}
$$

Proof. The representations are provided by Theorem 5.2. Several relations $\sigma^{n_j}\alpha_i = \alpha_i$ (see 5.2 (4c)) give $\sigma^n\alpha_i = \alpha_i$, $n = \text{g.c.d.}(n_i)$, which is equivalent to (5.5.2) if $n = 2r + 1$ or (5.5.5) if $n = 2r$. Similarly, several relations $[\alpha_1^{p_i}, \alpha_2] = 1$ (see 5.2 (4d)) give $[\alpha_1^p, \alpha_2] = 1$, $p = \text{g.c.d.}(p_i)$. If (5.5.2) is present, (5.5.4) is equivalent to (5.5.1) with $p = \text{g.c.d.}(p, q)$. Finally, (5.5.3) and (5.5.6) are obtained from 5.2 (2) after replacing each γ_j with its expressions in α_1, α_2. (If at least one fiber is a component of C, one can take it for L_0, and 5.2 (2) does not appear.) Since the powers of α_i appear only from 5.2 (4b) or (4d), together with the commutativity relations, one can collect them all together and, if (5.5.2) is present, replace α_2 with α_1. This also implies the divisibility conditions $p\,|u$ and $q\,|v$. The other two divisibility conditions are proved as follows: If 5.2 (4b) is present, then $q = 0$ in (5.5.2), and the condition $(2q+1)|w$ is trivial. Otherwise, since $u + v$ is the sum of $(p_j - q_j)$ over all the singular fibers L_j with $p_j \geqslant q_j$ and $2w = \deg C + u + v = \sum(q_j - p_j) + u + v$, it is the sum of $(q_j - p_j)$ over all the fibers with $q_j > p_j$, i.e., those which give the relations $\alpha_i = \sigma^{q_j - p_j}\alpha_i$ resulting in (5.5.2) or (5.5.5). Hence, the greatest common divisor of these numbers divides $2w$. ☐

Below we list some elementary properties of the groups obtained, assuming fixed some particular values of the parameters $p, q, r, \ldots$. Note, by the way, that E' and E obviously do not change under a permutation of (p, q, r) (respectively, a simultaneous permutation of (p, q, r) and (u, v, w)).

5.5.9. *One has* $O'_k \cong O' \times F_k$. *If all the* s_j *in* (5.5.7) *are even, then also* $E'_k = E' \times F_k$; *otherwise one can let* $s_1 = 1$ *and* $s_j = 0$ *for* $j \geqslant 2$.

Proof. The statement follows from the fact that σ is an inner automorphism of O' and σ^2 is an inner automorphism of E'. ☐

5.5.10. *One has*:

1. *if* $p \neq 0$, *then* $KO = KO' = K\,\mathrm{Gr}\langle\langle p, p \,|\, (2r+1)/2\rangle\rangle$. *In particular, the commutant is trivial only when* $p = 1$ *or* $r = 0$, *and it is finite only for the values of* (p, r) *listed in Table 1*;
2. *if* $p = 0$, *then* $O' = T_{2,2r+1}$ *and* $KO = KO' = F_{2r}$;
3. *if* $\text{g.c.d.}(p, 4r+2) = 1$, *then* $O' = \mathbb{Z} \times KO'$;

5.5.11. *One has*:

1. *if* $p, q, r \neq 0$, *then* $KE = KE' = K\,\mathrm{Gr}\langle\langle p, q \,|\, r\rangle\rangle$. *In particular, the commutant is trivial only when one of these numbers is* 1, *and it is finite only for the values of* (p, q, r) *listed in Table 2*;
2. *if* $r = 0$, *then* $KE = KE' = F_{(p-1)(q-1)}$. *If, besides,* $\text{g.c.d.}(p, q) = 1$, *then* $E' = \mathbb{Z} \times T_{p,q}$;
3. *if* $p = q = 0$ *and* $r > 1$, *then* $KE = KE' = F_\infty$;
4. *if* $\text{g.c.d.}(p, q) = \text{g.c.d.}(p, r) = \text{g.c.d.}(q, r) = 1$, *then* $E' = \mathbb{Z} \times \mathbb{Z} \times KE'$. *If, besides,* $\text{g.c.d.}(w - u, w - v) = 1$, *then* $E = \mathbb{Z} \times KE$;

TABLE 1

(p, r)	KG	$\mathrm{ord}\, KG$
$(2, r)$	$\mathbb{Z}_{2r+1}$	$2r+1$
$(3, 1)$	$\mathrm{Gr}\langle 2, 2, 2\rangle$	8
$(3, 2)$	$\mathrm{Gr}\langle 2, 3, 5\rangle$	120
$(4, 1)$	$\mathrm{Gr}\langle 2, 3, 3\rangle$	24
$(5, 1)$	$\mathrm{Gr}\langle 2, 3, 5\rangle$	120

TABLE 2

(p, q, r)	KG	$\mathrm{ord}\, KG$
$(2, 2, r)$	$\mathbb{Z}_r$	r
$(2, 3, 3)$	$\mathrm{Gr}\langle 2, 2, 2\rangle$	8
$(2, 3, 4)$	$\mathrm{Gr}\langle 2, 3, 3\rangle$	24
$(2, 3, 5)$	$\mathrm{Gr}\langle 2, 3, 5\rangle$	120

5.5.12. *If $p = q = 2$, then $O' = G_{2p+1}(t+1)$ and there are split exact sequences*

$$1 \to \mathbb{Z}_{2p+1}[t]/(t+1) \to O \to \mathbb{Z}_m \to 1,$$
$$1 \to \mathbb{Z}[t]/\{r(t+1), (t^2-1)\} \to E \to \mathbb{Z} \to 1.$$

Proof. 5.5.10 and 5.5.11 follow from trivial Lemma 5.5.13 below. 5.5.12 is proved by a direct calculation. Details are left to the reader. □

5.5.13. Lemma. *If $H \subset G$ is a central subgroup such that the projection $H \to G/KG$ is mono, then $KG = K(G/H)$. If, besides, the image of H is a direct summand in G/KG, then $G = H \times G/H$.*

5.6 Proof of Proposition 3.1. The result follows from 5.5.8 and 5.5.10: one can choose for C a curve whose formula has a pairs of type $(p, 0)$ and $2b$ pairs of type $(0, 2r+1)$. □

5.7 Proof of Proposition 3.2. In [D2] it is shown that there is a transformation $T(O_1 \leftarrow O_2 \leftarrow O_3)$ (see 4.6 and Fig. 3) which takes C to another curve $\overline{C}$ with a singular point of multiplicity k. According to 4.6, $\pi_1(\mathbb{C}p^2 \smallsetminus C)$ is the quotient of $\pi_1(\mathbb{C}p^2 \smallsetminus \overline{C} \cup \overline{E})$ by the relation $[\partial \bar{b}] = 1$, where $\bar{b}$, the transform of an analytical branch transversal to E, is tangent to $\overline{E}$. The following three cases are possible:

Case 1: $\deg \overline{C} = k+1$. In this case $\pi_1(\mathbb{C}p^2 \smallsetminus \overline{C} \cup \overline{E})$ is abelian due to Proposition 4.6.3.

Case 2: $\deg \overline{C} = k+2$, and $\overline{C}$ is inflection tangent to $\bar{b}$. Then the additional relation is $\gamma_1^2 \alpha_1 \alpha_2 \alpha_1 = 1$, and, taking into account the relations $\gamma_1^{-1} \alpha_i \gamma_1 = \sigma \alpha_i$, one obtains $(\alpha_1 \alpha_2 \alpha_1) \alpha_2 (\alpha_1 \alpha_2 \alpha_1)^{-1} = \gamma_1^{-2} \alpha_2 \gamma_1^2 = \sigma^2 \alpha_2 = \alpha_1 \alpha_2 \alpha_1^{-1}$, which implies $\alpha_1 \alpha_2 \alpha_1^{-1} = \alpha_2$.

Case 2: $\deg \overline{C} = k+2$, and $\overline{C}$ intersects $\overline{E}$ transversally at $\bar{b} \cap \overline{E}$. The additional relation is $\gamma_1^2 \alpha_2 = 1$, and there also are relations $\gamma_1^{-1} \alpha_i \gamma_1 = \sigma^{-2p} \alpha_i = (\alpha_1 \alpha_2)^{-p} \alpha_i (\alpha_1 \alpha_2)^p$ for some $p \geqslant 0$, which imply that γ_1 and, hence, α_2 commute with $\alpha_1 \alpha_2$. Thus, α_2 also commutes with α_1. □

6. Irreducible quintics

6.1. The quintics of type $C_5(3A_4)$. A curve of type $C_5(3A_4)$ can be obtained by $T(\overline{O}_1, \overline{O}_2, \overline{O}_3)$ from a 3-cuspidal quadric $\overline{C}$, see Fig. 6. Thus, its fundamental group is the quotient of $\pi_1(\mathbb{C}p^2 \smallsetminus \overline{C} \cup \overline{E}_k)$ by the relations $[\partial \bar{b}_k] = 1$, $k = 1, 2, 3$. The projection and the generators $a, b, c, d, \gamma_3, \gamma$ are shown in Fig. 6 (where γ_3 and γ are loops in M about the singular fibers L_3

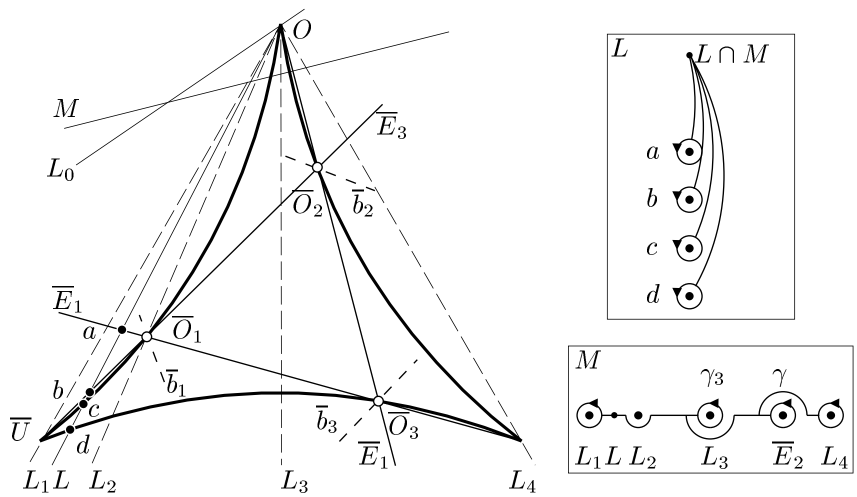


FIGURE 6

and $\overline{E}_1$ respectively), as well as the paths connecting L and L_j. The 'global' braid monodromy can easily be seen as all the intersection points remain real.

The relations $[\partial\bar{b}_1] = 1$ and $[\partial\bar{b}_2] = [\partial\bar{b}_3] = 1$ give, respectively,

$$abc = 1 \quad \text{and} \tag{6.1.1}$$

$$\gamma dcd^{-1}b = \gamma b^{-1}dc^{-1} = 1. \tag{6.1.2}$$

((6.1.2), as well as all the other relations, is written using (6.1.1).) Patching the other singular fibers gives

$$L_1\colon \quad [b, cd] = 1, \tag{6.1.3}$$

$$bcdc = dbcd, \tag{6.1.4}$$

$$L_3\colon \quad [c, (c^{-1}d)^2] = 1, \tag{6.1.5}$$

$$L_4\colon \quad [a, c^{-1}dcd^{-1}cad] = 1, \tag{6.1.6}$$

$$ada^{-1}c^{-1}dcd^{-1}cad = c^{-1}dcd^{-1}cadc^{-1}dcd^{-1}c. \tag{6.1.7}$$

Finally, patching L_0 gives $abcd\gamma\gamma_3 = 1$. Using (6.1.1), $\gamma = b^{-1}dc^{-1}d^{-1}$ from (6.1.2), and $\gamma_3 = (abcd)^2(c^{-1}abcd)^{-2} = dcd^{-1}c$ from 5.3 (and (6.1.1) again), this transforms into

$$b = cd. \tag{6.1.8}$$

6.2. To simplify the representation obtained, we use (6.1.1) and (6.1.8) to get $c = b^{-1}a^{-1}$ and $d = ab^2$; then (6.1.4) and (6.1.2), from which we eliminate γ, give

$$ab^4a = b, \tag{6.2.1}$$

$$bab = (ab^2a)^2. \tag{6.2.2}$$

Let us prove that these two relations imply the rest, i.e., (6.1.3) and (6.1.5–7). The first one obviously follows from (6.1.8). The three others after replacing c and d give

$$[bab, ab^2a] = 1,$$
$$[a, \underline{bab} \cdot a^{-1}b^{-2}a^{-1}b^{-1}ab^2] = 1, \quad \text{and}$$
$$ab^2 \cdot \underline{bab} \cdot a^{-1}b^{-2}a^{-1}b^{-1}ab^2abab^2a = \underline{bab} \cdot a^{-1}b^{-2}a^{-1}b^{-1}(ab^2a)(bab).$$

Now the first relation follows immediately from (6.2.2), and the others, after replacing the underlined expressions with $(ab^2a)^2$ and transposing the two factors in parentheses, transform to $[a, b^2ab^{-1}ab^2] = 1$ and $b^2ab^{-1}ab^2 = 1$, which follow from (6.2.1).

Finally, to get the representation announced in §3, we use (6.2.1) to transform (6.2.2) into $bab = \underline{ab^2} \cdot a^2 \cdot \underline{b^2a} = \underline{ba^{-1}b^{-2}} \cdot a^2 \cdot \underline{b^{-2}a^{-1}b}$, equivalent to

$$a^2 = b^2a^3b^2. \tag{6.2.3}$$

A standard calculation shows that the commutant K of this group is generated by $\alpha = a^5$ and $\delta_i = a^iba^{-(i+1)}$, $i = 0, \dots, 4$, and (6.2.1,3) take the form

$$\delta_1\delta_2\delta_3\delta_4\alpha = \delta_0, \tag{6.2.4}$$

$$(\delta_0\delta_1)^2 = \alpha^{-1}. \tag{6.2.5}$$

Besides, one can apply the automorphism $T\colon x \mapsto a^{-1}xa$ to any relation in K. In particular, one has $(\delta_3\delta_4)^2 = \alpha^{-1}$, and together with (6.2.4,5) this implies $[\alpha, \delta_i] = 1$. Now (6.2.4) and T^{-1}(6.2.4) can be rewritten in the form $\delta_1^{-1}\delta_0 = \delta_2\delta_3\delta_4\alpha = \delta_1\delta_0^{-1}$, which shows that $\delta_0^2 = \delta_1^2$ and, hence, $\delta_i^2 = \text{const}$. Denote $\delta_i^2 = \beta$. Obviously, this is a central element of the group; (6.2.5) implies $[\delta_i, \delta_{i+1}] = \beta^{-2}\alpha^{-1}$, and then the product of (6.2.4) and T^{-2}(6.2.4) gives $\delta_0\delta_2 = \delta_1\delta_2(\delta_3\delta_4\alpha\delta_3\delta_4)\delta_0\delta_1\alpha = (\delta_1\delta_2)(\delta_0\delta_1)\alpha = \delta_2\delta_0\beta\alpha$, i.e., $[\delta_i, \delta_{i+2}] = \beta^{-3}\alpha^{-1}$. Thus, the second commutant is generated by α, β and is central. Finally, substituting δ_0 from (6.2.4) to the other relations and using the commutators obtained gives $\alpha^2 = \beta^2 = 1$.

6.3. The quintics of type $C_5(A_6 \sqcup 3A_2)$**.** Such a curve can be obtained by $T(\overline{O}_1 \leftarrow \overline{O}_2, \overline{O}_3)$ from a 3-cuspidal quartic $\overline{C}$ (see Fig. 7, which shows two real forms of $\overline{C}$: either two cusps of $\overline{C}$ or the two tangency points of $\overline{C}$ and $\overline{E}_3$ have to be imaginary). The projection, singular fibers, branches $\overline{b}_k$ which give additional relations, and paths connecting L and L_j are shown in Fig. 7. (The paths go along one of the two real parts of M, which are denoted by $\Re_{1,2}$.) γ_2 and γ are the generators corresponding to L_2 and $\overline{E}_1$ respectively; the other generators a, b, c are some standard loops in L about the points a, b, c shown in Fig. 7.

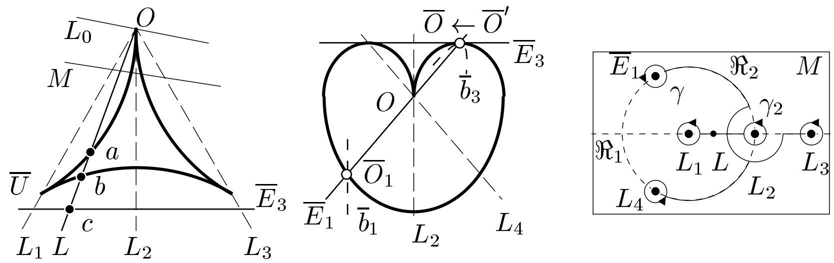


FIGURE 7

The singular fibers give the following relations:

$$L_1\colon \quad aba = bab, \tag{6.3.1}$$

$$L_2\colon \quad [a, (bc)^2] = 1, \tag{6.3.2}$$

$$L_3\colon \quad cac^{-1}bcac^{-1} = bcac^{-1}b, \tag{6.3.3}$$

$$L_4\colon \quad caca = acac, \tag{6.3.4}$$

$[\partial\overline{b}_1] = [\partial\overline{b}_2] = 1$ has the form

$$\gamma b = \gamma(b^{-1}abc)^2 = 1, \tag{6.3.5}$$

and patching L_0 gives the relation $abc\gamma_2\gamma = 1$, which, due to $\gamma = b^{-1}$ from (6.3.5) and $\gamma_2 = (abc)^2(bc)^{-2}$ (see 5.3), transforms into

$$abcabca = b^2c. \tag{6.3.6}$$

6.4. Let $u = c^{-1}$ and $v = b^{-1}abc$. From (6.3.5) it follows that $b = v^2$, and then $c = u^{-1}$ and $a = v^3uv^{-2}$, i.e., u and v generate the group. Relations (6.3.1) and (6.3.6) in u, v are

$$uv^3u = v^2uv^2, \quad \text{and} \tag{6.4.1}$$

$$uv^5u = v^2. \tag{6.4.2}$$

Given (6.4.2), the first relation is equivalent to either

$$v^2u^2v^2 = u, \quad \text{or} \tag{6.4.1$'$}$$

$$u^3v^7 = 1. \tag{6.4.1$''$}$$

(Just represent (6.4.1) as $v^3 = \underline{u^{-1}v^2} \cdot u \cdot \underline{v^2u^{-1}}$ and replace one or both underlined expressions using (6.4.2).) Prove that these relations imply (6.3.2–4), which in u, v are as follows:

$$\underline{v^{-2}uv^3} = u^{-1} \cdot \underline{v^3uv^{-2}} \cdot u,$$
$$u^{-1} \cdot \underline{v^3uv^{-2}} \cdot uv^2u^{-1} \cdot \underline{v^3uv^{-2}} \cdot uv^{-2} = v^2u^{-1} \cdot \underline{v^3uv^{-2}} \cdot u,$$
$$u^{-1}v^5 \cdot (v^{-2}uv^{-2}) \cdot u^{-1}v^3uv^{-2} = \underline{v^3uv^{-2}} \cdot u^{-1}v^3uv^{-2}u^{-1}.$$

After replacing the underlined expressions using (6.4.1) and the expression in parentheses using (6.4.1$'$), the first relation converts to $[u^3, v^2] = 1$, which follows from (6.4.1$''$), the third one is equivalent to (6.4.2), and the second one gives $u^{-2} \cdot \underline{v^2u^2} \cdot v^2u^{-2}v^2u^2v^{-2} = v^2u^{-2} \cdot \underline{v^2u^2}$; the substitution $v^2u^2 = uv^{-2}$ from (6.4.1$'$) transforms this to $[u^3, v^2] = 1$, which follows from (6.4.1$''$).

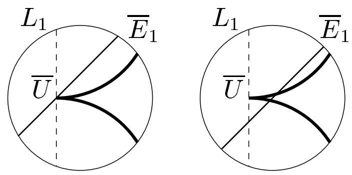
One can easily see that, given (6.4.1$''$), relation (6.4.2) is equivalent to $(uv^{-2})^2 = u^3$. Thus, replacing v with v^{-1}, one gets the second representation from 3.3.1, which shows that the group is infinite, as it factors through $\mathrm{Gr}(2, 3, 7)$.

6.5. Other irreducible quintics. If a curve has a triple or a quadruple singular point, its fundamental group can be found using Theorem 5.2 or Corollary 4.6.2 respectively. All these groups are abelian. Thus, it suffices to only consider curves of type $C_5(\sum a_pA_p)$. From Nori's theorem [N] it follows that the group of such a curve is abelian if $2a_1 + \sum_{p>1}(2p+2)a_p < 25$. All other curves, not covered by Nori's theorem, are adjacent to one of those considered in 6.5.1–4 below; hence, their groups are abelian as well. (The fact that the curves are adjacent follows from the way they are constructed in [D2].)

6.5.1. $C_5(A_{12})$ **and** $C_5(A_8 \sqcup A_4)$**.** These curves have abelian fundamental groups due to Proposition 3.2.

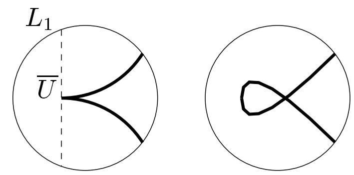
6.5.2. $C_5(2A_4 \sqcup A_2 \sqcup A_1)$**.** The curve can be obtained by a perturbation of $C_5(3A_4)$: the exceptional divisor $\overline{E}_3$ in Fig. 6 should not pass through $\overline{U}$ (see Fig. 8). In addition to (6.1.1–8) this gives relations $[b, c] = [b, d] = 1$. Then (6.1.1) implies $[a, b] = 1$, and we know that a and b generate the group.

6.5.3. $C_5(A_6 \sqcup 2A_2 \sqcup A_1)$**.** The curve is obtained from $C_5(A_6 \sqcup 3A_2)$ by perturbing $\overline{U}$ in Fig. 7 to a node (see Fig. 9). The additional relation is $a = b$, and (6.3.6) implies then $c = a^{-3}$.



$3A_4$ $2A_4 \sqcup A_2 \sqcup A_1$

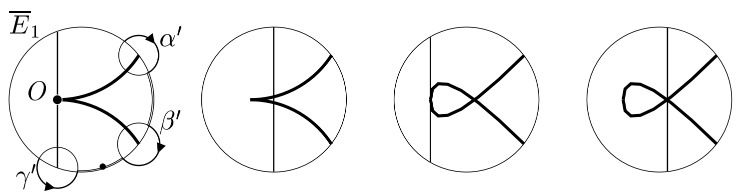
FIGURE 8



$A_6 \sqcup 3A_2$ $A_6 \sqcup 2A_2 \sqcup A_1$

FIGURE 9

6.5.4. $C_5(A_3 \sqcup 4A_2)$**,** $C_5(A_4 \sqcup 3A_2 \sqcup A_1)$**, and** $C_5(A_5 \sqcup 3A_2)$**.** All these curves are obtained by perturbing $C_5(A_6 \sqcup 3A_2)$, see Fig. 10, which shows the perturbation in a small neighborhood B of O. Choose some generators α', β', γ' of $\pi_1(\partial B \smallsetminus \overline{C} \cup \overline{E}_1)$ so that the inclusion homomorphism be given by $\alpha' \mapsto a$, $\beta' \mapsto a^{-1}\gamma_2 = (bc)a(bc)^{-1}$, and $\gamma' \mapsto \gamma = b^{-1}$ (see (6.3.5)). Below we consider all the three cases and prove that the additional relations caused by the perturbation make the group abelian.



$A_6 \sqcup 3A_2$ $A_3 \sqcup 4A_2$ $A_4 \sqcup 3A_2 \sqcup A_1$ $A_5 \sqcup 3A_2$

FIGURE 10

The type $C_5(A_3 \sqcup 4A_2)$. The additional relation $[\alpha', \gamma'] = 1$ gives $[a, b] = 1$. Then (6.3.1) implies $a = b$, and from (6.3.6) it follows that $c = a^{-3}$, i.e., a generates the group.

The type $C_5(A_4 \sqcup 3A_2 \sqcup A_1)$. The additional relation $[\alpha', \beta'] = 1$ gives $[a, bca(bc)^{-1}] = 1$. From (6.3.6) one has $bca(bc)^{-1} = (b^{-1}abc)a$; hence, a commutes with $b^{-1}abc$ and, due to (6.3.4), also with b, and the group is abelian (see previous case).

The type $C_5(A_5 \sqcup 3A_2)$. The additional relation $\alpha' = \beta'$ implies $[\alpha', \beta'] = 1$, and the previous case applies.

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