

COUNTING LINES WITH VINBERG'S ALGORITHM

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ABSTRACT. We combine classical Vinberg's algorithms [39] with the lattice-theoretic/arithmetical approach from [9] to give a method of classifying large line configurations on complex quasi-polarized $K3$ -surfaces. We apply our method to classify all complex $K3$ -octic surfaces with at worst Du Val singularities and at least 32 lines. The upper bound on the number of lines is 36, as in the smooth case, with at most 32 lines if the singular locus is non-empty.

1. INTRODUCTION

In the last decade, a substantial progress towards the complete understanding of configurations of lines on *smooth polarized* $K3$ -surfaces has been made (see §1.3 below). Unfortunately, the methods that led to that development do not yield optimal results (if any) when the surfaces in question have singularities. The main aim of this paper is to address this problem. More precisely, we combine classical Vinberg's algorithms [39] with the lattice-theoretic/arithmetical approach from [9] to create a uniform framework for the study of configurations of lines on complex projective $K3$ surfaces with at worst Du Val (*aka* simple or **A–D–E**) singularities.

Among other things, we discover a new phenomenon unthinkable in the realm of smooth polarized $K3$ -surfaces: a surface with a larger Néron–Severi lattice N' may have fewer lines than that with a smaller lattice $N \subset N' \subset H_2(X)$; we discuss this phenomenon in §3.3. In particular, unlike the smooth case, it is no longer sufficient to confine oneself to surfaces X with $\text{NS}(X)$ spanned by lines. (By an abuse of the language, *lines* in the minimal resolution of singularities $X \rightarrow X_{2d}$ are the strict transforms of those in $X_{2d} \subset \mathbb{P}^{d+1}$, and we say that $\text{NS}(X)$ is *spanned by lines* if it is a finite index extension of its sublattice generated by the classes of lines on X and the *quasi-polarization* h . The same convention applies to abstract polarized hyperbolic lattices.) It *is*, however, sufficient to assume that $\text{NS}(X)$ is spanned by lines and *exceptional divisors* (and the distinguished isotropic class in the case of degenerations as in Theorems 3.16, 3.17). For this reason, in most statements, we make an assumption on the number of lines, *i.e.*, the size of the *plain* Fano graph $\text{Fn}(X, h)$ (the dual adjacency graph of lines on X), but we classify *extended* Fano graphs $\text{Fn}^{\text{ex}}(X, h)$, *i.e.*, bi-colored graphs of both lines and exceptional divisors. Obviously, these extended graphs also give us more detailed information about the lines and singular points of the original projective surface $X_{2d} \subset \mathbb{P}^{d+1}$ itself.

To test our approach, we classify $K3$ -octics with many lines and at worst Du Val singularities. The principal results on octics are stated in Theorems 1.1 and 1.2,

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TABLE 1. $K3$ -octics with at least 32 lines (see [Theorem 1.1](#) and [§1.1](#))

Γ^{ex}	$\text{Sing } X_8$	$ \text{Aut } \Gamma^{\text{ex}} $	\det	(r, c)	$ \text{Aut } X_8 $	$T := \text{NS}(X)^\perp$	Ref
Θ'_{36}		64	32	(1, 0)	16	$[4, 0, 8]$	7.9 , 7.10
Θ''_{36}		576	36	(1, 0)	72	$[6, 0, 6]$	7.9 , 7.10
Θ'_{34}		96		$\dim = 1$		$\mathbf{U}_2 \oplus [12]$	7.9 , 7.10
Θ_{33}		192	80	(1, 0)	24	$[8, 4, 12]$	6.10 , 7.10
Ψ_{33}		6912	36	$\dim = 2$		\mathbf{U}_3^2	5.5
Θ_{32}		96	60	(1, 0)	24	$[4, 2, 16]$	7.10
Θ'_{32}		384		$\dim = 2$		$\mathbf{U}_2 \oplus [-4] \oplus [4]$	7.9
Θ''_{32}		512		$\dim = 2$		$\mathbf{U}_2 \oplus \mathbf{U}_4$	7.9
Θ'''_{32}		768	36	$\dim = 2$		\mathbf{U}_3^2	7.9
$\Theta_{32}^{\mathbf{K}}$		23040	32	$\dim = 3$		$\mathbf{U}_2^2 \oplus [-4]$	6.7 , 7.9
Θ_{32}^4	$4\mathbf{A}_1$	256	16	(1, 0)	128	$[4, 0, 4]$	6.9 , 7.9 , 7.10
$\Theta_{32}^{2'}$	$2\mathbf{A}_1$	48	27	(1, 0)	24	$[6, 3, 6]$	7.9
$\Theta_{32}^{2''}$	$2\mathbf{A}_1$	64	36	(1, 0)	16	$[6, 0, 6]$	7.9
$\Theta_{32}^{2'''}$	$2\mathbf{A}_1$	128		$\dim = 1$		$\mathbf{U}_2 \oplus [8]$	7.9 , 7.10
$\Theta_{32}^{1'}$	\mathbf{A}_1	16	32	(1, 0)	8	$[6, 2, 6]$	7.9 , 7.10
$\Theta_{32}^{1''}$	\mathbf{A}_1	96	32	$\dim = 1$		$\mathbf{U}_3 \oplus [6]$	7.9

where we distinguish *triquadrics* (ideal theoretical intersections of three quadrics) *vs.* *special octics* (see [Definition 2.28](#)). Recall that a complex $K3$ -surface is called *singular* if its Picard rank is maximal: $\rho(X) = 20$; when mapped to a projective space, such surfaces are projectively rigid. In general, when speaking about an “ s -parameter family”, we always mean the dimension $20 - \rho(X)$ modulo the projective group.

Theorem 1.1 (see [§8](#)). *Let $X_8 \subset \mathbb{P}^5$ be a degree 8 $K3$ -surface with at worst Du Val singularities. Then X_8 contains at most 36 (at most 32 if $\text{Sing } X_8 \neq \emptyset$) lines. Moreover, if X_8 contains at least 32 lines, then it is one of the surfaces listed in [Table 1](#). Thus, if X_8 has 32 lines and $\text{Sing } X_8 \neq \emptyset$, then X_8 is a triquadric: there are two connected 1-parameter families and four singular surfaces.*

We discover (see [Theorem 6.4](#)) that there are two disjoint 3-parameter families of *Kummer octics* (with the distinguished Kummer divisors mapped to lines). Each consists of triquadrics, and each contains octics with 32 lines, *viz.* $\Theta_{32}^{\mathbf{K}}$ (generic in one family) and Θ_{32}^4 (rigid in the other one) in [Table 1](#). See [§6.2](#) for details.

Special octics constitute a codimension 1 family in the space of all octics and are subject to a stronger bound.

Theorem 1.2 (see [§5.3](#)). *There are two connected families of special octics with at worst Du Val singularities and at least 30 lines: a 2-parameter family of smooth surfaces with 33 lines (see Ψ_{33} in [Table 1](#)) and a 1-parameter family of surfaces with three \mathbf{A}_1 -type points and 30 lines (see Ψ_{30}^3 in [Table 2](#)).*

The rows of [Tables 1](#) and [2](#) refer to the extended Fano graphs of octics; the other entries are explained in [§1.1](#) below. For obvious reasons, we do not try to depict these graphs; the precise descriptions are available electronically (in the form of a

TABLE 2. Other $K3$ -octics with many lines (see §1.1)

Γ^{ex}	Sing X_8	$ \text{Aut } \Gamma^{\text{ex}} $	det	(r, c)	$ \text{Aut } X_8 $	$T := \text{NS}(X)^\perp$	Ref
Φ'_{30}		240	140	(1, 1)	60	[12, 2, 12]	7.7
Φ''_{30}		40	135	(1, 1)	10	[12, 3, 12]	7.7
Φ'''_{30}	$5\mathbf{A}_1$	40	15	(1, 0)	20	[2, 1, 8]	7.7
Ψ_{30}^3	$3\mathbf{A}_1$	864			dim = 1	$\mathbf{U}_3 \oplus [6]$	5.5
Δ'_{28}		576	44	(1, 0)	96	[4, 2, 12]	6.9
$\Theta_{28}^{2,1}$	$2\mathbf{A}_2, \mathbf{A}_1$	16	16	(1, 0)	16	[2, 0, 8]	7.9
$\Theta_{25}^{1,4}$	$\mathbf{A}_2, 4\mathbf{A}_1$	2	16	(1, 0)	2	[2, 0, 8]	7.10

GRAPE [21, 22, 37] records) in [13]. Most graphs do not have a known “name” and, therefore, *de facto* these descriptions are their definitions.

Thus, we obtain a complete picture of the large line configurations on a class of varieties (*viz.* complete intersections of quadrics) that have been a subject of research since the XIX-th century. As long as $K3$ -surfaces are concerned, it is well understood that the larger the integer d , the smaller the maximal Fano graphs of smooth complex $K3$ -surfaces of degree $2d$ are (see [9]). Moreover, if d is sufficiently large, no Fano graph of a degree $2d$ $K3$ -surface can be hyperbolic (*cf.* §4). Thus, our approach can as well be applied to classify large Fano graphs of quasi-polarized $K3$ -surfaces (X, h) with $h^2 > 8$. On the other hand, the case of sextics, quartics, and, especially, double planes ramified at sextic curves would require a considerably more thorough treatment of the configurations containing a triangle or a quadrangle.

As yet another justification of our interest in the problems above, we recall that $K3$ -surfaces are 2-dimensional hyperkähler varieties. We hope that our new algorithms/methods developed in dimension 2, apart from being of interest on their own, may contribute to a better understanding of the higher-dimensional case.

1.1. Classification of $K3$ -octics with many lines. We use the girth (*cf.* §2.6) to subdivide *plain* Fano graphs of quasi-polarized $K3$ -surfaces into several classes (see §4) and obtain a more refined classification/bound for each class. A Fano graph $\Gamma := \text{Fn}(X, h)$ is called

- *triangular* (the Ψ_* -series), if $\text{girth}(\Gamma) = 3$; such graphs appear only as Fano graphs of special octics (see Lemma 5.2),
- *quadrangular* (the Θ_* -series), if $\text{girth}(\Gamma) = 4$,
- *pentagonal* (the Φ_* -series), if $\text{girth}(\Gamma) = 5$,
- *astral* (the Δ_* -series), if $\text{girth}(\Gamma) \geq 6$ and Γ has a vertex of valency ≥ 4 .

All other graphs are *locally elliptic* (the Λ_* -series), *i.e.*, one has $\text{val } v \leq 3$ for each vertex $v \in \Gamma$ (and we still assume $\text{girth}(\Gamma) \geq 6$ to exclude a few trivial cases).

Principal properties of $K3$ -octics with large line configurations are collected in Tables 1 and 2, where, inevitably, we have to restate some results of [9] concerning smooth octics. The first column refers to the isomorphism classes of the extended Fano graphs introduced elsewhere in the paper; the subscript always stands for the number of lines. Then, for each graph $\Gamma^{\text{ex}} := \text{Fn}^{\text{ex}}(X, h)$, we list,

- the number and types of the singular points of the corresponding octics;
- the order $|\text{Aut } \Gamma^{\text{ex}}|$ of the full automorphism group of Γ^{ex} ,

- the transcendental lattice $T := \mathrm{NS}(X)^\perp \subset H_2(X; \mathbb{Z})$ of generic $K3$ -surfaces X with $\mathrm{Fn}^{\mathrm{ex}}(X, h) \cong \Gamma^{\mathrm{ex}}$; this lattice determines a family of *abstract $K3$ -surfaces*,
- references to the parts of the text where the graph appears.

For the rigid configurations ($\mathrm{rk} T = 2$), we list, in addition,

- the determinant $\det T$,
- the numbers (r, c) of, respectively, real projective isomorphism classes and pairs of complex conjugate projective isomorphism classes of octics (X, h) with $\mathrm{Fn}^{\mathrm{ex}}(X, h) \cong \Gamma^{\mathrm{ex}}$,
- the order $|\mathrm{Aut} X_8|$ of the group of *projective* automorphisms of X_8 .

Each of the nine non-rigid configurations Γ listed in the tables is realized by a single connected equilinear deformation family $\mathcal{M}(\Gamma)$; we indicate the dimension $\dim(\mathcal{M}(\Gamma)/\mathrm{PGL}(6, \mathbb{C})) = \mathrm{rk} T - 2$ and the minimum of the discriminants $\det T$ of the singular $K3$ -surfaces $X(T) \in \mathcal{M}(\Gamma)$ (whenever we know its value).

The proof of [Theorem 1.1](#) is based on the study of various types of Fano graphs of $K3$ -octics. In particular, [Theorem 1.1](#) implies that each triquadric with at least 32 lines contains a quadrangle (*i.e.*, an $\tilde{\mathbf{A}}_3$ configuration of lines) and its singularities (if any) are of type \mathbf{A}_1 . As part of the proof, we obtain a classification of maximal pentagonal configurations (see [§7.3](#)), maximal astral configurations (see [§7.2](#)) and examples of large line configurations on octics with \mathbf{A}_2 -singularities (the entries $\Theta_{28}^{2,1}$, $\Theta_{25}^{1,4}$ in [Table 2](#)). Combined with Nikulin's theory [[26](#)] (*cf.* [§8](#)), this yields the following extra bounds.

Remark 1.3 (see [§8](#)). Let $X_8 \subset \mathbb{P}^5$ be a degree 8 $K3$ -surface with at worst Du Val singularities and Γ its Fano graph. Then:

- (1) if Γ is pentagonal, then $|\Gamma| \leq 30$ and the maximum is attained at the three singular surfaces with the extended Fano graphs Φ_* in [Table 2](#);
- (2) if Γ is astral, then $|\Gamma| \leq 28$ and the maximum is attained at a unique singular Kummer octic with the extended Fano graph Δ'_{28} in [Table 2](#).

For completeness, the extremal locally elliptic graphs (Λ_* of size 24 or 25) are described in [§7.1](#): they are not listed in [Table 2](#) as they are realized by too many $K3$ -octics with non-isomorphic transcendental lattices.

1.2. Contents of the paper. The paper splits into two parts. The first one, *viz.* [§2–§4](#) and [Appendix A](#), describes a general strategy for the classification of large configurations of lines on projective models of complex $K3$ -surfaces with at worst Du Val singularities. The second part, [§5–§8](#) and [Appendices B, C](#), demonstrates the effectiveness of our approach in the case of $K3$ -octics.

In [§2](#) we lay a theoretical foundation for the computation of the configuration of lines on a quasi-polarized $K3$ -surface (X, h) in terms of the lattice $\mathrm{NS}(X) \ni h$. Unlike the smooth case [[9, 12](#)], we have to use Vinberg's algorithm [[39](#)] and compute two layers of the fundamental polyhedron (*cf.* [\(2.7\)](#)). The result depends on the choice of a Weyl chamber, and we discuss the extent to which it is well defined (see [Lemmas 2.9](#) and [2.10](#)). Then, in [§2.3](#), we recall the geometry of the nef cone of X and relate our abstract construction to the geometric set of lines on X ([Theorem 2.19](#)). The section concludes with a discussion of Saint-Donat's conditions [[33](#)] for various degenerations of the quasi-polarization (see [§2.4, §2.5](#)).

In §3 we study polarized lattices generated by lines and, thus, constructed from graphs. The principal innovation here is the concept of extensibility (Definition 3.2) which is to replace the admissibility conditions used in the smooth case. A simple criterion is given by Lemma 3.3, which also asserts that, in spite of the ambiguity in the choice of a Weyl chamber, each extensible graph has a well-defined saturation. The condition for the geometric realizability of a graph is given by Theorems 3.8 and 3.9; we also state a version for special octics (Theorem 3.17, our primary concern) and hyperelliptic polarizations (Theorem 3.16, very similar). Finally, we address the new phenomenon mentioned at the beginning (see Warning 3.5 and a detailed discussion in §3.3) and explain how it affects our proof strategy.

In §4 we recall (after [9]) the taxonomy of hyperbolic graphs and combinatorial counterparts of elliptic pencils on $K3$ -surfaces. In §4.4 we explain our approach to the classification of large configurations of lines. After a thorough examination of the local properties specific to Fano graphs of octics (and proving Theorem 1.2) in §5, this general approach, mostly computer aided, is illustrated in human readable form in §6, on the example of (almost) Kummer octics. In §6.4 we also announce a few new results (mostly examples) concerning spatial quartics.

In §7 we state the results of the computation in the form of a number of bounds for various types of graphs. These statements are used in §8 to prove Theorem 1.1. The computation leading to §7 is heavily computer aided (we used GAP [16]); the algorithms and a few technical details are outlined in Appendices A, B, C.

1.3. History of the problem. Configurations of rational curves on surfaces have been a subject of intensive research since the very beginning of algebraic geometry. The question what the maximal number of lines on surfaces in a given family is (e.g., on smooth degree s hypersurfaces in \mathbb{P}^3 for a fixed integer $s > 2$) has a long history. There are quite a few approaches to this question.

In the case of a hypersurface in $\mathbb{P}^3(\mathbb{K})$, one can study the geometry of the so-called *flecnodal divisor*, i.e., the locus of fourfold contact of lines with the surface in question. This idea goes back to Salmon and Clebsch [6] and yields the bound of at most $s(11s - 24)$ lines on a smooth complex projective surfaces of any degree $s > 2$. Combined with certain properties of fibrations, it was used in Segre's proof [34] of the fact that 64 is the maximal number of lines on a smooth quartic in \mathbb{P}^3 . (A minor gap in this proof was recently bridged in [31].) The flecnodal divisor appears also in the proof of the sharp bound on the number of lines on quartics in $\mathbb{P}^3(\mathbb{K})$ for algebraically closed fields \mathbb{K} such that $\text{char}(\mathbb{K}) \neq 2, 3$ (see [31]), the sharp bound for complex affine quartics (see [17]), and the best known bounds for surfaces of degree $s > 4$ (see [2, 32]).

One can use the orbibundle Bogomolov-Miyaoka-Yau inequality to obtain bounds on the number of lines (more generally, rational curves of a bounded degree) on complex smooth degree $2d$ $K3$ -surfaces for $d > 2$ (see [23]).

One can try to find an appropriate hyperplane section of X and count the lines that meet each of its components separately. This approach, combined with the study of elliptic fibrations and Segre's surfaces of principal lines, yields the sharp upper bound for projective quartics when $\text{char}(\mathbb{K}) = 3$ (see [30]). It is also used in the arguments in [31, 34, 38].

In the case of rational surfaces embedded *via* a linear system $|h|$, one can try to classify all solutions to the equation $v \cdot h = 1$. In general, this method fails for surfaces of non-negative Kodaira dimension.

There is a very elegant approach of Elkies [15], which is very efficient when the number of lines on a surface is large in comparison with its Picard number.

Finally, one can try to classify the potential sublattices generated by the classes of lines in the Néron-Severi lattice (resp. second homology) of the surface in question. In presence of Torelli-type theorems this method not only leads to sharp bounds but provides examples of surfaces with large line configurations as well. In the case of $K3$ -surfaces, such approach was pioneered in [12]. It gave the complete classification of smooth complex quartics with at least 52 lines, the sharp bounds for $\mathbb{K} = \mathbb{R}$, and a bound for $\mathbb{K} = \mathbb{Q}$. Its refinements led to the sharp bound of at most 60 lines on smooth quartics when $\text{char}(\mathbb{K}) = 2$ (see [8]), sharp bounds for supersingular quartics when $\text{char}(\mathbb{K}) = 2, 3$ (see [8]), and sharp bounds for (smooth minimal) complex degree $2d$ $K3$ -surfaces for $d > 2$ or $d = 2$ (see [9, 11]). Besides, large configurations of lines are classified in [8, 9, 11], too. Further generalizations of this approach resulted in sharp bounds on the number of rational curves of a given degree on smooth high-degree $K3$ - and Enriques surfaces and a classification of maximal configurations (see [28, 29]).

In contrast, hardly anything is known about line configurations on projective surfaces with non-empty singular locus. The case of complex cubic surfaces is the only one with a complete classification (see [4] for a modern exposition), whereas for complex quartic surfaces with singularities there are only partial results (see [10, 20]) and bounds that seem not to be sharp (see [17, 38]). In particular, neither for complex hypersurfaces of degree $s > 3$ nor for complex degree $2d$ $K3$ -surfaces it is known whether the maximal number of lines can be attained by a surface with non-empty singular locus.

Recall that a projective complete intersection $K3$ -surface is of degree 4, 6 or 8. The configurations of lines on octic models of Kummer surfaces have a long history (see [14, § 10]). A complete classification of large line configurations on smooth $K3$ -octic can be found in [9]. In the present paper, we complete this picture in the $K3$ -case, whereas our approach sheds no light on the line configurations on ruled octic surfaces in \mathbb{P}^5 (for the general classification of projective octics see [5, Remark 1.7] and [19, § 4.2]). In particular, we show that complex $K3$ -octic with more than 32 lines are always smooth.

1.4. Common notation and conventions. We work over the field \mathbb{C} . Every elliptic fibration is assumed to have a section. Otherwise, we speak of a genus-one fibration (*i.e.*, a base-point free elliptic pencil).

As in [9], we use the following notation for common integral lattices:

- \mathbf{A}_p , $p \geq 1$, \mathbf{D}_q , $q \geq 4$, \mathbf{E}_6 , \mathbf{E}_7 , \mathbf{E}_8 are the *positive definite* root lattices generated by the indecomposable root systems of the same name (see [3]);
- $[a] := \mathbb{Z}u$ is the lattice of rank 1 given by the condition $u^2 = a$;
- $[a, b, c] := \mathbb{Z}u + \mathbb{Z}v$, $u^2 = a$, $u \cdot v = b$, $v^2 = c$, is a lattice of rank 2; when it is positive definite, we assume that $0 < a \leq c$ and $0 \leq 2b \leq a$: then, u is a shortest vector, v is a next shortest one, and the triple (a, b, c) is unique;
- $\mathbf{U} := [0, 1, 0]$ is the unimodular even lattice of rank 2;
- $L(n)$, denotes the lattice obtained by the scaling of a given lattice L by a fixed integer $n \in \mathbb{Z}$;
- $L^\vee := \text{Hom}(L, \mathbb{Z})$ denotes the dual group; if L is nondegenerate, there is a natural inclusion $L^\vee \subset L \otimes \mathbb{Q}$, equipping L^\vee with a \mathbb{Q} -valued bilinear form,
- the inertia indices of the quadratic form $L \otimes \mathbb{R}$ are denoted by $\sigma_{\pm, 0}(L)$.

In general, we maintain the standard notation for various objects associated to a lattice (the determinant, the discriminant group, *etc.*) — see, *e.g.*, [7], [26].

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2. LATTICES AND FANO GRAPHS

All lattices considered in this paper are even: $v^2 \in 2\mathbb{Z}$ for each $v \in L$.

2.1. Root lattices (see [3]). A *root lattice* is a negative definite lattice R generated by *roots*, *i.e.*, vectors $r \in R$ of square (-2) . Given a negative definite lattice S , we denote by

$$\text{root}_0 S := \{r \in S \mid r^2 = -2\}$$

the set of roots in S ; then the sublattice $\text{rt} S$ generated by $\text{root}_0 S$ is a root lattice.

Let R be a root lattice and Δ a Weyl chamber for (the group generated by reflections in) R . We denote by $\{\Delta\}$ the set of the “outward” roots orthogonal to the walls of Δ . A subset $B \subset R$ is of the form $\{\Delta\}$ if and only if it is a “standard” Dynkin basis for R . Recall also that a Weyl chamber Δ gives rise to a partition

$$\text{root}_0 R = P_\Delta \cup (-P_\Delta), \quad P_\Delta \cup (-P_\Delta) = \emptyset,$$

with the set P_Δ of *positive roots closed*:

$$\text{if } u, v \in P_\Delta \text{ and } u + v \text{ is a root, then also } u + v \in P_\Delta.$$

The positive roots $r \in P_\Delta$ are the linear combinations $\sum n_e e$, $e \in \{\Delta\}$, with all $n_e \in \mathbb{N}$, whereas the *negative* roots $r \in -P_\Delta$ have all coefficients in $-\mathbb{N}$. Conversely, any partition

$$(2.1) \quad \text{root}_0 R = P \cup (-P), \quad P \cap (-P) = \emptyset, \quad P \text{ is closed,}$$

defines a unique Weyl chamber Δ such that $P = P_\Delta$: the elements of $\{\Delta\}$ are those positive roots $e \in P$ that are *indecomposable*, *i.e.*, cannot be represented as a sum of two or more positive roots. Needless to say that any partition as above has the form

$$P := \{r \in \text{root}_0 R \mid \ell(r) > 0\},$$

where $\ell: R \rightarrow \mathbb{R}$ is a linear functional *generic* in the sense that $\ell(r) \neq 0$ for each root $r \in \text{root}_0 R$.

2.2. Polarized lattices. A nondegenerate lattice S is called *hyperbolic* if $\sigma_+ S = 1$. A *polarized lattice* $S \ni h$ is a hyperbolic lattice S equipped with a distinguished vector h of positive square; the square h^2 is called the *degree* of the polarization and S is said to be h^2 -polarized. We often use the following consequence of the requirement that $\sigma_+ S = 1$:

$$(2.2) \quad \text{for any pair } u, v \in S \text{ one has } \det \text{Gram}(\mathbb{Z}h + \mathbb{Z}u + \mathbb{Z}v) \geq 0;$$

moreover, the determinant is 0 if and only if h, u, v are linearly dependent.

Given a polarized lattice $S \ni h$, we let

$$\text{root}_n(S, h) := \{r \in S \mid r^2 = -2, r \cdot h = n\}, \quad n \in \mathbb{N},$$

and denote by $\mathfrak{rt}(S, h) \subset h^\perp \subset S$ the sublattice generated by $\text{root}_0(S, h)$. If the polarization h is understood, it is omitted from the notation. We define the *positive cone* as

$$\mathcal{C}^+(S, h) := \{v \in S \otimes \mathbb{R} \mid v^2 > 0, v \cdot h > 0\}.$$

We follow [39] and call every connected component of

$$\mathcal{C}^+(S, h) \setminus \bigcup_{r^2=-2} r^\perp$$

a *fundamental polyhedron*. As in the case of root lattices (see (2.1)) each fundamental polyhedron corresponds to a partition of the set of all roots into the positive and the negative ones.

Given a lattice S (negative definite or polarized), sublattice $F \subset S$ (negative definite or hyperbolic containing the polarization), and Weyl chamber Δ for $\mathfrak{rt} S$, we denote by $\Delta|_F$ the Weyl chamber for $\mathfrak{rt} F$ defined by the partition of $\text{root}_0 F$ into $\pm P_\Delta \cap F$. Note that we do *not* assert that $\{\Delta|_F\} = \{\Delta\} \cap F$.

A fixed Weyl chamber Δ for $\mathfrak{rt}(S, h)$ gives rise to a distinguished fundamental polyhedron Δ^\sharp for the group generated by reflections of S . Here, we regard Δ (resp. Δ^\sharp) as a polyhedral (resp. locally polyhedral) subcone of the cone $\mathcal{C}^+(S, h)$ and we require that $h \in \bar{\Delta}^\sharp \subset \bar{\Delta}$, *i.e.*, by an abuse of notation, Δ also denotes the cone

$$\{v \in \mathcal{C}^+(S, h) \mid v \cdot e > 0 \text{ for all } e \in \{\Delta\}\}.$$

In the sequel we will say that the Weyl chamber $\Delta \subset \mathfrak{rt}(S, h)$ *extends* to the distinguished fundamental polyhedron $\Delta^\sharp \subset S$ (resp. Δ^\sharp *restricts* to Δ).

Remark 2.3. We are mainly interested in the sets of walls $\{\Delta\}$ and $\{\Delta^\sharp\}$ (see below), thus treating Δ and Δ^\sharp as combinatorial objects. However, when necessary, we follow the tradition and regard them as *open* subsets of $\mathcal{C}^+(S, h)$; their closures $\bar{\Delta}$ and $\bar{\Delta}^\sharp$ are referred to as the *closed* Weyl chamber and fundamental polyhedron, respectively. Note that any one of the seven sets Δ , $\bar{\Delta}$, $\{\Delta\}$, P_Δ , Δ^\sharp , $\bar{\Delta}^\sharp$, $\{\Delta^\sharp\}$ determines the six others. We express this relation by using the same letter Δ .

The set $\{\Delta^\sharp\}$ of (the ‘‘outward’’ roots orthogonal to) the walls of Δ^\sharp can be found by Vinberg’s algorithm [39]: $\{\Delta^\sharp\} = \bigcup_{n \geq 0} \{\Delta^\sharp\}_n$, where $\{\Delta^\sharp\}_0 := \{\Delta\}$ and the other sets are defined recursively:

$$\{\Delta^\sharp\}_n := \{r \in \text{root}_n(S, h) \mid r \cdot e \geq 0 \text{ for all } e \in \{\Delta^\sharp\}_k, 0 \leq k < n\}.$$

Denoting $\{\Delta^\sharp\}_+ := \bigcup_{n > 0} \{\Delta^\sharp\}_n$, it is immediate that

$$(2.4) \quad v \cdot e \geq 0 \text{ for each } e \in P_\Delta \text{ and } v \in \bar{\Delta}^\sharp \text{ or } v \in \{\Delta^\sharp\}_+;$$

in particular, two distinct vectors $v_1, v_2 \in \bar{\Delta}^\sharp \cup \{\Delta^\sharp\}_+$ are never *separated by a root*, *i.e.*, $(v_1 \cdot e)(v_2 \cdot e) \geq 0$ for each $e \in \text{root}_0(S, h)$, and

$$(2.5) \quad u \cdot v \geq 0 \text{ for any two distinct vectors } u, v \in \{\Delta^\sharp\}.$$

The last assertion follows directly from the construction unless $u \cdot h = v \cdot h > 0$. In the latter case, by (2.2), the only alternative is $u \cdot v = -1$, and then u and v would be separated by the root $(u - v) \in \text{root}_0(S, h)$.

The (plain) *Fano graph* of a polarized lattice (S, h) with a distinguished Weyl chamber Δ for $\mathfrak{rt}(S, h)$ is defined as the set of vertices

$$(2.6) \quad \text{Fn}_\Delta(S, h) := \{\Delta^\sharp\}_1 = \{l \in \text{root}_1(S, h) \mid l \cdot e \geq 0 \text{ for all } e \in \{\Delta\}\},$$

with two vertices $l_1 \neq l_2$ connected by an edge of multiplicity $l_1 \cdot l_2$, cf. (2.5). On a few occasions (usually, as the ultimate result of the computation), we also use the bi-colored *extended Fano graph*

$$(2.7) \quad \text{Fn}_{\Delta}^{\text{ex}}(S, h) := \{\Delta^{\#}\}_1 \cup \{\Delta\},$$

with the same convention about the multiplicities of the edges and vertices v colored according to the value $v \cdot h \in \{0, 1\}$. Most of the time, it is the plain graphs that are used in the algorithms, whereas their bi-colored extended counterparts play a crucial rôle at the end of the proof and in the statements. We discuss the relation between the two categories in §3.3 below.

The vertices of $\text{Fn}_{\Delta}(S, h)$ are called *lines*, whereas the vectors $e \in \{\Delta\}$ are called *exceptional divisors*. Due to (2.4), one can also define lines as vectors $l \in \text{root}_1(S, h)$ such that $l \cdot e \geq 0$ for all $e \in P_{\Delta}$. As a consequence, we have the following lemma.

Lemma 2.8. *Let $S \ni h$ be a polarized lattice, Δ a Weyl chamber for $\mathfrak{rt}(S, h)$, and F the primitive hull of the sublattice of S generated by h and all $l \in \text{Fn}_{\Delta}(S, h)$. Then, there is a canonical inclusion $\text{Fn}_{\Delta}(S, h) \subset \text{Fn}_{\Delta|_F}(F, h)$. \triangleleft*

Thus, as long as we are interested in *maximizing* the number of lines, it suffices to consider polarized lattices $S \ni h$ spanned by lines.

At first sight, the graphs Fn_{Δ} , $\text{Fn}_{\Delta}^{\text{ex}}$ depend on the choice of the Weyl chamber Δ . However, we have the following simple observation.

Lemma 2.9. *For any pair Δ', Δ'' of Weyl chambers for $\mathfrak{rt}(S, h)$, there are canonical isomorphisms $\text{Fn}_{\Delta'}(S, h) = \text{Fn}_{\Delta''}(S, h)$ and $\text{Fn}_{\Delta'}^{\text{ex}}(S, h) = \text{Fn}_{\Delta''}^{\text{ex}}(S, h)$.*

Proof. Let $R := \mathfrak{rt}(S, h)$. Since the Weyl group W of R acts simply transitively on the set of Weyl chambers, Δ' and Δ'' are related by a unique element $\sigma \in W$. As a product of reflections, σ admits a canonical (identical on $R^{\perp} \ni h$) extension to an automorphism of $S \ni h$, which induces isomorphisms of the Fano graphs. \square

We conclude with a lemma playing a crucial rôle in our algorithms. A Weyl chamber Δ is called *compatible* with a subset $\Gamma \subset \text{root}_1(S, h)$ if $\Gamma \subset \text{Fn}_{\Delta}(S, h)$. A root $r \in \text{root}_0(S, h)$ is called *separating* (with respect to a subset Γ as above) if there is a pair of vectors $u, v \in \Gamma$ such that $r \cdot u > 0$ and $r \cdot v < 0$.

Lemma 2.10. *A subset $\Gamma \subset \text{root}_1(S, h)$ admits a compatible Weyl chamber if and only if there is no separating root $r \in \text{root}_0(S, h)$. In this case, each Weyl chamber Δ' for the lattice $S' := (\mathbb{Z}h + \mathbb{Z}\Gamma)^{\perp} \subset S$ is a face of a unique Weyl chamber Δ for S compatible with Γ .*

Proof. The necessity is given by (2.4), and for the sufficiency and uniqueness we observe that the set P_{Δ} of positive roots (see §2.1) is the set

$$P_{\Delta} = \{r \in \text{root}_0 \mathcal{F} \mid r \cdot l_* + \varepsilon \varphi(r) > 0\}, \quad \text{where } l_* := \sum_{l \in \Gamma} l, \quad 0 < \varepsilon \ll 1,$$

and $\varphi: S' \rightarrow \mathbb{R}$ is any functional positive on $P_{\Delta'}$ (and extended by 0 on the span of h and Γ). We choose ε so small that $|\varepsilon \varphi(r)| < 1$ for all $r \in \text{root}_0 S$; then, the functional $\ell: r \mapsto r \cdot l_* + \varepsilon \varphi(r)$ is generic and defines a partition (see §2.1). Indeed, if $\ell(r) = 0$, then $r \cdot l_* = \varphi(r) = 0$ (since $r \cdot l_* \in \mathbb{Z}$). The former implies that $r \cdot l = 0$ for all $l \in \Gamma$ (as all products $r \cdot l$ are assumed to be of the same sign); since also $r \cdot h = 0$, we conclude that $r \in S'$, contradicting to $\varphi(r) = 0$. \square

Algorithm 2.11. The unique compatible Weyl chamber Δ can be constructed by Vinberg’s algorithm: $\{\Delta\} = \bigcup_{n \geq 0} \{\Delta\}_n$, where $\{\Delta\}_0 := \{\Delta'\}$, $l_* := \sum_{l \in \Gamma} l$, and

$$\{\Delta\}_n = \{e \in \text{root}_0(S, h) \setminus S' \mid e \cdot l_* = n \text{ and } e \cdot r \geq 0 \text{ for all } r \in \{\Delta\}_k, k < n\}.$$

Since the set $\text{root}_0(S, h)$ is finite, the algorithm terminates. (In practice, since the elements of $\{\Delta\}$ are linearly independent, one can as well terminate the algorithm as soon as $\text{rk } \text{rt}(S, h)$ vectors have been collected.) Note also that, once $\{\Delta'\}$ is known, we do not need to use the original functional α and adjust constant ε .

2.3. Lines on projective $K3$ -surfaces. Recall that, since a $K3$ -surface X is simply connected, the map $D \mapsto [D]$ factors to an isomorphism $\text{Pic}(X) = \text{NS}(X)$; therefore, we freely identify classes of divisors on X with their images in $\text{NS}(X)$. Given an integer $d > 0$, a *degree $2d$ quasi-polarized $K3$ -surface* is a pair (X, h) , where X is a (minimal) $K3$ -surface and $h \in \text{NS}(X)$ a big and nef class such that

$$(2.12) \quad h^2 = 2d \text{ and the system } |h| \text{ is base-point free.}$$

Remark 2.13. Usually, a quasi-polarized $K3$ -surface is defined as a pair (X, h) , where h is a big and nef line bundle on the $K3$ -surface X . Since we work mostly in $\text{NS}(X)$, we prefer to consider h as a class in $\text{NS}(X)$.

Some authors allow the linear system $|h|$ of a quasi-polarization to have fixed components. However, since we are interested in the geometry of the projective surface $f_h(X)$, the base-point freeness is a natural assumption.

By (2.12), the system $|h|$ defines a morphism $f_h: X \rightarrow \mathbb{P}^{d+1}$. It is well known (see [33, p. 615]) that the restricted map $f_h: X \rightarrow f_h(X)$ is either birational or of degree 2 (*aka* hyperelliptic). In the former case, we say that the quasi-polarization h of X is *birational*, or that (X, h) is *birationally* quasi-polarized. In this case, the image $X_{2d} := f_h(X)$ is a surface of degree $2d$ with at worst isolated singularities, all of which are Du Val, and $f_h: X \rightarrow X_{2d}$ is the minimal resolution of the singularities of X_{2d} (see [33, Theorem 6.1]). In the latter case (with f_h of degree 2), we say that (X, h) is *hyperelliptic*. In this case, the image $f_h(X)$ is \mathbb{P}^2 , a scroll or a cone (see [33, Proposition 5.7]).

In what follows the Néron-Severi group (resp. Picard number) of the surface X is denoted by $\text{NS}(X)$ (resp. $\rho(X)$). Recall that, by the Hodge Index Theorem, the intersection form on $V := \text{NS}(X) \otimes \mathbb{R}$ is a non-degenerate quadratic form of index $(1, \rho(X) - 1)$, so that the set $\{x \in V \mid x^2 > 0\}$ consists of two components, one of which (denoted by \mathcal{C}_X) contains all ample classes on X . Apart from the positive cone \mathcal{C}_X , the vector space V contains also the *nef cone* $\text{Nef}(X)$ (resp. *ample cone* $\text{Nef}(X) \setminus \partial \text{Nef}(X)$), *i.e.*, the set of classes in $\text{NS}(X) \otimes \mathbb{R}$ that have non-negative intersection with all curves on X (resp. the cone spanned by the ample classes). Moreover, by the Riemann–Roch theorem,

$$(2.14) \quad \text{every root in } \text{NS}(X) \text{ is either effective or anti-effective.}$$

In the former case we speak of an effective root.

By the adjunction formula, for each irreducible curve $C \subset X$ with $C^2 = -2$, we have $p_a(C) = 0$, so that C is smooth rational. Such curves are called (-2) -curves (in particular (-2) -curves are always assumed to be irreducible, *i.e.*, they define nodal classes in $\text{NS}(X)$). We have the following well-known fact, that we will use in the sequel (see [18, § 8.1]).

Lemma 2.15. *The following statements hold.*

- (1) Let $\alpha \in \text{Nef}(X)$. Then $\alpha \in \partial \text{Nef}(X)$ (i.e., α is not ample) if and only if either $\alpha^2 = 0$ or $\alpha \cdot C = 0$ for a smooth rational curve $C \subset X$.
- (2) Every (-2) -curve $C \subset X$ defines a codimension 1 wall of $\text{Nef}(X)$.
- (3) For every (-2) -curve $C \subset X$ there exists a nef class α such that C is the only smooth rational curve in α^\perp . \triangleleft

In particular, the above lemma shows that there is a *one-to-one correspondence* between the set of codimension 1 walls of the nef cone and the set of smooth rational curves in X . Moreover, by [18, Corollary 8.2.11], the cone $\text{Nef}(X) \cap \mathcal{C}_X$ is a closed fundamental polyhedron for the action of the group generated by reflections on the cone \mathcal{C}_X , cf. $\bar{\Delta}^\sharp$ in §2.2.

The irreducible curves contracted by the map f_h are called *exceptional divisors*. Obviously, all exceptional divisors are (-2) -curves, so that each exceptional divisor defines a root in $\mathfrak{rt}(\text{NS}(X), h)$. Indeed, the Grauert contractibility criterion implies that each exceptional divisor C has negative self-intersection. Since $p_a(C) \geq 0$, the adjunction formula yields the equality $C^2 = -2$. Furthermore, the dual adjacency graph of the exceptional divisors is a union of simply laced Dynkin diagrams, cf. [33, Proposition 5.7 and Theorem 6.1].

Given a quasi-polarized $K3$ -surface (X, h) , we call a (-2) -curve $C \subset X$ a *line* on (X, h) if $C \cdot h = 1$. This definition is justified by the fact that, whenever the quasi-polarization is birational and C is an irreducible curve, we have

$$(2.16) \quad C \cdot h = 1 \text{ and } C^2 = -2 \text{ if and only if } f_h(C) \text{ is a degree one curve on } X_{2d}.$$

We follow [9] and interpret the set

$$(2.17) \quad \text{Fn}(X, h) := \{(-2)\text{-curves } C \subset X \text{ with } C \cdot h = 1\}$$

as a graph without loops, where a pair of vertices $v, w \in \text{Fn}(X, h)$ is connected by a $(v \cdot w)$ -fold edge. We call this set the (*plain*) *Fano graph* of the surface (X, h) . As in the case of lattices, we can also consider the bi-colored *extended Fano graph*

$$(2.18) \quad \text{Fn}^{\text{ex}}(X, h) := \{(-2)\text{-curves } C \subset X \text{ with } C \cdot h \leq 1\},$$

with the vertices colored according to their projective degree $2C \cdot h$. The relations between the two classes of graphs are discussed in §3.3 below.

Consider the definite root lattice $\mathfrak{rt}(\text{NS}(X), h)$ and define the positive roots as the effective ones, see (2.14). We denote by Δ_X the fundamental Weyl chamber given by this choice of positive roots. Then, by Lemma 2.15, the nef cone $\text{Nef}(X)$ is the distinguished closed fundamental polyhedron $\bar{\Delta}_X^\sharp \subset \text{NS}(X)$ extending the Weyl chamber Δ_X . In view of Lemma 2.15, identifying (-2) -curves with their classes in $\text{NS}(X)$, we conclude that

$$\text{Fn}(X, h) = \text{Fn}_{\Delta_X}(\text{NS}(X), h), \quad \text{Fn}^{\text{ex}}(X, h) = \text{Fn}_{\Delta_X}^{\text{ex}}(\text{NS}(X), h).$$

Then, using Lemma 2.9, we arrive at the following statement.

Theorem 2.19. *Let (X, h) be a quasi-polarized $K3$ -surface. Then, for any choice of the Weyl chamber Δ for $\mathfrak{rt}(\text{NS}(X), h)$, there are canonical isomorphisms*

$$\text{Fn}(X, h) = \text{Fn}_\Delta(\text{NS}(X), h), \quad \text{Fn}^{\text{ex}}(X, h) = \text{Fn}_\Delta^{\text{ex}}(\text{NS}(X), h). \quad \triangleleft$$

Remark 2.20. If h is ample, the condition on intersection with roots in (2.6) is void and finding lines reduces to solving the equations

$$(2.21) \quad v^2 = -2 \quad \text{and} \quad v \cdot h = 1.$$

Otherwise, (2.21) results in an overcount: *e.g.*, if C is a line, and E is a (-2) -curve such that $E \cdot C = 1$ and $E \cdot h = 0$, then both C and $C + E$ satisfy (2.21).

2.4. Admissible lattices (see [27, 33]). Let d be a fixed positive integer and let $S \ni h$ be a polarized lattice of degree $2d$. A vector $p \in S$ is called *m-isotropic*, where $m = 1, 2, 3$, if

$$p^2 = 0 \quad \text{and} \quad p \cdot h = m.$$

Let Δ be a fixed Weyl chamber for $\mathfrak{rt}(S, h)$. It extends to a unique fundamental polyhedron $\Delta^\sharp \subset \mathcal{C}^+(S, h)$ such that $h \in \bar{\Delta}^\sharp$ (see §2.2). In order to realize the lattice $S \ni h$ as the Néron–Severi lattice of a quasi-polarized $K3$ -surface (X, h) with $\bar{\Delta}^\sharp = \text{Nef}(X)$ and control the geometry of the map $f_h: X \rightarrow \mathbb{P}^{d+1}$, we need to impose certain extra conditions on $S \ni h$:

- (1) there is no 1-isotropic vector $p \in S$;
- (2) for $h^2 \geq 4$, there is no 2-isotropic vector $p \in S$;
- (3) for $h^2 = 8$, there is no 3-isotropic vector $p \in S$,

As we will explain below, for $h^2 = 8$, there is also the condition

- (4) $h \notin 2S$ (so that f_h does not factor through the Veronese embedding, see Lemma 2.27 below);

however, it holds automatically whenever $\text{Fn}_\Delta(S, h) \neq \emptyset$.

Definition 2.22. A polarized lattice $S \ni h$ is called *m-admissible* (or just *admissible*, if the parameter $m = 1, 2, 3$ is understood), if it satisfies conditions (1)–(m) above, with $h^2 \geq 4$ if $m = 2$ and $h^2 = 8$ if $m = 3$.

The admissibility of a lattice is easily established in the algorithm computing the Fano graph (see §A.1.1 below or [9, Algorithm 2.5]). By definition, the notion of *m-admissibility* is independent of the choice of a Weyl chamber Δ for $\mathfrak{rt}(S, h)$ (*i.e.*, the choice of a fundamental polyhedron Δ^\sharp). Still we have the following observation, that will be useful in the sequel.

Observation 2.23. Let $S \ni h$ be an $(m - 1)$ -admissible lattice with h^2 in the range of applicability of the corresponding condition. If S fails to be *m-admissible*, then there exists an *m-isotropic* vector $p \in \bar{\Delta}^\sharp$.

Proof. Let p be an *m-isotropic* vector in S . Since the Weyl group of $\mathfrak{rt}(S, h)$ is finite, $\bar{\Delta} \subset \bar{\mathcal{C}}^+(S, h)$ is a fundamental domain of the (extended to $\bar{\mathcal{C}}^+$) action of this group. Hence, after applying a sequence of reflections (which all preserve h), we can assume that $p \in \bar{\Delta}$. (Geometrically, if $S = \text{NS}(X)$ and $\Delta = \Delta_X$, this procedure corresponds to the passage to the moving part of the linear system $|p|$.)

We assert that then immediately $p \in \bar{\Delta}^\sharp$. Indeed, let $r \in \{\Delta^\sharp\}_n$, $n \geq 1$, be a wall such that $p \cdot r = -a \leq -1$. By (2.2), this implies

$$(2.24) \quad h^2 \leq \frac{2m(m - an)}{a^2},$$

and the cases to exclude are

- none, if $m = 1$, as the inequality would imply $h^2 \leq 0$;
- $h^2 = 4$, *cf.* (2), and $n = a = 1$ if $m = 2$;
- $h^2 = 8$, *cf.* (3), and $n = a = 1$ if $m = 3$.

In the last two cases, we have $(p - r)^2 = 0$ and $(p - r) \cdot h = m - 1$, *i.e.*, (S, h) is not $(m - 1)$ -admissible and the extra restriction makes no sense. \square

We will mainly consider 2-admissible lattices, (*i.e.*, those corresponding to the birational projective models of $K3$ -surfaces, see [Lemma 2.27](#) below).

Lemma 2.25. *Let $S \ni h$ be a polarized lattice, Δ a Weyl chamber, $e_1, e_2 \in \{\Delta\}$ distinct exceptional divisors, and $l_1, l_2 \in \text{Fn}_\Delta(S, h)$ distinct lines. Then:*

- (1) *if $S \ni h$ is 2-admissible, then $l_1 \cdot l_2 \in \{0, 1\}$;*
- (2) *if $S \ni h$ is 1-admissible, then $l_1 \cdot e_1 \in \{0, 1\}$;*
- (3) *for all lattices, $e_1 \cdot e_2 \in \{0, 1\}$.*

Proof. By [\(2.2\)](#) and [\(2.5\)](#), we have $0 \leq l_1 \cdot l_2 \leq 2(h^2 + 1)/h^2$ and, since $h^2 \geq 4$, the maximum is $l_1 \cdot l_2 = 2$. However, in this case $p := l_1 + l_2$ is a 2-isotropic vector and (S, h) is not 2-admissible. Similarly, $0 \leq l_1 \cdot e_1 \leq 2$ and, in the case $l_1 \cdot e_1 = 2$, the vector $p := l_1 + e_1$ is 1-isotropic. The last assertion is a well-known property of root systems with all roots of square (-2) . \square

To justify our interest in m -isotropic vectors, let us consider the polarized lattice $\text{NS}(X) \ni h$ for a $K3$ -surface X and a big and nef class $h \in \text{NS}(X)$. As before, we fix the Weyl chamber Δ_X given by the effective roots, *cf.* [\(2.14\)](#). Recall that the closure of its extension to \bar{C}_X is the nef cone $\text{Nef}(X)$.

Let us assume that $m \in \{1, 2, 3\}$, and (for $m > 1$) the lattice $\text{NS}(X)$ is $(m - 1)$ -admissible. We claim that

$$(2.26) \quad \text{NS}(X) \text{ is } m\text{-admissible iff } E \cdot h > m \text{ for every elliptic curve } E \subset X.$$

Indeed, assume that $\text{NS}(X)$ fails to be m -admissible. Then, by [Observation 2.23](#), the lattice $\text{NS}(X)$ contains a nef m -isotropic class p . It is well-known that $|p|$ is a base-point free elliptic pencil on X , and one can take for E any smooth fiber of $|p|$. The converse statement is obvious: take $p = [E]$.

The above discussion, combined with [\[33\]](#), yields the following well-known lemma, which we state for the reader's convenience.

Lemma 2.27. *Let X be a $K3$ -surface and let $h \in \text{NS}(X)$ be big and nef. Then:*

- (1) *the linear system $|h|$ is base-point-free if and only if the polarized lattice $\text{NS}(X) \ni h$ is 1-admissible;*
- (2) *assuming that $h^2 \geq 4$, $h^2 \neq 8$, and $\text{NS}(X) \ni h$ is 1-admissible, the map f_h is birational (onto its image) if and only if $\text{NS}(X) \ni h$ is 2-admissible;*
- (3) *assuming that $h^2 = 8$ and $\text{NS}(X) \ni h$ is 1-admissible, the map f_h is birational if and only if $\text{NS}(X) \ni h$ is 2-admissible and $h \notin 2\text{NS}(X)$;*
- (4) *assuming that $h^2 = 8$ and f_h is birational, *cf.* [\(3\)](#), the image $X_8 := f_h(X)$ is an intersection of three quadrics if and only if $\text{NS}(X) \ni h$ is 3-admissible.*

Proof. By [\[27\]](#) (see also [\[33\]](#), the proof of Proposition 8.1) the linear system $|h|$ is base-point free if and only if the surface X contains no irreducible genus-one curve E such that $E \cdot h = 1$. Thus, Statement [\(1\)](#) follows from [\(2.26\)](#).

Statements [\(2\)](#) and [\(3\)](#) (resp. Statement [\(4\)](#)) follow from [\[33\]](#), Theorem 5.2] (resp. [\[33\]](#), Theorem 7.2]) and [\(2.26\)](#). \square

Definition 2.28. Let (X, h) be a degree 8 birationally quasi-polarized $K3$ -surface. We say that (X, h) (or the image $X_8 := f_h(X)$) is a *triquadric* (resp. a *special octic*) if it satisfies the equivalent conditions of [Lemma 2.27\(4\)](#) (resp. otherwise).

2.5. Hyperelliptic models and special octics (see [9]). Given a quasi-polarized $K3$ -surface (X, h) it is natural to consider the situation when the map f_h degenerates (*i.e.*, the hyperelliptic case) or its image is a special octic. In lattice-theoretic terms one has to study a polarized lattice $S \ni h$, with a fixed Weyl chamber Δ for $\text{rt}(S, h)$, that fails to be m -admissible (see Lemma 2.27). Moreover, in view of Observation 2.23, such a degeneration amounts to *imposing* the existence of an m -isotropic vector $p \in \bar{\Delta}^\sharp$ while assuming that $S \ni h$ is $(m - 1)$ -admissible for $m = 2, 3$. We have the following lemma, slightly different from the smooth case.

Lemma 2.29. *Assume that the polarized lattice $S \ni h$ is $(m - 1)$ -admissible and $p \in \bar{\Delta}^\sharp$ is an m -isotropic vector, $m = 2, 3$. Then:*

- (1) *if $h^2 = 4$ and $m = 2$, there are at most two 2-isotropic vectors $p, \bar{p} \in \bar{\Delta}^\sharp$ and, if there are two such vectors (*i.e.*, $p \neq \bar{p}$), then $p + \bar{p} = h$;*
- (2) *otherwise, p is the unique m -isotropic vector in $\bar{\Delta}^\sharp$.*

Furthermore, one has $l \cdot p \in \{0, 1\}$ for each line $l \in \text{Fn}_\Delta(S, h)$ and $e \cdot p \in \{0, 1\}$ for each exceptional divisor $e \in \{\Delta\}$.

Proof. Let \bar{p} be an m -isotropic vector, $\bar{p} \neq p$. Then, by (2.2), $0 < p \cdot \bar{p} \leq 2m^2/h^2$, implying, with two exceptions, that $p \cdot \bar{p} = 1$ and, hence, p and \bar{p} are separated by the root $(p - \bar{p})$, so that $\bar{p} \notin \bar{\Delta}^\sharp$, see §2.2.

Exceptionally, one may have $p \cdot \bar{p} = 2$ for $h^2 = 4$, $m = 2$ or $h^2 = 8$, $m = 3$. In the latter case, $(h - p - \bar{p})$ is a 2-isotropic vector, so that this case is excluded and the proof of Statement (2) is complete.

For $h^2 = 4$, $m = 2$, the linear dependence given by (2.2) is $p + \bar{p} = h$ and both vectors are in $\bar{\Delta}^\sharp$ *unless they are separated by a root*. This proves Statement (1).

For the last statement, one has $0 \leq l \cdot p \leq 1$ by (2.4) and (2.24), with the only exception $h^2 = 4$, $m = 2$, $l \cdot p = 2$. However, in this exceptional case, $(h - p - l)$ is a 1-isotropic vector. Similarly, the inequality $e \cdot p \leq 1$ follows from (2.24). \square

The unique vector $p \in \bar{\Delta}^\sharp$ (or pair $p, \bar{p} \in \bar{\Delta}^\sharp$) given by Lemma 2.29 constitutes an extra structure on the lattice. For example, there is a canonical partition

$$(2.30) \quad \text{Fn}_\Delta(S, h) = C_0 \cup C_1, \quad C_i := \{l \mid l \cdot p = i\}, \quad i = 0, 1.$$

(In case (1), the numbering of C_i depends on the choice of one of the two vectors.) Conversely, if we assume that S is rationally generated by h , p , and all lines (*cf.* Lemma 2.8 and remark thereafter), the class $p \in \bar{\Delta}^\sharp$ (or the pair $p, \bar{p} \in \bar{\Delta}^\sharp$), if it exists, is uniquely recovered from the above partition.

Remark 2.31. Although graph-theoretically the two notions are identical, we will distinguish between *partitioned* (*via* $v \mapsto v \cdot p$) and *bi-colored* (*via* $v \mapsto v \cdot h$) graphs, reserving the two words as separate terms. Occasionally, we even consider bi-colored partitioned graphs, so that each *part* C_i , $i = 0, 1$, is bi-colored.

Remark 2.32. Case (1) of Lemma 2.29 is of limited interest when dealing with line configurations on $K3$ -surfaces. Indeed, let us consider a quasi-polarized $K3$ -surface (X, h) of degree 4 and assume that $\text{Nef}(X)$ contains two 2-isotropic vectors. Each nef 2-isotropic class defines a genus-one fibration on X . Thus, X is endowed with two genus-one fibrations and, by (2.30), each line on X is a component of a fiber of exactly one of them. This gives us an obvious upper bound of 48 lines, which is sharp for smooth models (see [10]), extends to all models, and fails to be sharp (by at least two) when h is not ample.

The other cases with $m = 2$ are not interesting either: essentially, we are speaking about a single genus-one fibration.

2.6. Miscellaneous definitions. For the reader's convenience, we collect below a few common definitions used in the sequel.

The *girth* $\text{girth}(\Gamma)$ of a graph Γ is the length of a shortest cycle in Γ (with the convention that the girth of a forest is ∞), and the *independence number* $\alpha(\Gamma)$ is the cardinality of the largest independent vertex set. Given a bi-colored graph Γ' , we denote by $\text{sp}_c \Gamma'$ the *plain* subgraph of Γ' induced by the vertices of Γ' of the chosen color $c \in \{0, 1\}$.

Definition 2.33. Let $S \ni h$ be a 1-admissible lattice and Δ a distinguished Weyl chamber. A *pseudo-vertex* of the Fano graph $\Gamma := \text{Fn}_\Delta(S, h)$ is either an exceptional divisor $e \in \{\Delta\}$ or a 2- or 3-isotropic vector $p \in \Delta^\sharp$.

Given a subgraph $\Pi \subset \Gamma$, we define the *support* of a (pseudo-)vertex v of Γ (relative to the subgraph Π) as the set

$$\|v\|_\Pi = \text{supp}_\Pi v := \{l \in \Pi \mid l \cdot v = 1\}.$$

As usual, if the subgraph Π is understood, the subscript in $\text{supp}_\Pi v$ is omitted. The support of a (pseudo-)vertex relative to the whole graph is called the *star*

$$\text{star}(v) := \text{supp}_\Gamma v.$$

We use these notions in the discussion of the local structure of Fano graphs (§5.1) and in various algorithms, where we *identify* an extra vertex of a graph extension with its support (see *e.g.* §A.3).

3. ABSTRACT GRAPHS

This section is devoted to the question whether a given graph is (isomorphic to) the Fano graph of a quasi-polarized $K3$ -surface. We start with a discussion of various properties of polarized lattices spanned by lines.

3.1. Extensible graphs. Given a graph Γ , let $\mathbb{Z}\Gamma$ be the lattice freely generated by the vertices $v \in \Gamma$, $v^2 = -2$, with $u \cdot v = n$ whenever u and v are connected by an n -fold edge. Then, we associate to Γ the polarized lattice

$$(3.1) \quad \mathcal{F}_{2d}(\Gamma) := (\mathbb{Z}\Gamma + \mathbb{Z}h) / \ker, \quad h^2 = 2d > 0, \quad h \cdot v = 1 \text{ for } v \in \Gamma.$$

Usually, the even integer $2d$ is fixed in advance and omitted from the notation. We speak of *2d-polarized* graph Γ when it is needed to avoid ambiguity.

In a similar manner, we define the lattice $\mathcal{F}_{2d}(\Gamma')$ for a bi-colored graph Γ' ; the last condition in (3.1) is replaced by the equality $h \cdot v = c(v)$, where $c(v) \in \{0, 1\}$ is the color of the vertex $v \in \Gamma'$.

We treat vertices of Γ as vectors in $\mathcal{F}(\Gamma)$ and freely use mixed terminology:

- if v_1, v_2 are adjacent in Γ , we say that they *intersect* or $v_1 \cdot v_2 = 1$;
- otherwise, we say that v_1, v_2 are *disjoint* or $v_1 \cdot v_2 = 0$;
- vertices are (linearly) independent if so are the corresponding vectors.

We also apply to Γ other lattice theoretic terminology; *e.g.*, we speak about the *rank* $\text{rk} \Gamma := \text{rk}_{2d} \Gamma$ or *inertia indices* $\sigma_\pm \Gamma := \sigma_{2d, \pm} \Gamma$, referring to the lattice $\mathcal{F}(\Gamma)$. Note, though, that we only consider graphs with $\sigma_+ \Gamma = 1$, *i.e.*, we *always assume that $\mathcal{F}(\Gamma)$ is hyperbolic.*

Occasionally, we also pick an isotropic subgroup $\mathcal{K} \subset \text{discr } \mathcal{F}(\Gamma)$ and consider the extension $\mathcal{F}(\Gamma, \mathcal{K}) := \mathcal{F}_{2d}(\Gamma, \mathcal{K})$ of $\mathcal{F}(\Gamma)$ by \mathcal{K} ; in this notation, $\mathcal{F}(\Gamma)$ is an abbreviation for $\mathcal{F}(\Gamma, 0)$. (Recall that the discriminant group $\text{discr } L := L^\vee/L$ of a non-degenerate even lattice L is endowed with a $\mathbb{Q}/2\mathbb{Z}$ -valued quadratic form q , and the isomorphism classes of even finite index extensions of L are in a one-to-one correspondence with the q -isotropic subgroups of $\text{discr } L$, see [26].)

Definition 3.2. As in §2.5, a Weyl chamber Δ for $\text{rt}(\mathcal{F}(\Gamma, \mathcal{K}), h)$ is called *compatible* with Γ (assuming a fixed kernel \mathcal{K}) if $\Gamma \subset \text{Fn}_\Delta \mathcal{F}(\Gamma, \mathcal{K})$. The graph Γ or pair (Γ, \mathcal{K}) is called *extensible* (in degree $2d$) if it admits a compatible Weyl chamber.

The concept of extensibility replaces the requirement $\text{root}_0 \mathcal{F}(\Gamma, \mathcal{K}) = \emptyset$ used in the smooth case to rule out the majority of “bad” graphs. It is effective due to a simple criterion given by the next statement and heredity given by Corollary 3.4.

Lemma 3.3 (a corollary of Lemma 2.10). *A pair (Γ, \mathcal{K}) is extensible if and only if the lattice $\mathcal{F}(\Gamma, \mathcal{K})$ has no separating roots. If this is the case, there is a unique Weyl chamber Δ for $\text{rt}(\mathcal{F}(\Gamma, \mathcal{K}), h)$ compatible with Γ . \triangleleft*

Given a pair (Γ, \mathcal{K}) and an induced subgraph $\Gamma' \subset \Gamma$, denote

$$\mathcal{K}|_{\Gamma'} := (\mathcal{F}(\Gamma') \otimes \mathbb{Q}) \cap \mathcal{F}(\Gamma, \mathcal{K}) \text{ mod } \mathcal{F}(\Gamma') \subset \text{discr } \mathcal{F}(\Gamma'),$$

so that $\mathcal{F}(\Gamma', \mathcal{K}|_{\Gamma'}) \subset \mathcal{F}(\Gamma, \mathcal{K})$ is the primitive sublattice rationally generated by h and the vertices of Γ' . We say that a pair (Γ', \mathcal{K}') is *subordinate* to (Γ, \mathcal{K}) , denoted $(\Gamma', \mathcal{K}') \prec (\Gamma, \mathcal{K})$, if $\Gamma' \subset \Gamma$ is an induced subgraph and $\mathcal{K}' \subset \mathcal{K}|_{\Gamma'}$.

Corollary 3.4 (of Lemma 3.3). *Extensibility is a hereditary property: if (Γ, \mathcal{K}) is extensible, then so is any subordinate pair $(\Gamma', \mathcal{K}') \prec (\Gamma, \mathcal{K})$. \triangleleft*

As another consequence, an extensible pair (Γ, \mathcal{K}) has a well-defined *saturation* and *extended saturation*

$$\text{sat}_{2d}(\Gamma, \mathcal{K}) := \text{Fn}_\Delta \mathcal{F}_{2d}(\Gamma, \mathcal{K}), \quad \text{sat}_{2d}^{\text{ex}}(\Gamma, \mathcal{K}) := \text{Fn}_\Delta^{\text{ex}} \mathcal{F}_{2d}(\Gamma, \mathcal{K}),$$

where $\Delta \subset \text{rt } \mathcal{F}(\Gamma, \mathcal{K})$ is the unique Weyl chamber compatible with Γ . As with the other notation, we usually omit the subscript $2d$.

Warning 3.5. *A priori*, unlike the case of smooth models, we cannot claim that the inequality $(\Gamma', \mathcal{K}') \prec (\Gamma, \mathcal{K})$ implies the inclusion $\text{sat}(\Gamma', \mathcal{K}') \subset \text{sat}(\Gamma, \mathcal{K})$, as the smaller lattice may contain extra lines that are inhibited by some extra exceptional divisors in the larger one. As a simple example, we have

$$\text{sat}(\tilde{\mathbf{A}}_2 \oplus \mathbf{A}_2) = 2\tilde{\mathbf{A}}_2 \not\subset \tilde{\mathbf{A}}_2 \oplus \mathbf{A}_3 = \text{sat}(\tilde{\mathbf{A}}_2 \oplus \mathbf{A}_3):$$

the second saturation has an exceptional divisor completing \mathbf{A}_3 to $\tilde{\mathbf{A}}_4$. Even more striking Example 6.8 below shows that the number of lines may decrease when the lattice is extended. This renders the study of surfaces with singular points more difficult. For example, any validity criteria based on the absence of extra lines of a certain kind (*e.g.*, the type of the full graph) are *not* hereditary and must be avoided. Besides, *until the very end of the computation*, we cannot try to reduce the overcounting by using any sorting based on the full configuration $\text{sat}(\Gamma, \mathcal{K})$: only the lines explicitly contained in Γ can be taken into account. We give more details of these technical problems in Appendices A, B, C.

3.2. Admissible and geometric graphs. We call a graph Γ (resp. pair (Γ, \mathcal{K})) *m-admissible* (in a fixed degree $2d$) if it is extensible and the lattice $\mathcal{F}_{2d}(\Gamma)$ (resp. $\mathcal{F}_{2d}(\Gamma, \mathcal{K})$) is *m-admissible*, where $m = 1, 2, 3$. Clearly, this property is hereditary, cf. [Corollary 3.4](#).

Obviously, if a given configuration can be realized as the Fano graph of a *K3*-surface (X, h) , there must exist a primitive isometric embedding of $\mathcal{F}(\Gamma, \mathcal{K})$ into the *K3*-lattice $H_2(X; \mathbb{Z})$. That is why we introduce the following notion.

Definition 3.6. Let $d \geq 1$ be a fixed integer and $m \in \{1, 2, 3\}$.

- (1) A $2d$ -polarized hyperbolic lattice $S \ni h$ is called *m-geometric* if it is *m-admissible* and there exists a primitive isometric embedding

$$S \hookrightarrow \mathbf{L} := 2\mathbf{E}_8 \oplus 3\mathbf{U}.$$

- (2) Given a *plain* graph Γ , an isotropic subgroup $\mathcal{K} \subset \text{discr } \mathcal{F}_{2d}(\Gamma)$ is an *m-geometric kernel* (in degree $2d$) if the lattice $\mathcal{F}_{2d}(\Gamma, \mathcal{K})$ is *m-geometric*. We put

$$\mathfrak{G}_{2d}^m(\Gamma) := \{\mathcal{K} \subset \text{discr } \mathcal{F}_{2d}(\Gamma) \mid \mathcal{K} \text{ is } m\text{-geometric}\}.$$

- (3) A *plain* graph Γ is *m-subgeometric* (in degree $2d$) if $\mathfrak{G}_m(\Gamma)$ is non-empty.
(4) An *m-subgeometric* graph Γ is called *m-geometric* (in degree $2d$) if

$$\Gamma \cong \text{sat}_{2d}(\Gamma', \mathcal{K})$$

for a graph $\Gamma' \subset \Gamma$ and kernel $\mathcal{K} \in \mathfrak{G}_{2d}^m(\Gamma')$.

- (5) A *bi-colored* graph Γ' is *m-geometric* (in degree $2d$) if

$$\Gamma' = \text{Fn}_{\Delta}^{\text{ex}}(S, h)$$

for some *m-geometric* $2d$ -polarized lattice $S \ni h$.

Whenever this leads to no ambiguity, we omit the prefix “*m*” and/or degree $2d$ and speak about (sub-)geometric graphs *etc.*

Remark 3.7. The fact that $\mathcal{F}(\Gamma, \mathcal{K})$ is *m-geometric* is *not* a hereditary property of pairs (Γ, \mathcal{K}) as one can lose the primitivity by passing to a smaller subgroup $\mathcal{K}' \subset \mathcal{K}$. In contrast, the *existence* of an *m-geometric kernel* is a hereditary property of graphs: if $\mathcal{K} \in \mathfrak{G}_m(\Gamma)$, then $\mathcal{K}|_{\Gamma'} \in \mathfrak{G}_m(\Gamma')$ for any induced subgraph $\Gamma' \subset \Gamma$.

Nikulin’s results [26] give us precise criteria for a lattice to admit a primitive embedding into the *K3*-lattice \mathbf{L} (cf. [12, Theorem 3.2]). Thus, *m*-(sub-)geometric graphs are easily detected using **GAP** [16] (see §A.1.2 below).

As an immediate consequence of [Definition 3.6](#) and general theory of lattice-polarized *K3*-surfaces (Nikulin [25], Saint-Donat [33]; cf. also [12, Theorem 3.11] and [10, Theorem 7.3]), we have the following theorem.

Theorem 3.8. *Let $d \geq 2$ be a fixed integer. A graph Γ is 2-geometric in degree $2d$ if and only if $\Gamma \cong \text{Fn}(X, h)$ for a degree $2d$ birationally quasi-polarized *K3*-surface (X, h) such that $\text{NS}(X)$ is spanned by lines. If $2d = 8$, then Γ is 3-geometric if and only if (X, h) above can be chosen to be a triquadric. \triangleleft*

[Theorem 3.8](#) is a “classical” statement dealing with lines only. However, in view of [Warning 3.5](#), it may fail to describe *all* configurations of lines: unlike the smooth case, we cannot assert that $\Gamma \subset \text{Fn}(X, h)$ if the extension $\text{NS}(X) \supset \mathcal{F}(\Gamma, \mathcal{K})$ is of positive corank. We address this phenomenon in the next section; for now, we make a precise statement, taking into account the exceptional divisors.

Theorem 3.9. *A bi-colored graph Γ' is 2-geometric in degree $2d$ if and only if $\Gamma' \cong \text{Fn}^{\text{ex}}(X, h)$ for a degree $2d$ birationally quasi-polarized K3-surface (X, h) . It is 3-geometric in degree 8 if and only if (X, h) can be chosen to be a triquadric. \triangleleft*

Remark 3.10. As in the smooth case (cf. [9, Theorem 2.2]) the general theory gives us the dimension of the family of K3-surfaces that realize a given graph Γ : modulo the projective group, it equals $20 - \text{rk}_{2d} \Gamma$.

Note that, for $d \neq 4$, the geometric interpretation of the fact that a bi-colored graph is 3-geometric can be derived from [33, Theorem 7.2]. We omit the discussion of the general case to maintain our exposition compact.

3.3. Extended vs. plain Fano graphs. At this point, we need to emphasize the fact that we thoroughly distinguish between two categories:

- plain graphs—typically, the input for $\mathcal{F}(-)$ and output of $\text{Fn}(-)$, and
- bi-colored graphs—the output of $\text{Fn}^{\text{ex}}(-)$ (even if all colors are 1).

(Occasionally, as in §3.4 below, we also consider partitioned graphs, both plain and bi-colored, cf. Remark 2.31.) Apart from the fact that we are interested in large configurations of *lines* (plain graphs) but strive to provide more complete geometric information (bi-colored extended graphs), there is a deeper reason that makes the presence of both categories essential for the paper:

- for plain graphs (regarded as graphs of lines), we have a simple hereditary extensibility criterion given by Lemma 3.3, whereas
- for extended graphs, we have a complete geometric realizability criterion given by Theorem 3.9 (vs. Theorem 3.8, which needs extra assumptions).

The difficulty is that Theorem 3.9 applies to *saturated* bi-colored graphs only, but, unless $\text{sp}_0 \Gamma' = \emptyset$, we have no means of detecting whether a given bi-colored graph Γ' is saturable and, hence, almost no means of constructing saturated bi-colored graphs other than saturating (via sat^{ex}) *plain* ones. For this reason, we do not introduce the notion of subgeometric or even admissible bi-colored graph.

One may try to relate the two categories by functors

$$\Gamma' \mapsto \text{sp}_1 \Gamma' \text{ (bi-colored} \rightarrow \text{plain)} \quad \text{and} \quad \Gamma \mapsto \Gamma^{\text{ex}} \text{ (plain} \rightarrow \text{bi-colored)}$$

(for the latter, see Convention 3.12 below), but this does not work quite well: the latter, if properly defined, turns out to be multi-valued, whereas the former does not need to take geometric graphs to geometric (see Example 6.8 below; in general, even if Γ' itself is geometric, we can only assert that $\text{sp}_1 \Gamma'$ is *subgeometric*).

Remark 3.11. In other words, unless $\text{NS}(X)$ is spanned by lines, the plain graph $\text{Fn}(X, h)$ does not need to be geometric in the sense of Definition 3.6. Still our definition of the term “geometric” for plain graphs is both consistent with the existing literature on the smooth case and convenient (it is used in most algorithms below). Consequently, whenever we speak of a geometric plain graph, we mean *geometric in the sense of Definition 3.6*. Otherwise (when $\Gamma \cong \text{Fn}(X, h)$ for a K3-surface (X, h)), we informally use the term “geometrically realizable”.

Convention 3.12. Recall that, given an m -geometric plain graph Γ and a kernel $\mathcal{K} \in \mathfrak{G}_{2d}^m(\Gamma)$, one can define the bi-colored *extended graph* $\text{sat}_{2d}^{\text{ex}}(\Gamma, \mathcal{K})$. However, this extended graph is not uniquely determined by Γ alone (see, e.g., the extended Fano graphs Φ_{30}' and Φ_{30}'' in §7.3), and we are using the vague notation Γ^{ex} to refer to *any* of these extensions (usually, when listing the exceptions).

In view of these subtleties, the classification of large Fano graphs, *even plain*, is more involved than in the smooth case: it is no longer enough to work with abstract graphs and lattices of the form $\mathcal{F}(-)$. Here, by “large” we still mean graphs with many *lines*, *i.e.*, we fix a threshold M and try to list all geometrically realizable plain (resp. geometric bi-colored) Fano graphs $\Gamma^* = \text{Fn}^*(X, h)$ satisfying $|\Gamma| \geq M$ (resp. $|\text{sp}_1 \Gamma^{\text{ex}}| \geq M$). To localize the problem, we break the proofs into two stages: the first (the bulk of the proof) runs essentially as in the smooth case, whereas the second, new one boils down to a more thorough analysis of but a few extremal graphs.

3.3.1. *Stage 1: lattices spanned by lines.* We list all geometric plain graphs Γ satisfying $|\Gamma| \geq M$. The algorithms are essentially those used in the smooth case, with a few minor modifications (*cf.* [Warning 3.5](#)) and [Lemma 3.3](#) replacing the requirement that $\text{root}_0 \mathcal{F}(\Gamma, \mathcal{K}) = \emptyset$. The result of this “classical” part of the proof is the sharp upper bound on the number of lines (see [Lemma 2.8](#) and remark thereafter) and some (but not all) geometrically realizable large configurations of lines.

3.3.2. *Stage 2: the general case.* To complete the *classification*, we still need to consider the polarized lattices $S \ni h$ that fail to be spanned by lines. (For example, the configuration of 16 pairwise disjoint lines on the elements of family \mathfrak{R}_{64} would be missing after Stage 1, see [Example 6.8](#) below.) To this end, we need to consider all geometric extensions $\bar{S} \supset S$ of all geometric lattices $S := \mathcal{F}(\Gamma, \mathcal{K})$, $|\Gamma| \geq M$, found in Stage 1. We may assume that

- (1) $\text{rk } \bar{S} > \text{rk } S$ and, in particular, $\text{rk } S \leq 19$;
- (2) $\bar{\Gamma} := \text{Fn}(\bar{S}, h) \subset \Gamma$ (as otherwise either $\mathcal{F}(\bar{\Gamma}) \supset S$ is not geometric or it is a proper overlattice that can be used instead of S) and $|\bar{\Gamma}| \geq M$;
- (3) \bar{S} is rationally generated over S by a number of exceptional divisors e_i .

In other words, for each sufficiently large subgraph $\bar{\Gamma} \subset \Gamma$ (to play the rôle of $\text{Fn}(\bar{S}, h)$) we add to S , one-by-one, a number of exceptional divisors and analyze the result. We use essentially the same algorithms (see [§A.4](#) below) as for adding lines (*cf.* the proof of [Proposition 6.7](#)), with [Lemma 3.3](#) replaced by more general [Lemma 2.10](#). Note also that by an “exceptional divisor” we merely mean an extra vector e satisfying $e^2 = -2$, $e \cdot h = 0$, and certain formal geometric conditions (*cf.* [Lemmas 5.1](#) and [5.4](#) below for octics); we do *not* insist that e remain an exceptional divisor in $\text{Fn}^{\text{ex}}(\bar{S})$, mainly because we do not know a usable criterion to control the behaviour of exceptional divisors.

Remark 3.13. In practice, instead of $\bar{\Gamma}$ we use an even smaller subgraph of Γ known to generate S over \mathbb{Q} : this reduces the number of choices for e . Then, for each “good” vector e found, we analyze all geometric finite index extensions of the lattice $S + \mathbb{Z}e$ and check conditions (1), (2) above.

3.4. **Partitioned graphs.** Let $\Gamma = C_0 \cup C_1$, $C_0 \cap C_1 = \emptyset$, be a partitioned graph, *cf.* (2.30) and [Remark 2.31](#). For such a graph, we redefine the lattice $\mathcal{F}_{2d}(\Gamma)$ in order to *impose* the existence of an m -isotropic vector p :

$$(3.14) \quad \mathcal{F}_{2d}^m(\Gamma) := (\mathbb{Z}\Gamma + \mathbb{Z}h + \mathbb{Z}p) / \ker, \quad h^2 = 2d > 0, \quad p^2 = 0, \quad p \cdot h = m,$$

where the other intersections are

$$h \cdot v = 1 \text{ for } v \in \Gamma, \quad p \cdot v = i \text{ for } v \in C_i, \quad i = 0, 1.$$

As above, the degree $2d$ is usually fixed and omitted from the notation.

Since we want to use partitioned graphs to study configurations of rational curves in hyperelliptic models/special octics, we always assume $\mathcal{F}^m(\Gamma)$ hyperbolic.

Obviously, all notions introduced in §3.2 (e.g., extension $\mathcal{F}^m(\Gamma, \mathcal{K})$, compatible Weyl chamber with $p \in \bar{\Delta}^\sharp$, p -extensibility, to name a few) can be generalized to partitioned graphs and the lattices that they define; details are left to the reader. A root $r \in \text{root}_0 \mathcal{F}^m(\Gamma, \mathcal{K})$ is called p -separating if $r \cdot p > 0$ and $r \cdot v < 0$ for some vertex $v \in \Gamma$. Then, Lemma 3.3 (and its proof) extends almost literally.

Lemma 3.15. *A partitioned pair (Γ, \mathcal{K}) is p -extensible if and only if the lattice $\mathcal{F}^m(\Gamma, \mathcal{K})$ has neither separating nor p -separating roots. If this is the case, there is a unique Weyl chamber for $\text{rt}(\mathcal{F}^m(\Gamma, \mathcal{K}), h)$ compatible with Γ . \triangleleft*

Crucial are the concepts of an m' -admissible and m' -(sub-)geometric partitioned graph: in Definition 3.6, we always refer to the lattice \mathcal{F}^m , $m = m' + 1$. Observe the relation between the two parameters: when imposing the presence of an m -isotropic vector, we always assume that *this m is minimal possible!*

For partitioned graphs, Theorem 3.8 is recast as follows (where, by definition, “spanned by lines” includes lines, h , and the distinguished isotropic class).

Theorem 3.16. *A partitioned graph Γ is 1-geometric in degree $2d \geq 4$ if and only if $\Gamma \cong \text{Fn}(X, h)$ for a hyperelliptic degree $2d$ quasi-polarized K3-surface (X, h) such that $\text{NS}(X)$ is spanned by lines. \triangleleft*

Theorem 3.17. *A partitioned graph Γ is 2-geometric in degree 8 if and only if $\Gamma \cong \text{Fn}(X, h)$ for a special octic (X, h) such that $\text{NS}(X)$ is spanned by lines. \triangleleft*

The more precise and general counterparts of Theorems 3.16 and 3.17 in terms of bi-colored partitioned graphs (cf. Theorem 3.9 vs. 3.8) are left to the reader.

4. THE TAXONOMY OF HYPERBOLIC GRAPHS

In this section we introduce necessary tools to study the configurations of lines on certain K3-surfaces. Our approach is a generalization of [9, § 4].

Given a graph Γ , we can consider the lattice $\mathbb{Z}\Gamma$ and distinguish three cases:

- (1) elliptic — if both $\sigma_+(\mathbb{Z}\Gamma)$ and $\sigma_0(\mathbb{Z}\Gamma)$ vanish,
- (2) parabolic — if $\sigma_+(\mathbb{Z}\Gamma) = 0$ and $\sigma_0(\mathbb{Z}\Gamma) > 0$,
- (3) hyperbolic — if $\sigma_+(\mathbb{Z}\Gamma) = 1$.

In view of the Hodge Index Theorem, we never consider graphs with $\sigma_+(\mathbb{Z}\Gamma) > 1$.

4.1. Parabolic graphs. A connected elliptic (resp. parabolic) graph is called a (simply laced) *Dynkin diagram* (resp. *affine Dynkin diagram*). A detailed account of the properties of such diagrams (in particular, their classification) can be found in [3]. When Γ is parabolic or elliptic, we use $\mu(\Gamma) := \text{rk}(\mathbb{Z}\Gamma/\ker)$ to denote its *Milnor number*. Given a connected parabolic graph Σ , one can use the primitive positive annihilator of the lattice $\mathbb{Z}\Sigma$ to define *the fundamental cycle*

$$(4.1) \quad \mathfrak{l}_\Sigma := \sum_{v \in \Sigma} n_v v,$$

where all n_v are positive (see [1, Lemma I.2.12], [9, § 3.1]). Since each parabolic/elliptic graph Γ is a disjoint union of simply laced Dynkin diagrams and affine Dynkin diagrams, with at least one affine component in the parabolic case, such a graph can be described by a formal sum of the **A-D-E** types of its components.

Following [9, §4], we introduce an order on the set of isomorphism classes of connected parabolic graphs (*i.e.*, affine Dynkin diagrams): for graphs Σ' , Σ'' with $\mu(\Sigma') = \mu(\Sigma'')$ we let

$$\tilde{\mathbf{A}}_\mu < \tilde{\mathbf{D}}_\mu < \tilde{\mathbf{E}}_\mu;$$

otherwise,

$$\Sigma' < \Sigma'' \quad \text{if and only if} \quad \mu(\Sigma') < \mu(\Sigma'').$$

Finally, since $\sigma_0(\Sigma) = 1$ for each affine Dynkin diagram Σ , we have the equality

$$(4.2) \quad |\Pi| = \mu(\Pi) + \#\{\text{parabolic components of } \Pi\}.$$

that holds for parabolic graphs (*cf.* [9, p. 614]).

4.2. The type of a hyperbolic graph. Now, we can discuss the hyperbolic case, which is of our primary interest. It is well-known that any hyperbolic graph Γ contains a connected parabolic subgraph, so we can make the following definition.

Definition 4.3. Let Γ be a hyperbolic or parabolic graph. A *minimal fiber* $\Sigma \subset \Gamma$ is a connected parabolic subgraph of Γ that is minimal with respect to the order “ $<$ ” introduced in §4.1. All minimal fibers are of the same type, and this common type is called the *type* of Γ . Alternatively, Γ is called a Σ -*graph*.

This taxonomy is applied to plain graphs only (mostly, plain Fano graphs).

Recall that we are only interested in graphs Γ such that $\sigma_+ \mathcal{F}_{2d}(\Gamma) = 1$ for some fixed $d > 0$. Under this assumption, if a graph has a hyperbolic component, all its other components are elliptic. Hence, given a hyperbolic graph Γ with a connected parabolic subgraph Σ , we can consider the maximal parabolic subgraph of Γ that contains Σ :

$$\Pi := \Sigma \cup \{v \in \Gamma \mid v \cdot l = 0 \text{ for all } l \in \Sigma\}.$$

This subgraph is called the *pencil* containing Σ (*cf.* §4.3 below). We define the sets

$$\text{secl} := \{v \in \Gamma \setminus \Sigma \mid v \cdot l = 1\}, \quad l \in \Sigma, \quad \text{and} \quad \text{sec } \Sigma := \bigcup_{l \in \Sigma} \text{secl}$$

of *sections* of Π . All parabolic components of Π are called its *fibers*. We define the *multiplicity* of a section $w \in \text{sec } \Sigma$ as $\sum n_v$, see (4.1), where the summation runs over all vertices $v \in \Sigma$ that intersect w . A section of multiplicity 1 is called *simple*. It is obvious geometrically and easily follows from the assumption $\sigma_+ \mathcal{F}_{2d}(\Gamma) = 1$ that the multiplicity of a section is independent of the choice of a parabolic component $\Sigma \subset \Pi$; thus, a section is adjacent to at least one vertex in each such component.

It may appear that the assumption that a graph is of a given type is relatively weak. Below we discuss a simple example to show its importance/usefulness.

Example 4.4. Let Γ be a $\tilde{\mathbf{D}}_4$ -graph and let a_j , $j = 1, \dots, 5$, be the components of the $\tilde{\mathbf{D}}_4$ -fiber Σ . Since Γ contains no triangles (*i.e.*, subgraphs $\tilde{\mathbf{A}}_2$), distinct sections that meet the same component a_j must be disjoint. Similarly, from the absence of quadrangles ($\tilde{\mathbf{A}}_3$) or pentagons (by definition, $\tilde{\mathbf{A}}_4 < \tilde{\mathbf{D}}_4$, see §4.1), we conclude that all sections in $\text{sec } \Sigma$ are pairwise disjoint. Thus, the incidence matrix of the union $\Sigma \cup \text{sec } \Sigma$ is determined by the five integers $|\text{sec } a_j|$, $j = 1, \dots, 5$.

4.3. Elliptic pencils on $K3$ -surfaces. Given a quasi-polarized $K3$ -surface (X, h) such that the Fano graph $\text{Fn}(X, h)$ is either elliptic or hyperbolic, it is easy to see (see, *e.g.*, [9]) geometrically that $|\text{Fn}(X, h)| \leq 24$. Thus, we have

(4.5) the graph $\text{Fn}(X, h)$ of a $K3$ -surface with at least 25 lines is hyperbolic.

In particular, the existence of many lines on a $K3$ -surface implies the existence of a genus-one fibration (given by the linear system $|\mathbb{I}_\Sigma|$, see (4.1), where $\Sigma \subset \text{Fn}(X, h)$ is a connected parabolic subgraph) with at least one reducible fiber consisting entirely of lines. On the other hand, if $X \rightarrow \mathbb{P}^1$ is a genus-one fibration on a $K3$ -surface X , the dual adjacency graph $\tilde{\Pi}$ of the components of its reducible fibers is a union of affine Dynkin diagrams, $\tilde{\Pi} = \Sigma_0 \cup \dots \cup \Sigma_k$, and we have

$$(4.6) \quad \mu(\tilde{\Pi}) \leq 18, \quad |\tilde{\Pi}| \leq 24, \quad \alpha(\tilde{\Pi}) \leq 16,$$

where the first (resp. last) inequality follows from [36] (resp. [24]). Obviously, the projective degree of a curve is a function $\text{deg}: \tilde{\Pi} \rightarrow \mathbb{N}$, $v \mapsto v \cdot h$, and the *linear* fiber components constitute the subgraph

$$\Pi := \{v \in \tilde{\Pi} \mid \text{deg } v = 1\},$$

which we call a *combinatorial pencil*.

Assuming that $\Pi \supset \Sigma_0 =: \Sigma$ is a Σ -graph, the degree function has the following properties:

$$(4.7) \quad \text{deg}(\Sigma_k) = \text{deg}(\Sigma), \quad \text{deg}|_\Sigma \equiv 1, \quad \text{deg}|_{\Sigma_k} \not\equiv 1 \text{ unless } \Sigma_k \geq \Sigma,$$

where $0 \leq s \leq k$ and $\text{deg}(\Sigma_k) := \sum_{a \in \Sigma_k} n_a \text{deg } a$ is the total degree of a fiber (see (4.1) for the definition of the multiplicities n_a).

All configurations of singular fibers of elliptic fibrations on complex $K3$ -surfaces were enumerated in [35]. Given a genus-one fibration on an algebraic $K3$ -surface, its Jacobian fibration has singular fibers of the same type and it also is a $K3$ -surface (see [18, § 11.4]). (In fact, a combinatorial pencil in the graph of lines of a quasi-polarized $K3$ -surface with sufficiently many lines does have a simple section; the particular case of octics is discussed in Lemma 5.3 below.) Thus, all combinatorial pencils that we are to consider can be derived from the list in [35], with the extra constraints given by (4.7) taken into account.

4.4. The approach. For the reader's convenience, before passing to the technical details, we sketch the reasoning that will lead us to the proof of Theorem 1.1. As explained in §1, similar strategy should yield sharp upper bounds on the number of lines on quasi-polarized degree- $2d$ $K3$ -surfaces for $d > 4$.

In view of (4.5), we focus our attention on hyperbolic graphs. For Stage 1 (see §3.3.1), we fix a *type* Σ and consider Σ -graphs in the form

$$\Gamma \supset \Pi \supset \Sigma,$$

where $\Sigma \subset \Gamma$ is a distinguished *fiber* (any fixed subgraph of Γ within the chosen isomorphism class) and Π is the pencil containing Σ . Formally, both Π and $\text{sec } \Sigma$ depend on Γ , but we omit this fact in the notation since, in fact, we usually construct Γ *starting* from a given pencil Π or collection of sections $\text{sec } \Sigma$.

We fix a threshold M and prove that, with a few exceptions that are listed explicitly, the inequality

$$(4.8) \quad |\Gamma| < M$$

holds for any geometric Σ -graph Γ . To justify this claim, we choose another pair of thresholds M_Σ, M_Π such that $M_\Sigma + M_\Pi \leq M + 1$ and show that, with the same exceptions, the following inequalities hold

$$(4.9) \quad |\Gamma| < M \text{ whenever } |\text{sec } \Sigma| \geq M_\Sigma, \quad \text{and} \quad |\Gamma| < M \text{ whenever } |\Pi| \geq M_\Pi.$$

Since, obviously,

$$\Gamma = \Pi \cup \text{sec } \Sigma \quad (\text{disjoint union}),$$

the two inequalities imply (4.8). Roughly speaking, the choice of the type Σ imposes certain extra constraints on the Gram matrix of the sections (*cf.* Example 4.4). Thus, adding many sections to Σ rules out (almost) all configurations of singular fibers allowed by [35]. On the other hand, once we choose many fiber components, the resulting lattice has high rank and cannot accommodate many sections.

Thus, typically, we start with a *sufficiently large* reasonably “standard” graph Γ_0 and extend it by adding a number of extra vertices. Each extension $\Gamma \supset \Gamma_0$ that is not subgeometric is disregarded immediately, *cf.* Remark 3.7. Crucial is the fact that the rank of a (sub-)geometric graph does not exceed 20 — the maximal Picard rank of a $K3$ -surface. Hence, once an extension $\Gamma \supset \Gamma_0$ of this maximal rank has been obtained (usually, after adding but a few extra vertices), there remains to apply Nikulin’s theory and study the finite index extensions of the lattice $\mathcal{F}(\Gamma)$: we select those that are geometric and compute the respective saturations of Γ . It is this fact that makes our algorithms converge reasonably fast.

The last stage of the proof (see §3.3.2) boils down to the study of but a few extremal configurations. The general approach is outlined in §3.3.2, and we give the necessary explanations on a case-by-case basis.

The general computer aided algorithms are described in the appendices, and a more detailed exposition and numerous intermediate statements for the case of degree 8 $K3$ -surfaces are the subject of the rest of the paper. A human readable example of this reasoning —the case of Kummer octics— is found in §6.

5. A FEW EXTREME CASES

Starting from this section, we confine ourselves to the Fano graphs of octic $K3$ -surfaces (X, h) ; thus, we fix the degree $2d = h^2 = 8$ and consider m -admissible 8-polarized lattices/graphs, where $m \geq 2$.

5.1. The star of a pseudo-vertex. We start with discussing the local structure, ruling out a few very simple subgraphs of the Fano graphs of octics. We refer to Definition 2.33 for the definition of a pseudo-vertex and its star.

Lemma 5.1. *The star of a (pseudo-)vertex e of a 2-admissible graph Γ can be as follows:*

- (1) $\text{star}(e) \cong \mathbf{A}_2 \oplus a\mathbf{A}_1$, $a \leq 3$, or $a\mathbf{A}_1$, $a \leq 8$, if $e \in \Gamma$ is a line;
- (2) $\text{star}(e) \cong a\mathbf{A}_1$, $a \leq 7$, if $e \in \Gamma$ is a line and Γ is 3-admissible;
- (3) $\text{star}(e) \cong a\mathbf{A}_1$, $a \leq 5$, if $e \in \{\Delta\}$ is an exceptional divisor;
- (4) $\text{star}(e) \cong \mathbf{A}_2$ or $a\mathbf{A}_1$, $a \leq 9$, if $e \in \Delta^\sharp$ is a 3-isotropic vector.

Proof. We change the notation and confine ourselves to the subgraph $\Gamma := \text{star}(e)$ and sublattice $S := \mathbb{Z}\Gamma + \mathbb{Z}h + \mathbb{Z}e$, with $v \cdot e = 1$ for each $v \in \Gamma$. Assuming that Γ contains a subgraph \mathbf{A}_2 , generated by a pair u_1, u_2 , and/or $a\mathbf{A}_1$, generated by v_1, v_2, \dots , we make the following observations:

- if $e \in \Gamma$ is a line, then $u_1 + u_2 + e$ is a 3-isotropic vector;

- if $e \in \Gamma$ and $a \geq 8$, then $3e - h + v_1 + \dots + v_8$ is a 3-isotropic vector;
- if $e \in \Gamma$ and $a \geq 4$, then $2e - h + u_1 + u_2 + \dots + v_4$ is a separating root;
- if $e \in \{\Delta\}$ is an exceptional divisor, then $u_1 + u_2 - e$ is 2-isotropic;
- if $e \in \Delta^\sharp$ is 3-isotropic, then $2e - h + u_1 + u_2 + v_1$ is a e -separating root.

All other border cases, *i.e.*, $\Gamma \supset \tilde{\mathbf{A}}_2$, or $\Gamma \supset \mathbf{A}_3$, or $\Gamma \supset b\mathbf{A}_2 \oplus a\mathbf{A}_1$ with the pair (a, b) exceeding that announced in the statement, are ruled out by $\sigma_+ S \geq 2$. \square

Lemma 5.2. *Let $v_1, v_2 \in \Gamma$ be two vertices in a 2-admissible graph Γ . Then the intersection $\text{star}(v_1) \cap \text{star}(v_2)$ is discrete and:*

- (1) *if $v_1 \cdot v_2 = 1$, then $|\text{star}(v_1) \cap \text{star}(v_2)| \leq 1$;*
- (2) *if $v_1 \cdot v_2 = 0$, then $|\text{star}(v_1) \cap \text{star}(v_2)| \leq 3$.*

*If, in addition, Γ is 3-admissible, then both inequalities above are strict, *i.e.*, Γ is both triangle- and biquadrangle free.*

Proof. The first inequality is a restatement of Lemma 5.1(1) and (2). For the second one, let $u_1, \dots, u_k \in \text{star}(v_1) \cap \text{star}(v_2)$. If $k \geq 4$ or $k \geq 2$ and $u_1 \cdot u_2 = 1$, one has $\sigma_+ \Gamma > 1$. Thus, all vertices are pairwise disjoint and $k \leq 3$. In the latter case $k = 3$, the vector $h - v_1 - v_2 - u_1 - u_2 - u_3$ is 3-isotropic. \square

As an immediate consequence, we show that, in a large 2-geometric graph, any pencil has a simple section.

Lemma 5.3. *Let $\Gamma \supset \Pi \supset \Sigma$ be a 2-geometric Σ -graph, where Π is a pencil, and assume that $|\Gamma| > 24$. Then there is at least one simple section of Π in Γ .*

Proof. Clearly, $\Gamma \setminus \Pi \neq \emptyset$, and we only need to show that *multiple* sections would not suffice to achieve the goal $|\Gamma| > 24$.

If $\Sigma = \tilde{\mathbf{A}}_4$ or $\Sigma \geq \tilde{\mathbf{A}}_5$, there are no multiple sections by definition.

If $\Sigma = \tilde{\mathbf{A}}_2$, multiple sections are ruled out by Lemma 5.1(1).

If $\Sigma = \tilde{\mathbf{A}}_3$, then, by Lemma 5.2(2), there can be at most two (in fact, at most one) double sections. However, in the presence of a double section, the pencil Π may have at most 4 parabolic components, see Lemma 5.1(1); therefore, by (4.2) and (4.6), we have $|\Gamma| = |\Pi| + 2 \leq 24$.

Finally, if $\Sigma = \tilde{\mathbf{D}}_4$, by Lemma 5.1(1), there may be $m \leq 4$ double sections, intersecting the central fiber $a_1 \in \Sigma$. Let $p \geq 1$ and $0 \leq q \leq 2$ be the numbers of, respectively, type $\tilde{\mathbf{D}}_4$ and type $\tilde{\mathbf{A}}_5$ fibers in Π . We have $p + q \leq 4$ and, hence, $|\Pi| \leq 22$, see (4.2) and (4.6). Consider the following possibilities:

- if $m = 4$, then $p + q = 1$ and $|\Gamma| = |\Pi| + 4 \leq 23$, see (4.2);
- if $2 \leq m \leq 3$, then $p = 1$; hence, $p + q \leq 3$ and $|\Gamma| \leq |\Pi| + 3 \leq 24$;
- if $m \leq 1$, then $|\Gamma| \leq |\Pi| + 1 \leq 23$. \square

The following lemma holds for any polarization $h^2 \geq 4$. It is obvious for graphs of lines on $K3$ -surfaces — we state it merely for future reference. We formulate it for bi-colored graphs $\tilde{\Gamma}$, but it can be applied to plain graphs as well. In this case, we endow all vertices of a plain Γ with color 1.

Lemma 5.4. *Let $\tilde{\Gamma}$ be a bi-colored graph such that the lattice $\mathcal{F}_{2d}(\tilde{\Gamma})$ is 2-admissible and hyperbolic. Then*

- *each triangle in $\tilde{\Gamma}$ consists of lines only, and*
- *a quadrangle in $\tilde{\Gamma}$ contains at most one exceptional divisor.*

More generally, each cycle in $\tilde{\Gamma}$ contains at least three lines (at least four, if $\mathcal{F}_{2d}(\tilde{\Gamma})$ is also required to be 3-admissible).

Proof. Assume the contrary and consider $e := \sum v$, the summation running over the cycle in question. We have $e \cdot h = \#(\text{lines in the cycle})$ and $e^2 \geq 0$. Hence, either e is 1- or 2-isotropic or $\sigma_+(\mathbb{Z}h + \mathbb{Z}e) = 2$. \square

According to [Lemma 5.1\(2\)](#), the presence of a triangle (*i.e.*, an $\tilde{\mathbf{A}}_2$ subgraph) in a 2-geometric graph Γ automatically implies that $\mathcal{F}(\Gamma)$ has a 3-isotropic vector. For such graphs, we have the following lemma (recall [Convention 3.12](#)).

Lemma 5.5 (see [§C.3](#)). *Let Π be an $\tilde{\mathbf{A}}_2$ -pencil and $|\Pi| \geq M_\Pi := 21$. Then, for any 2-geometric extension $\Gamma \supset \Pi$, one has either*

- $\Gamma^{\text{ex}} \cong \Psi_{33}$ (see [Table 1](#)) and $\Pi \cong 8\tilde{\mathbf{A}}_2$, or
- $\Gamma^{\text{ex}} \cong \Psi_{30}^3$ (see [Table 1](#)) and $\Pi \cong 6\tilde{\mathbf{A}}_2 \oplus 3\mathbf{A}_1$,

or $|\Gamma| < 30$. In the case $\Gamma^{\text{ex}} \cong \Psi_{30}^3$, the three \mathbf{A}_1 -components and three exceptional divisors constitute a single $\tilde{\mathbf{A}}_5$ type fiber of the elliptic pencil.

5.2. Special octics. In this section we study line configurations on special octics (*cf.* [Definition 2.28](#)). We fix a degree 8 quasi-polarized $K3$ -surface (X, h) and a 3-isotropic vector $p \in \text{Nef}(X)$. Moreover, we assume that $\text{NS}(X) \ni h$ is 2-admissible.

The linear system $|p|$ endows X with a genus-one fibration $f_p: X \rightarrow \mathbb{P}^1$. As in [§3.4](#), we obtain a partition of the Fano graph

$$\text{Fn}(X, h) = C_0 \cup C_1,$$

where C_1 (resp. C_0) are sections (resp. fiber components) of the fibration f_p .

Lemma 5.6. *If C_0 contains no triangle, then $|C_0| \leq 18$.*

Proof. Assume that $|C_0| > 18$. Since each singular fiber of f_p contains at most three lines, at least seven singular fibers contain lines. By the assumption, each such fiber contains an extra component, *viz.* the one of degree other than 1. (Recall that, by the 2-admissibility, two lines cannot form an $\tilde{\mathbf{A}}_1$ -fiber.) Thus, the singular fibers of f_p contain at least 26 rational curves. Contradiction. \square

Lemma 5.7. *If $C_0 \supset 6\tilde{\mathbf{A}}_2$, then any exceptional divisor on X is orthogonal to p .*

Proof. If e is an exceptional divisor and $e \cdot p > 0$, then e intersects each fiber of f_p . Hence, e intersects at least six lines, contradicting to [Lemma 5.1\(3\)](#). \square

Lemma 5.8. *If $C_0 \supset 7\tilde{\mathbf{A}}_2$, then X is smooth.*

Proof. Let e be an exceptional divisor in X . By [Lemma 5.7](#), e is a fiber component of f_p and, hence, e is orthogonal to the sublattice $\mathcal{F}(7\tilde{\mathbf{A}}_2)$. On the other hand, it is easily seen that, up to automorphism, $7\tilde{\mathbf{A}}_2$ has a unique geometric kernel and the corresponding transcendental lattice $T \cong \mathbf{U}(3)^2 \oplus \mathbf{A}_2(3)$ is root free. \square

5.3. Proof of [Theorem 1.2](#). We keep the notation of [§5.2](#) and follow the strategy outlined in [§3.3](#). By [Lemma 5.1\(4\)](#),

$$(5.9) \quad |C_1| \leq 9.$$

If C_0 contains no triangles, then [Lemma 5.6](#) yields $|\text{Fn}(X, h)| \leq 27$. Thus, we can assume that $|C_0| \geq 21$ and C_0 contains a triangle, in which case, by [Lemma 5.5](#),

the statement follows from [Theorem 3.17](#) *provided that* $\mathrm{NS}(X)$ *is spanned by lines*. This completes Stage 1 of the proof (*cf.* [§3.3.2](#)).

For Stage 2, we need to extend $\Gamma := \mathrm{sp}_1 \Gamma^{\mathrm{ex}}$, where $\Gamma^{\mathrm{ex}} \cong \Psi_{30}^3$ or Ψ_{33} , by an extra exceptional divisor e . In the notation therein, for $\Gamma^{\mathrm{ex}} \cong \Psi_{30}^3$ we have to take $\bar{\Gamma} = \Gamma$. By [Lemma 5.7](#), the new divisor e would be a fiber component of f_p ; hence, by [Lemma 5.5](#), it would coincide with one of the three already present. (Note that we do *not* assert that the existing exceptional divisors remain such. However, the *curves* do not disappear: they may only become reducible, adding even more fiber components to the elliptic pencil which already has 24 components.)

Thus, assume that $\bar{\Gamma} \subset \Psi_{33}$ and $|\bar{\Gamma}| \geq 30$. Obviously, the new octic X contains at least five $\tilde{\mathbf{A}}_2$ connected components of Π ; hence, the new part C_0 must be one of $8\tilde{\mathbf{A}}_2$, $7\tilde{\mathbf{A}}_2 \oplus \mathbf{A}_1$, or $7\tilde{\mathbf{A}}_2$ (see [Lemma 5.5](#) and observe that, because of the degree, a *single* line cannot be removed from an $\tilde{\mathbf{A}}_2$ component.) In each case, by [Lemma 5.8](#), the octic would have to remain smooth, contradicting the fact that an exceptional divisor has been added. \square

6. KUMMER OCTICS

In this section, we treat separately the Kummer and so-called *almost Kummer* (see [§6.3](#) below) octics. We have two reasons to single out these two classes. On the one hand, the corresponding lattices have very large geometric kernels and our standard naïve algorithms fail due to the lack of simplification (*cf.* [Warning 3.5](#)). On the other hand, these classes serve as an example where our approach can be explained in detail in a human readable form.

6.1. The Golay code (see [\[7\]](#)). Consider the extended binary Golay code \mathcal{C}_{24} , pick a codeword \mathfrak{o} of length 16, and denote by \mathcal{C} the code $\{o \in \mathcal{C}_{24} \mid o \subset \mathfrak{o}\}$: it consists of \emptyset , \mathfrak{o} , and 30 octads. For a subset $s \subset \mathfrak{o}$, we let $[s] := \sum_{l \in s} l \in \mathbb{Z}\mathfrak{o}$.

We distinguish certain subsets of \mathfrak{o} . Namely, define

$$\mathcal{S} := \{o \cap \mathfrak{o} \mid o \in \mathcal{C}_{24}\} \setminus \mathcal{C} = \mathcal{S}_4 \cup \mathcal{S}_6 \cup \mathcal{S}_8 \cup \mathcal{S}_{10} \cup \mathcal{S}_{12},$$

where the last splitting is according to the length of a subset $s \in \mathcal{S}$. The involution $s \mapsto \bar{s} := \mathfrak{o} \setminus s$ sends \mathcal{S}_n to \mathcal{S}_{16-n} . Define also a $\bar{}$ -invariant equivalence relation

$$r \sim s \text{ if and only if } r \Delta s \in \mathcal{C},$$

where Δ is the symmetric difference. Then,

- $\mathcal{S}_4 \cup \mathcal{S}_8 \cup \mathcal{S}_{12}$ splits into 35 equivalence classes of size $4 + 24 + 4$;
- $\mathcal{S}_6 \cup \mathcal{S}_{10}$ splits into 28 equivalence classes of size $16 + 16$.

The equivalence class of a set $o \in \mathcal{S}$ is denoted by $\llbracket o \rrbracket$, and we let $\llbracket o \rrbracket_n := \llbracket o \rrbracket \cap \mathcal{S}_n$. For a fixed element $o \in \mathcal{S}$ and residue $m \in \mathbb{Z}/2$, define

$$\mathcal{S}_n(o, m) := \{s \in \mathcal{S}_n \mid |s \cap o| = m \pmod{2}\};$$

this set depends on the class $\llbracket o \rrbracket$ only.

As an alternative description, for a subset $s \subset \mathfrak{o}$ we have

$$(6.1) \quad |s \cap o| = 0 \pmod{2} \text{ for all } o \in \mathcal{C} \text{ iff } s \in \mathcal{S} \cup \mathcal{C}.$$

The group $\mathrm{Aut} \mathcal{C}$ of automorphisms of \mathcal{C} is the stabilizer of \mathfrak{o} in M_{24} : it has order 322560, preserves $\bar{}$ and \sim , and acts transitively on each \mathcal{S}_n .

According to Nikulin [\[24\]](#), sixteen is the maximal number of pairwise disjoint smooth rational curves in a $K3$ -surface, not necessarily polarized. Identifying the

16 rational curves with the elements of \mathfrak{o} , up to isomorphism, the only finite index extension of $\mathbb{Z}\mathfrak{o}$ admitting a primitive embedding to \mathbf{L} has kernel

$$(6.2) \quad \mathcal{K}_{\mathfrak{o}} := \left\{ \frac{1}{2}[o] \bmod \mathbb{Z}\mathfrak{o} \mid o \in \mathcal{C} \right\};$$

indeed, this is the only subspace $V \subset \text{discr } \mathbb{Z}\mathfrak{o} \cong \mathbb{F}_2^{16}$ of dimension $\dim V \geq 5$ and minimal Hamming distance ≥ 8 . This statement is easily taken down to 13 lines. For each $n = 1, 2, 3$, there is a unique, up to $\text{Aut } \mathcal{C}$, subset $\star \subset \mathfrak{o}$, $|\star| = n$; fixing such a subset and identifying the $(16 - n)$ lines with the elements of $\mathfrak{o}^{\star} := \mathfrak{o} \setminus \star$, we conclude that the only geometric finite index extension of $\mathbb{Z}\mathfrak{o}^{\star}$ has kernel

$$(6.3) \quad \mathcal{K}_{\mathfrak{o}^{\star}} := \left\{ \frac{1}{2}[o] \bmod \mathbb{Z}\mathfrak{o}^{\star} \mid o \in \mathcal{C}^{\star} \right\}, \quad \mathcal{C}^{\star} := \{o \in \mathcal{C} \mid o \subset \mathfrak{o}^{\star}\}.$$

Note, though, that if $|\star| > 1$, the automorphism group $\text{Aut } \mathcal{C}^{\star}$ is larger than the mere restriction of the stabilizer of \mathfrak{o}^{\star} in M_{24} .

6.2. Kummer octics. In this paper, by a *Kummer octic* we mean a degree-8 birationally quasi-polarized $K3$ -surface (X, h) with a distinguished collection of 16 lines pairwise disjoint in X . We identify the 16 lines with the elements of \mathfrak{o} , regarding the latter as a graph without edges.

To our knowledge, the following statement is new.

Theorem 6.4. *There are two disjoint deformation families \mathfrak{K}_d , $d = 64$ or 256 , of Kummer octics; they are distinguished by the determinant*

$$d := |\det \text{NS}(X)|, \quad \text{where } (X, h) \in \mathfrak{K}_d \text{ is a generic member.}$$

(Alternatively, for any $(X, h) \in \mathfrak{K}_d$, the parameter d is recovered as the determinant of $\text{NS}(X) \cap (\mathbb{Q}\mathfrak{o} + \mathbb{Q}h)$.) Both families have dimension 3 and consist of triquadrics.

Proof. The lattice $\mathcal{F}(\mathfrak{o}, \mathcal{K}_{\mathfrak{o}})$, see (6.2), is not geometric. However, starting from $\mathcal{K}_{\mathfrak{o}}$, it is easy to list all geometric kernels $\mathcal{K} \supset \mathcal{K}_{\mathfrak{o}}$ of \mathfrak{o} . There are two:

$$(6.5) \quad \mathcal{K}_{64}, \quad \text{generated over } \mathcal{K}_{\mathfrak{o}} \text{ by } \frac{1}{4}h + \frac{1}{8}[\mathfrak{o}] + \frac{1}{2}[o], \quad o \in \mathcal{S}_6, \text{ or}$$

$$(6.6) \quad \mathcal{K}_{256}, \quad \text{generated over } \mathcal{K}_{\mathfrak{o}} \text{ by } \frac{1}{2}h + \frac{1}{4}[\mathfrak{o}] + \frac{1}{2}[o], \quad o \in \mathcal{S}_8$$

(see §6.1 for the notation), which give rise to the two families in the statement.

To show that all Kummer octics are triquadrics, we try to add to \mathfrak{o} a 3-isotropic vector p . Lemma 5.1(4) asserts that $\|p\|_{\mathfrak{o}} \leq 9$; however, in the presence of all 16 lines, only $\|p\|_{\mathfrak{o}} \in \{0, 1, 2\}$ passes the Sylvester test (see Lemma A.4). By (6.1), none of these options passes the kernel test (see Lemma A.3). \square

Proposition 6.7. *If $(X, h) \in \mathfrak{K}_{64}$, then either*

- X_8 is smooth and $\text{Fn}(X, h) \cong \Theta_{32}^{\mathbb{K}}$, see [9] and Table 1, or
- $\text{Sing}(X_8) = 8\mathbf{A}_1$ and $\text{Fn}(X, h) = \mathfrak{o}$.

The $K3$ -octics in \mathfrak{K}_{64} with $\text{Fn}(X, h) = \mathfrak{o}$ (i.e., such that $\text{Sing } X_8 \neq \emptyset$) constitute a family of dimension 2.

Proof. Fix an element $o \in \mathcal{S}_6$ and consider the corresponding kernel \mathcal{K}_{64} , see (6.5). The stabilizer $G_{64} := \text{stab } \mathcal{K}_{64}$ is an order 11520 subgroup of $\text{Aut } \mathcal{C}$.

Using Lemmas 5.1(1) and A.3, if an extra line v can be added to \mathfrak{o} , then

$$\|v\|_{\mathfrak{o}} \in \mathcal{S}_6(o, 0) = \llbracket o \rrbracket_6 \cup \{\text{another } G_{64}\text{-orbit}\}.$$

The former orbit is already present in $\text{sat}(\mathfrak{o}, \mathcal{K}_{64})$, resulting in $\Theta_{32}^{\mathbb{K}}$. If $\|v\|_{\mathfrak{o}} \notin \llbracket o \rrbracket_6$, the pair $(\mathfrak{o} \cup v, \mathcal{K}_{64})$ is not extensible.

Similarly, if an exceptional divisor e can be added to \mathfrak{o} , then $\|e\|_{\mathfrak{o}} \in \mathcal{S}_4(o, 1)$ (a single G_{64} -orbit). On this set, we have a coarser equivalence relation

$$r \approx s \quad \text{whenever} \quad r \sim s \text{ or } r \sim s \Delta o,$$

and the lattice $\mathcal{F}(\mathfrak{o}, \mathcal{K}_{64}) + \mathbb{Z}e$ contains the whole octuple $\llbracket e \rrbracket_{\approx}$. On the other hand, if $\|e_1\|_{\mathfrak{o}} \not\approx \|e_2\|_{\mathfrak{o}}$, the bi-colored graph $\mathfrak{o} \cup e_1 \cup e_2$ violates the Sylvester test (Lemma A.4), no matter whether $e_1 \cdot e_2 = 0$ or 1 (where the latter option needs to be considered only if $\|e_1\|_{\mathfrak{o}} \cap \|e_2\|_{\mathfrak{o}} = \emptyset$, see Lemma 5.4). \square

Example 6.8. The surfaces in the family \mathfrak{K}_{64} (see Proposition 6.7) demonstrate the phenomenon discussed in Warning 3.5: extending the geometric lattice $\mathcal{F}_8(\mathfrak{o}, \mathcal{K}_{64})$ by an exceptional divisor results in $K3$ -surfaces with fewer ($32 \mapsto 16$) lines!

Note that, *accidentally*, \mathfrak{o} still is geometric, but with a different kernel, as a member of the other family \mathfrak{K}_{256} , see Proposition 6.9 below.

Proposition 6.9. *If $(X, h) \in \mathfrak{K}_{256}$, then either*

- X_8 is smooth and $|\text{Fn}(X, h)| = 16, 20, 24$, or 28 (with $\text{Fn}(X, h) \cong \Delta'_{28}$), or
- $\text{Sing}(X_8) = 4\mathbf{A}_1$ and $|\text{Fn}(X, h)| = 16, 24$, or 32 (with $\text{Fn}^{\text{ex}}(X, h) \cong \Theta_{32}^4$).

The configurations Δ'_{28} (see [9]) and Θ_{32}^4 are described in Tables 1 and 2.

Proof. Fix an element $o \in \mathcal{S}_8$ and consider the corresponding kernel \mathcal{K}_{256} , see (6.6). The stabilizer $G_{256} := \text{stab } \mathcal{K}_{256}$ is an order 9216 subgroup of $\text{Aut } \mathcal{C}$. By Lemmas 5.1(1) and A.3, if an extra line v can be added to \mathfrak{o} , then

$$\|v\|_{\mathfrak{o}} \in \mathcal{S}_4(o, 1) \cup \mathcal{S}_6(o, 0)$$

(two G_{256} -orbits). In the former case, the saturation $\text{sat}(\mathfrak{o} \cup v, \mathcal{K}_{256})$ contains the whole quadruple $\llbracket v \rrbracket_4$; in the latter case, the pair $(\mathfrak{o} \cup v, \mathcal{K}_{256})$ is not extensible. Adding (in the progressive mode, see §A.4.4 below) up to three independent lines, we obtain all but one configurations in the statement (those spanned by lines).

Similarly, if an exceptional divisor e can be added to $(\mathfrak{o}, \mathcal{K}_{256})$, then

$$\|e\|_{\mathfrak{o}} \in \mathcal{S}_4(o, 0) = \llbracket o \rrbracket_4 \cup \{\text{another } G_{256}\text{-orbit}\}.$$

In the former case, the lattice $\mathcal{F}_e := \mathcal{F}(\mathfrak{o}, \mathcal{K}_{256}) + \mathbb{Z}e$ has no geometric extensions; in the latter case, the quadruple $\llbracket e \rrbracket_4$ gives rise to four nodes in \mathcal{F}_e . An attempt to add an extra exceptional divisor to any of the configurations obtained above either fails or produces a configuration that is already on the list. \square

6.3. Almost Kummer octics. We define an *almost Kummer octic* (X, h) (resp. $X_8 \subset \mathbb{P}^5$) as a degree-8 birationally quasi-polarized $K3$ -surface (resp. the image $X_8 := f_h(X)$) containing at least 15 pairwise disjoint lines.

Proposition 6.10. *There is a unique deformation family \mathfrak{K}^* of almost Kummer octics; they are all triquadrics. For any $(X, h) \in \mathfrak{K}^*$, one has $|\text{Fn}(X, h)| \leq 29$ unless (X, h) is Kummer (see §6.2) or $\text{Fn}(X, h) \cong \Theta_{33}$, see [9] and Table 1.*

Proof. We fix a one-element set $\star \in \mathfrak{o}$ and start with the pair $(\mathfrak{o}^*, \mathcal{K}_{\mathfrak{o}^*})$, see (6.3). It is immediate that the lattice $\mathcal{F}(\mathfrak{o}^*, \mathcal{K}_{\mathfrak{o}^*})$ is geometric and it has no further geometric finite index extension compatible with \mathfrak{o}^* ; thus, there is a single deformation family.

Similar to §6.1, introduce the sets

$$\mathcal{S}^* := \{o \cap \mathfrak{o}^* \mid o \in \mathcal{C}_{24}\} \setminus \mathcal{C}^* = \mathcal{S}_3^* \cup \mathcal{S}_4^* \cup \dots \cup \mathcal{S}_{11}^* \cup \mathcal{S}_{12}^* \cup \mathcal{S}_{15}^*.$$

For any 3-isotropic vector p that can be added to \mathfrak{o}^* , the Sylvester test (Lemma A.4) implies that $\|\|p\|_{\mathfrak{o}^*}\| \leq 2$, and only $\|p\|_{\mathfrak{o}^*} = \emptyset$ passes the kernel test (Lemma A.3);

however, the partitioned pair $(\sigma^* \cup \emptyset, \mathcal{K}^*)$ is not subgeometric. Thus, any almost Kummer octic is a triquadric. Similarly, by Lemmas 5.1(2) and A.3, if an extra line v can be added to σ^* , then

$$\|v\|_{\sigma^*} \in \mathcal{S}_3^* \cup \dots \cup \mathcal{S}_7^* \cup \mathcal{C},$$

and only $\mathcal{S}_3^* \cup \dots \cup \mathcal{S}_6^*$ pass the Sylvester test. Now, adding to σ^* (in the progressive mode, see §A.4.4 below) up to four independent lines, after saturating and sorting the results we arrive at 107 configurations spanned by lines. Seven of them are Kummer (see §6.2), one is Θ_{33} , and the others have at most 29 lines. There remains to observe that $\text{rk } \Theta_{33} = 20$ and Stage 2 of the proof (see §3.3.2) is void. \square

6.4. Digression: Kummer quartics. Certainly, the arguments used above apply to other degrees h^2 as well. Without going into detail (to be published elsewhere), we merely announce a few interesting findings. First of all,

- if $2d = 2 \pmod{4}$, then $\alpha(X) \leq 12$ for any birationally quasi-polarized $K3$ -surface (X, h) of degree $h^2 = 2d$;

thus, (almost) Kummer $K3$ -surfaces may exist only in degrees $0 \pmod{4}$. Henceforth, we confine ourself to the most interesting case of spatial quartics.

Theorem 6.11. *There are but eight equilinear families of Kummer quartics (X, h) with $\text{NS}(X)$ spanned by lines. Among them, there is one with 48 lines and 4 nodes.*

Concerning the last statement, recall that the maximal number of lines on a quartic $X_4 \subset \mathbb{P}^3$ with $\text{Sing } X_4 \neq \emptyset$ is still an open problem. The known upper bound is 64 (see [38]). There is a single example of a quartic with 52 lines and two nodes (see [10]), whereas all other known examples have at most 40 lines.

Another open problem is the maximal number of lines in a triangle free configuration of lines on a smooth quartic: the best bound is 52 (see [12]), and the best example is 37 (see [9]). We have discovered a larger example.

Proposition 6.12. *There exists a smooth almost Kummer quartic $X_4 \subset \mathbb{P}^3$ with a quadrangular configuration of 39 lines.*

7. OTHER TRIQUADRICS

In this section, we consider 8-polarized 3-admissible graphs, *i.e.*, we assume that $h^2 = 8$ and $m = 3$. In other words, we deal with triquadrics. Here, we state a number of technical lemmas and use them to derive Theorem 3.17 (in the next section); the GAP [16] aided proofs of these lemmas are explained in the appendices. We maintain the notation introduced in §3.3 (*cf.* Convention 3.12).

7.1. Locally elliptic configurations. From (4.2), (4.6), and obvious combinatorial bounds on the number of sections in a locally elliptic graph (see [9, Figure 1]), we conclude that, as in the smooth case,

$$(7.1) \quad |\Gamma| \leq (18 + 3) + 8 = 29 \quad \text{for any geometric locally elliptic graph } \Gamma,$$

i.e., any Σ -graph with $\mu(\Sigma) \geq 5$. This bound holds for any degree $h^2 \geq 4$. In fact, a simple computation with $\tilde{\mathbf{A}}_5$ - and $\tilde{\mathbf{D}}_5$ -graphs confirms that, in degree 8,

$$(7.2) \quad |\Gamma| \leq 23 \quad \text{except } \Gamma^{\text{ex}} \cong \Lambda_{25}, \Lambda_{24}^{\mathbf{A}}, \Lambda_{24}, \Lambda'_{24}, \Lambda''_{24}, \text{ or } \Lambda_{24}^6.$$

The first five exceptional configurations (the smooth ones) are introduced in [9], and $\mathcal{F}_8(\Lambda_{24}^6) \supset \mathcal{F}_8(\Lambda_{24}^{\mathbf{A}})$ is an index 4 extension with the same set of lines.

7.2. Astral configurations. A graph of type $\tilde{\mathbf{D}}_4$ is called *astral*. In other words, a graph Γ is astral if and only if $\text{girth } \Gamma \geq 6$ and Γ has a vertex v of $\text{val } v \geq 4$.

If Π is a parabolic $\tilde{\mathbf{D}}_4$ -graph, then, by (4.2) and (4.6), we have $|\Pi| \leq 22$.

Lemma 7.3 (see §B.3). *Let $\Gamma \supset \Sigma \cong \tilde{\mathbf{D}}_4$ be an astral geometric configuration, and let the central vertex of Σ be one of the maximal valency in Γ . Then, one has*

$$|\text{sec } \Sigma| \leq 12 \quad \text{and} \quad |\Gamma| < 28 \quad \text{whenever} \quad |\text{sec } \Sigma| \geq M_\Sigma := 11.$$

Lemma 7.4 (see §C.3). *Let Π be a $\tilde{\mathbf{D}}_4$ -pencil, $|\Pi| \geq M_\Pi := 18$, and $\alpha(\Pi) \leq 14$. Then*

$$|\Gamma| < 28$$

for any geometric astral extension $\Gamma \supset \Pi$.

Thus, we have (4.9) and, hence, (4.8), i.e., $|\Gamma| < M := 28$ for any astral graph Γ containing a $\tilde{\mathbf{D}}_4$ -pencil Π with $\alpha(\Pi) \leq 14$. However, there also is an astral Kummer configuration Δ'_{28} (see Proposition 6.9), containing three pencils of type $4\tilde{\mathbf{D}}_4$, and, summarizing, we obtain a sharp bound

$$(7.5) \quad |\Gamma| \leq 28 \quad \text{for any astral 3-geometric 8-polarized graph } \Gamma.$$

7.3. Pentagonal configurations. A graph of type $\tilde{\mathbf{A}}_4$ is called *pentagonal*. In other words, a graph Γ is pentagonal if and only if $\text{girth } \Gamma = 5$.

Lemma 7.6 (see §B.4). *Let $\Gamma \supset \Sigma \cong \tilde{\mathbf{A}}_4$ be a pentagonal geometric configuration. Then, one has*

$$|\text{sec } \Sigma| \leq 17 \quad \text{and} \quad |\Gamma| < 30 \quad \text{whenever} \quad |\text{sec } \Sigma| \geq M_\Sigma := 14.$$

Lemma 7.7 (see §C.3). *Let Π be an $\tilde{\mathbf{A}}_4$ -pencil, $|\Pi| \geq M_\Pi := 17$, and $\alpha(\Pi) \leq 14$. Then,*

$$\Gamma^{\text{ex}} \cong \Phi'_{30}, \Phi''_{30}, \text{ or } \Phi'''_{30} \quad (\text{see [9] or Table 2}) \quad \text{or} \quad |\Gamma| < 30$$

for any geometric pentagonal extension $\Gamma \supset \Pi$.

The lattice $\mathcal{F}(\Phi'''_{30})$ is an index 3 extension of $\mathcal{F}(\Phi''_{30})$; the two lattices have the same set of lines and differ by the exceptional divisors only.

Since Kummer or almost Kummer configurations (see §6) are never pentagonal, we conclude that

$$(7.8) \quad |\Gamma| \leq 30 \quad \text{for any pentagonal 3-geometric 8-polarized graph } \Gamma.$$

7.4. Quadrangular configurations. A graph of type $\tilde{\mathbf{A}}_3$ is called *quadrangular*. Thus, a graph Γ is quadrangular if and only if $\text{girth } \Gamma = 4$.

Lemma 7.9 (see §B.5). *Let $\Gamma \supset \Sigma \cong \tilde{\mathbf{A}}_3$ be a quadrangular geometric configuration. Then, one has*

$$|\text{sec } \Sigma| \leq 20 \quad \text{and} \quad |\Gamma| < 31 \quad \text{whenever} \quad |\text{sec } \Sigma| \geq M_\Sigma := 16$$

unless Γ^{ex} is one of the 13 bi-colored graphs listed in Table 1 as the entries with a reference to Lemma 7.9.

Lemma 7.10 (see §C.3). *Let Π be an $\tilde{\mathbf{A}}_3$ -pencil, $|\Pi| \geq M_\Pi := 17$ and $\alpha(\Pi) \leq 14$, and $\Gamma \supset \Pi$ a geometric quadrangular extension. Then*

$$|\Gamma| < 32$$

unless Γ^{ex} is one of the 8 bi-colored graphs listed in Table 1 as the entries with a reference to Lemma 7.10.

Observing that the Kummer configurations Θ_{32}^k (see Proposition 6.7) and Θ_{32}^4 (see Proposition 6.9) and almost Kummer configuration Θ_{33} (see Proposition 6.10) appear among the exceptions in Lemmas 7.9 and 7.10, we conclude that, with the 15 exceptions listed as Θ_* in Table 1, one has

$$(7.11) \quad |\Gamma| \leq 31 \text{ for any quadrangular 3-geometric 8-polarized graph } \Gamma.$$

8. PROOF OF THEOREM 1.1

For special octics the statement of theorem is given by Theorem 1.2 (see §5.3): we obtain a single configuration (*viz.* Ψ_{33}) with 33 lines on a smooth $K3$ -surface. Thus, we can assume that (X, h) is a triquadric (*i.e.*, the 8-polarized lattice $\text{NS}(X) \ni h$ is 3-admissible, see Theorem 3.9), and then Theorem 3.9 reduces the proof to the classification of the 8-polarized bi-colored graphs Γ' such that

$$\Gamma' \text{ is 3-geometric and } |\text{sp}_1 \Gamma'| \geq 32.$$

We follow the general strategy described in §3.3.

8.1. Stage 1: the case of $\text{NS}(X)$ spanned by lines. In this part of the proof we classify the extended saturations $\text{sat}^{\text{ex}} \Gamma$ assuming that Γ is 3-geometric and $|\Gamma| \geq 32$; by Theorem 3.8, this corresponds to the triquadrics (X, h) such that the lattice $\text{NS}(X)$ is spanned by lines. By (4.5) and Lemma 5.2, Γ is hyperbolic, *i.e.*, it is a Σ -graph for a certain affine Dynkin diagram $\Sigma > \tilde{\mathbf{A}}_2$ (see §4.2). If Γ is quadrangular (*i.e.*, $\Sigma \cong \tilde{\mathbf{A}}_3$), then, by (7.11), $\text{sat}^{\text{ex}} \Gamma$ is one of the fifteen Θ_* -graphs listed in Table 1. The other types are ruled out by

- (7.8), if Γ is pentagonal (*i.e.*, $\Sigma \cong \tilde{\mathbf{A}}_4$),
- (7.5), if Γ is astral (*i.e.*, $\Sigma \cong \tilde{\mathbf{D}}_4$), or
- (7.1), if Γ is locally elliptic (*i.e.*, $\Sigma > \tilde{\mathbf{D}}_4$).

8.2. Stage 2: the general case (see §3.3.2). There remains to analyze (extend by exceptional divisors) the seven graphs $\Gamma := \text{sp}_1 \Gamma^{\text{ex}}$ in Table 1 that have rank $\text{rk} \Gamma < 20$. Since the threshold is $|\bar{\Gamma}| \geq 32$, for the six configurations Θ_{32}^* it is the graph Γ itself that is to be extended, and a direct check shows that the resulting graph is always one of those listed in Table 1. (In fact, instead of Γ , we use smaller “natural” generating sets by means of which the graphs were constructed in the proof, *cf.* Remark 3.13.)

The configuration $\Gamma \cong \Theta'_{34}$ in Table 1 needs more work, as we have $|\Gamma \setminus \bar{\Gamma}| \leq 2$. We fix a certain “natural” generating set $\Lambda \subset \Gamma$ (*cf.* Remark 3.13), consisting of an $\tilde{\mathbf{A}}_3$ -fiber and 14 sections, and run the extension algorithm to check that there are no corank 1 extensions $\bar{S} \supset \mathcal{F}(\Lambda) = \mathcal{F}(\Gamma)$ satisfying conditions (1), (2) in §3.3.2. Next, we observe that the action of the group $G := \text{Aut} \Gamma$ has three orbits, $\Gamma = \bigcup_n \Omega_n$, where $n = |\Omega_n| \in \{2, 8, 24\}$, and each orbit has a representative $v \in \Gamma \setminus \Lambda$, so that the lattice $\mathcal{F}(\Gamma \setminus v)$ has no extensions other than those of $\mathcal{F}(\Lambda)$. Finally, the action of G on the set of unordered pairs of vertices has 19 orbits, 18 of which also have representatives in $\Gamma \setminus \Lambda$. The remaining orbit consists of a single pair $\{u, v\} = \Omega_2$, and the graph $\Gamma \setminus \Omega_2 \cong \Theta'_{32}$ has already been analyzed.

8.3. Graphs to octics. Finally, there remains to apply Nikulin's theory [26] and, for each of the sixteen Fano graphs Γ found, classify the isomorphism classes of primitive isometric embeddings $\mathcal{F}(\Gamma, \mathcal{K}) \hookrightarrow \mathbf{L}$, $\mathcal{K} \in \mathfrak{G}(\Gamma)$. In each case, we obtain a single connected deformation family. \square

APPENDIX A. BASIC ALGORITHMS

In this section we describe the most basic algorithms applicable to any degree $h^2 = 2d$. Essentially, they are those used in [9], with a few minor modifications adjusting them to the case of non-empty singular locus. Most notably, the “smooth” requirement $\text{root}_0(S, h) = \emptyset$ is replaced with [Lemma 2.10](#).

Note that, most of the time, we work with *plain* Fano graphs, even though computing such a graph starts with computing a Weyl chamber (*cf.* [§A.1.1](#) below). At the very end, when computing the saturation lists of the large plain graphs (see [§A.1.3](#) below), we keep the record of the exceptional divisors, thus obtaining the extended Fano graphs that appear in the final statements.

A.1. The master test. In the heart of all algorithms is a procedure detecting if a given graph Γ is geometric. More precisely, the input consists of a polarized lattice $S \ni h$ and distinguished subset $\Gamma \subset \text{root}_1(S, h)$. Typically, S is of the form $\mathcal{F}(\Gamma, \mathcal{K})$; however, occasionally we take for S an extension of this lattice by an m -isotropic vector and/or a few “potential” exceptional divisors.

A.1.1. Extensibility and admissibility. We assume S given by its Gram matrix in a certain basis $\{h, b_1, b_2, \dots\}$ containing h . Then, we consider the *rational* lattice

$$h_{\mathbb{Q}}^{\perp} := \sum \mathbb{Z} \left(b_k - \frac{b_k \cdot h}{2d} h \right),$$

multiply the form by $-2d$, and use GAP’s [16] function `ShortestVectors` to compute

$$\mathcal{V}_s := \{v \in h_{\mathbb{Q}}^{\perp}(-2d) \mid v^2 = s\} \quad \text{for all } 1 \leq s \leq 4d + 1.$$

Given an admissibility level $m \in \{1, 2, 3\}$, we check if there is

- $v \in \mathcal{V}_{m^2}$ such that $v + m(2d)^{-1}h \in S$;

if such a vector is found, the algorithm terminates as (S, h) is not m -admissible.

Next, we compute the set

- $\text{root}_0(S, h) = \mathcal{V}_{4d} \cap S$.

If there is a separating (with respect to Γ or Γ and a given isotropic vector) root $r \in \text{root}_0(S, h)$, the algorithm terminates as Γ is not extensible. Otherwise, we compute a Weyl chamber Δ' for $S' := (\mathbb{Z}h + \mathbb{Z}\Gamma)^{\perp} \subset S$ (*cf.* [Lemma 2.10](#)) using any generic functional $\alpha: S' \rightarrow \mathbb{R}$, and use [Algorithm 2.11](#) to extend Δ' to the unique Weyl chamber Δ compatible with Γ .

Remark A.1. On the few occasions where $\text{root}_0 S' \neq \emptyset$, a generic functional is found as follows. Pick any root $r \in \text{root}_0 S'$ and let $\alpha: v \mapsto -(v \cdot r)$. Whenever we can find a root $r \in \text{root}_0 S'$ such that $\alpha(r) = 0$, we change α to $v \mapsto 2\alpha(v) - (v \cdot r)$, continuing this process until α is generic. This works since, in an even lattice, we have $|r_1 \cdot r_2| \leq 1$ for any two roots $r_1 \neq \pm r_2$.

There remains to compute the set

- $\text{root}_1(S, h) = \{v + (2d)^{-1}h \mid v \in \mathcal{V}_{2d+1}\} \cap S$

and Fano graphs $\text{Fn}_{\Delta}(S, h)$ and $\text{Fn}_{\Delta}^{\text{ex}}(S, h)$ (directly as explained in [§2.2](#)).

A.1.2. *Detecting subgeometric sets.* To check if a given lattice $S \ni h$ is geometric, we compute the discriminant group $\text{discr } S$ and apply [26, Theorem 1.12.2] (see also [12, Theorem 3.2]). If the result is negative, we proceed as follows:

- list all isotropic ($\alpha^2 = 0 \pmod{2\mathbb{Z}}$) vectors $\alpha \in \text{discr } S$ of prime order;
- for each α , compute the finite index extension $S_\alpha \supset S$ and check whether it is admissible (via §A.1.1, using the same polarization h and subset Γ);
- if successful, repeat §A.1.2 (this algorithm) for S_α .

The algorithm terminates as soon as a geometric lattice has been found or all admissible finite index extensions have been tried.

Remark A.2. Primitive as it is, this algorithm serves our needs as the discriminant groups $\text{discr } S$ are usually reasonably small. We do use a couple of tricks to speed up the computation:

- if a large subgroup $G \subset O_h(S)$ preserving Γ is known (cf. §A.4 below), we use a single representative of each G -orbit of isotropic vectors;
- at the first step (for S itself), prime order isotropic vectors α resulting in non-admissible extensions S_α are recorded not to be used again.

For statistical purpose, whenever we establish that a graph Γ is geometric, we automatically record the counts $(|\Gamma|, |\text{Sing } \Gamma|)$ (but not Γ itself).

A.1.3. *The saturation lists.* The *saturation list* of a pair (Γ, \mathcal{K}) is the set

$$\text{Sat}_m(\Gamma, \mathcal{K}) := \{\text{sat}^{\text{ex}}(\Gamma, \mathcal{K}') \mid \mathcal{K}' \in \mathfrak{G}_m(\Gamma) \text{ and } \mathcal{K}' \supset \mathcal{K}\}.$$

If $\mathcal{K} = 0$, it is omitted from the notation. If a certain *global criterion* (not necessarily hereditary) $\text{NC}: \{\text{graphs}\} \rightarrow \{\text{false}, \text{true}\}$ is given, we denote

$$\text{Sat}_m(\Gamma, \mathcal{K}; \text{NC}) := \{\Gamma' \in \text{Sat}_m(\Gamma, \mathcal{K}) \mid \text{NC}(\Gamma') = \text{true}\}.$$

Typically, NC consists of a fixed type Σ in the taxonomy of graphs and a certain numeric bound, *e.g.*, $|\Gamma| \geq M$. In fact, in order to construct plenty of examples, we also retain all m -geometric graphs Γ satisfying

$$|\Gamma| \geq M_{\text{lines}} \quad \text{or} \quad |\text{Sing } \Gamma| \geq M_{\text{sing}} := 4,$$

where M_{lines} depends on the kind of the graphs considered.

The saturation lists are computed by the same algorithm as in §A.1.2 (applied to the lattice $S := \mathcal{F}(\Gamma, \mathcal{K})$ and Γ itself as the distinguished subset), except that we do not stop on the first hit, listing all geometric extensions. These lists are not intended for further processing (except sorting), and it is here that we keep track of the exceptional divisors, obtaining extended Fano graphs for the statements.

A.2. **Preliminary tests.** The algorithms in §A.1 are relatively expensive (mainly, due to the `ShortestVectors`); for this reason, they are usually preceded by a few preliminary test ruling out the vast majority of the possibilities.

As explained in §A.3 below, typically we extend a given graph Γ_0 or pair (Γ_0, \mathcal{K}) by a one or several (pseudo-)vertices v described by means of their supports $\|v\|$. The number of possibilities for a single extra vertex (essentially, a subset of Γ_0) is huge, even when restricted by a statement like Lemma 5.1. Most of them are ruled out by the following two obvious tests, for which the bulk of the computation (*i.e.* the computation of the inverse Gram matrix) depends on (Γ_0, \mathcal{K}) only and can be done once. Namely, recall that we speak of the support of a (pseudo-)vertex v

(cf. Definition 2.33) only within the range of applicability of Lemmas 2.25 and 2.29, so that $\|v\|$ determines the projection

$$\mathcal{F}(\Gamma_0) + \mathbb{Z}v \rightarrow \mathcal{F}(\Gamma_0)^\vee, \quad v \mapsto v^*,$$

and, hence, the one-vector extension $\mathcal{F}(\Gamma_0) + \mathbb{Z}v$ itself. (The projection v^* is easily computed in terms of $\|v\|$ and the inverse Gram matrix of the *original* graph Γ_0 .)

Lemma A.3 (the kernel test). *An extra (pseudo-)vertex v cannot be added to a pair (Γ_0, \mathcal{K}) if $v^* \cdot k \notin \mathbb{Z}$ for at least one element $k \in \mathcal{K}$.* \triangleleft

Lemma A.4 (the Sylvester test). *A (pseudo-)vertex v cannot be added to a graph Γ_0 if $v^2 > (v^*)^2$. If $v^2 = (v^*)^2$, the addition of v does not increase the rank.* \triangleleft

Besides, usually we fix a certain type Σ and assume that the *explicit part* (see Warning 3.5) of each new graph is a Σ -graph (or a similar condition like lack of bi-quadrangles *etc*). As a rule, we run a few quick tests directly in terms of the supports of the vertices added (cf. Remark A.6 below). Then, the Gram matrix of the new graph needs to be computed, and, before passing this matrix to §A.1, we check more thoroughly that it defines a Σ -graph.

A.3. Graph extensions. In most algorithms, we fix a certain *base graph* Γ_0 and consider its *extensions* $\Gamma \supset \Gamma_0$ by a few extra vertices. In this construction, each extra vertex $v \in \Gamma \setminus \Gamma_0$ can be represented by and, henceforth, is *identified* with its support $\text{supp}_{\Gamma_0} v$. Therefore, we merely regard $\mathbf{v} := \Gamma \setminus \Gamma_0$ as a *multiset* (as we do not assert that the correspondence is injective) of subsets of Γ_0 . We fix an ordering of Γ_0 and assume each extra vertex $v \in \mathbf{v}$ ordered and each multiset \mathbf{v} of extra vertices ordered lexicographically, $\mathbf{v} = \{v_1 \leq \dots \leq v_r\}$. For a set S and $n \in \mathbb{N}$, we denote by $S[n]$ the n -th symmetric power of S and by $C(S, n)$, the set of all n -combinations of S ; then, we abbreviate $C_*(S, n) := \bigcup_{i=0}^n C(S, i)$.

Given a multiset \mathbf{v} as above, we use the notation

$$(A.5) \quad \Gamma := \Gamma_0 \sqcup \mathbf{v} \quad \text{or} \quad \Gamma := \Gamma_0 \sqcup \mathbf{v}(\mathbf{m})$$

for the set theoretic union $\Gamma_0 \cup \mathbf{v}$ equipped with an extra edge connecting $u \in \Gamma_0$ and $v \in \mathbf{v}$ whenever $u \in v$. In the former case, \mathbf{v} itself is regarded as a discrete graph, whereas in the latter case, the graph structure on \mathbf{v} is an extra piece of data given by an adjacency matrix $\mathbf{m} = [m_{ij}] \in \text{Sym}(|\mathbf{v}|, \mathbb{F}_2)$, $m_{ij} = v_i \cdot v_j \pmod{2}$.

A Γ_0 -*isomorphism* between two graph extensions $\Gamma' \supset \Gamma_0$ and $\Gamma'' \supset \Gamma_0$ is a graph isomorphism $\Gamma' \rightarrow \Gamma''$ taking Γ_0 to Γ_0 as a set. The group of Γ_0 -automorphisms of $\Gamma \supset \Gamma_0$ is denoted by $\text{Aut}(\Gamma, \Gamma_0)$. The *explicit sorting* of a list of graph extensions is the procedure removing all but one representative of each Γ_0 -isomorphism class. In the special case where $\Gamma_0 = \emptyset$, this procedure is called the *ultimate sorting*. We use the GRAPE package [21, 22, 37] in GAP [16] (with a few minor performance enhancements), computing also, as a by-product, the groups $\text{Aut}(\Gamma, \Gamma_0)$.

A.4. The extension algorithm. In this section, we describe a procedure which is the essential part of all other algorithms considered below.

A.4.1. The input. Since, usually, the algorithm is part of a larger computation, we always have a certain *global criterion* $\text{NC}: \{\text{graphs}\} \rightarrow \{\text{false}, \text{true}\}$ in mind. In addition, the input consists of the following data:

- an affine Dynkin diagram Σ ,
- an *admissibility level* $1 \leq m \leq 3$,

- a choice of the *mode*, see §A.4.3 vs. §A.4.4 below,
- a local *numeric criterion* $nc: \{\text{graphs}\} \rightarrow \{\text{false}, \text{true}\}$,
- a base Σ -graph $\Gamma_0 \supset \Sigma$ and a *symmetry group* $G_0 \subset \text{Aut}(\Gamma_0, \Sigma)$, and
- a G_0 -invariant initial set \mathcal{S}_1 of extra vertices (as subsets of Γ_0).

The goal is finding all extensions $\Gamma \supset \Gamma_0$ as in (A.5), with $\mathbf{v} := \Gamma \setminus \Gamma_0 \in \mathcal{S}_1[r]$, $r \in \mathbb{N}$, satisfying, at least, the following conditions:

- (1) the graph Γ *itself* is a hyperbolic Σ -graph, and
- (2) one has $\mathfrak{G}_m(\Gamma) \neq \emptyset$.

Sometimes (e.g., if \mathbf{v} is not assumed discrete), we also insist that

- (3) the rank of Γ is as large as possible: $\text{rk } \Gamma = \text{rk } \Gamma_0 + |\Gamma \setminus \Gamma_0|$.

We proceed step by step, with formal Step 0 returning $\{\Gamma_0\}$ and $\bar{\mathcal{S}}_0 := \{\emptyset\}$.

A.4.2. *Step 1.* We compute the set

$$\bar{\mathcal{S}}_1 := \{v \in \mathcal{S}_1 \mid \Gamma := \Gamma_0 \sqcup \{v\} \text{ satisfies §A.4.1(1), (2) and } \text{rk } \Gamma < 20\}.$$

(For the last condition $\text{rk } \Gamma < 20$, see [Convention A.8](#) below.) To this end, we

- (1) run the Sylvester test ([Lemma A.4](#)), reducing \mathcal{S}_1 to a subset \mathcal{S}'_1 ;
- (2) pick a representative v of each G_0 -orbit on \mathcal{S}'_1 and let $\mathbf{v} := \{v\}$;
- (3) for each \mathbf{v} , run appropriate *preliminary tests* (to be specified below);
- (4) for each \mathbf{v} left, check that $\Gamma := \Gamma_0 \sqcup \mathbf{v}$ is a Σ -graph;
- (5) for each Γ left, run the master test ([§A.1](#)) to check [§A.4.1\(2\)](#), upon which we discard all graphs Γ of rank 20 (see [Convention A.8](#) below).

Remark A.6. Typical preliminary tests used (for speed) in item (3) are as follows (where $1 \leq p < q < r \leq |\mathbf{v}|$):

- if $\Sigma > \tilde{\mathbf{A}}_2$, then $u_1 \cdot u_2 = 0$ for $u_1, u_2 \in v_p$, $u_1 \neq u_2$ (no triangles);
- if $\Sigma > \tilde{\mathbf{A}}_3$, then $|v_p \cap v_q| \leq 1$ (no quadrangles);
- if $m = 3$, then $|v_p \cap v_q| \leq 2$ and $|v_p \cap v_q \cap v_r| \leq 1$ (no bi-quadrangles).

If applicable (and not covered by any of the above), we can also use [\(A.10\)](#) below, checking appropriate b_{**} -fold intersections of the vertices to be added. More subtle combinatorial tests are usually ignored as they are incorporated to item (4), where we thoroughly check the full Gram matrix of $\mathcal{F}(\Gamma)$.

A.4.3. *Step $r \geq 2$, the safe mode.* In this version of the algorithm, we assume that the graph \mathbf{v} is discrete. The goal is computing the set

$$\bar{\mathcal{S}}_r := \{\mathbf{v} \in \bar{\mathcal{S}}_1[r] \mid \Gamma := \Gamma_0 \sqcup \mathbf{v} \text{ satisfies §A.4.1(1), (2) and } \text{rk } \Gamma < 20\}.$$

We use the result of Step $(r-1)$ and start with the set of ordered r -tuples

$$(A.7) \quad \mathcal{S}_r := \{\mathbf{v} \in \bar{\mathcal{S}}_{r-1} \times \bar{\mathcal{S}}_1 \mid v_1 \leq \dots \leq v_{r-1} \leq v_r\}.$$

Then, if $r = 2$, we let $\mathcal{S}'_2 := \mathcal{S}_2$; otherwise, we reduce \mathcal{S}_r to

$$\mathcal{S}'_r := \{\mathbf{v} \in \mathcal{S}_r \mid (\dots, \hat{v}_i, \dots) \in \bar{\mathcal{S}}_{r-1} \text{ for each } i < r\};$$

apart from reducing the overcounting, this *must* be done to ensure a well-defined action of G_0 . Finally, we pick a representative \mathbf{v} of each G_0 -orbit on \mathcal{S}'_r and repeat operations (3)–(5) in [§A.4.2](#).

A.4.4. *Step $r \geq 2$, the progressive mode.* If \mathbf{v} is not assumed discrete, we insist that each step of the algorithm should increase the rank. Thus, we modify the output of Step 1 to

$$\bar{\mathcal{S}}'_1 := \{v \in \bar{\mathcal{S}}_1 \mid \text{rk}(\Gamma_0 \sqcup \{v\}) > \text{rk} \Gamma_0\},$$

compute the set

$$\begin{aligned} \bar{\mathcal{S}}_r^* &:= \{(\mathbf{v}, \mathbf{m}) \in \bar{\mathcal{S}}'_1[r] \times \text{Sym}(r, \mathbb{F}_2) \mid \\ &\quad \Gamma(\mathbf{m}) := \Gamma_0 \sqcup \mathbf{v}(\mathbf{m}) \text{ satisfies } \S\text{A.4.1(1)–(3)} \text{ and } \text{rk} \Gamma(\mathbf{m}) < 20\}, \end{aligned}$$

and define $\bar{\mathcal{S}}_r$ as the projection of $\bar{\mathcal{S}}_r^*$ to the first factor. We proceed as in §A.4.3, with a few alterations:

- starting from item (3), we deal with pairs (\mathbf{v}, \mathbf{m}) rather than sets \mathbf{v} (in the hope that items (3) and (4) would rule out most matrices \mathbf{m}), and
- in item (5), we check, in addition, that $\text{rk} \Gamma(\mathbf{m}) = \text{rk} \Gamma_0 + r$.

In both modes, the algorithm terminates as soon as $\bar{\mathcal{S}}_r = \emptyset$. Sometimes, we also terminate it after a preset number r_{\max} of steps given as part of the input.

Convention A.8 (graphs of the maximal rank). Since any geometric extension of any lattice $\mathcal{F}(\Gamma)$ of rank 20 is of finite index, *we systematically discard any graph Γ of the maximal rank $\text{rk} \Gamma = 20$* ; instead, we just store the set $\text{Sat}_m(\Gamma; \text{NC})$ in a separate list \mathcal{G}_{\max} . (Clearly, it suffices to compute the saturation lists only for the extensions $\Gamma \supset \Gamma_0$ satisfying $\text{rk} \Gamma = \text{rk} \Gamma_0 + |\Gamma \setminus \Gamma_0|$.)

A.4.5. *Plain vs. saturated output.* The *plain output* of the algorithm (usually, it is intended for further processing) is just the union of the outputs, *i.e.*, graphs

$$(A.9) \quad \Gamma := \Gamma_0 \sqcup \mathbf{v}, \quad \mathbf{v} \in \bar{\mathcal{S}}_r, \quad \text{or} \quad \Gamma := \Gamma_0 \sqcup \mathbf{v}(\mathbf{m}), \quad (\mathbf{v}, \mathbf{m}) \in \bar{\mathcal{S}}_r^*,$$

of all steps, filtered *via* nc: only the graphs Γ with $\text{nc}(\Gamma) = \text{true}$ are retained. The content of the list \mathcal{G}_{\max} (see Convention A.8) is *not* part of the output, although it *is* taken into account when drawing the final conclusion.

The *saturated output* (usually intended as the final result) is the union of the list \mathcal{G}_{\max} and the saturations $\text{Sat}_m(\Gamma; \text{NC})$ over all graphs Γ as in (A.9) and such that $\text{rk} \Gamma = \text{rk} \Gamma_0 + |\mathbf{v}|$. (In the actual implementation, searching for examples, we compute the saturated output in any case, even if only plain output is needed for further processing.)

A.4.6. *Two-phase computation.* In a few cases, when the group G_0 and, hence, intermediate lists \mathcal{S}_r are too large, we have to break the algorithm into two separate phases. Namely, we run up to a certain preset number r_{break} of steps, upon which start over and apply the same algorithm to each graph Γ in the output. To reduce the overcounting, we modify the initial set for phase 2 as follows:

- start with the set $\bar{\mathcal{S}}_1$ computed in phase 1, and
- assuming that shorter extra vertices are added first, reduce this set (for each graph $\Gamma \supset \Gamma_0$) to $\{u \in \bar{\mathcal{S}}_1 \mid |u| \geq |v| \text{ for all } v \in \Gamma \setminus \Gamma_0\}$.

If plain output is required, the combined output of phase 2 is subject to the explicit sorting (see §A.3).

A.5. *A priori* bounds and patterns. We fix an order $\Sigma = \{a_1, \dots, a_N\}$ of each affine Dynkin diagram Σ to be considered. Usually, for Σ -graphs, we have certain *a priori* bounds $b_i, b_{ij}, b_{i0}, b_{0i}, 1 \leq i, j \leq N$, so that

$$(A.10) \quad \begin{aligned} \#\{l \in \sec a_i \mid l \cdot s = 1\} &\leq b_{ij} \quad \text{for any } s \in \sec a_j, \\ \#\{l \in \sec a_i \mid l \cdot s = 1\} &\leq b_{i0} \quad \text{for any } s \in \sec \emptyset, \\ \#\{l \in \sec \emptyset \mid l \cdot s = 1\} &\leq b_{0i} \quad \text{for any } s \in \sec a_i, \\ |\sec a_i| &\leq b_i, \end{aligned}$$

where $\sec \emptyset := \sec_\Gamma \emptyset$ stands for the set of lines $l \in \Gamma \supset \Sigma$ disjoint from Σ . (For b_i , we usually take $b_i = v_{\max}(\Sigma) - \text{val}_\Sigma a_i$, where $v_{\max}(\Sigma)$ is a bound for the maximal valency of a vertex in an m -admissible Σ -graph.)

A Σ -*pattern*, or just *pattern*, is a function $\pi: \Sigma \rightarrow \mathbb{N}$ such that $\pi(a_i) \leq b_i$ for each $i \leq N$. The *size* of a pattern π is $|\pi| := \sum \pi(a), a \in \Sigma$. The group $\text{Aut } \Sigma$ acts on the set of patterns, and we denote by $\text{pat}(\Sigma)$ the set of the *lexicographically maximal* representatives of the $(\text{Aut } \Sigma)$ -orbits. Clearly, we can use the symmetry and confine ourselves to the Σ -graphs $\Gamma \supset \Sigma$ such that the *section count* $\pi_\Gamma: a \mapsto |\sec a|, a \in \Sigma$ is an element of $\text{pat}(\Sigma)$.

Given a function $\rho: D \rightarrow \mathbb{N}, D := \text{domain}(\rho) \subset \Sigma$, a subset $S \subset \Sigma \setminus D$, and a bound $M \in \mathbb{N}$, we define the *range* associated with these data as

$$\text{range}_S(\rho, M) := \left\{ \sum_{a \in S} \pi(a) \mid \pi \in \text{pat}(\Sigma), \pi|_D = \rho, |\pi| \geq M \right\}.$$

APPENDIX B. SECTIONS AT A SINGLE FIBER Σ

We start with a *fiber* (affine Dynkin diagram) $\Sigma := \{a_1, \dots, a_N\}$ and set a *goal* $|\sec \Sigma| \geq M_\Sigma$; thus, the global criterion is

$$\text{NC}(\Gamma) := (\Gamma \text{ is a } \Sigma\text{-graph}) \ \& \ (|\sec_\Gamma \Sigma| \geq M_\Sigma).$$

We make the following assumptions (which are proved separately):

- (1) each section to be added intersects exactly one of a_1, \dots, a_N ;
- (2) for each $1 \leq i \leq N$, all sections $s \in \sec a_i$ are pairwise disjoint.

The algorithm computing large sets of sections at Σ has N levels, starting from the graph $\Gamma_0 := \Sigma$. Each subsequent level k produces a list $\{\Gamma_k\}$ from each graph Γ_{k-1} obtained at level $(k-1)$, and the full output is the union of these lists.

B.1. Level $k \geq 1$ of the algorithm. For the input, we fix a graph

$$\Gamma_{k-1} := \Sigma \cup \sec a_1 \cup \dots \cup \sec a_{k-1}$$

with a known automorphism group G_{k-1} fixing Σ *pointwise*. Consider the function $\rho: \{a_1, \dots, a_{k-1}\} \rightarrow \mathbb{N}, a \mapsto |\sec a|$, and let $R_k := \text{range}_k(\rho, M_\Sigma)$. We run up to $r_{\max} := \max R_k$ steps of the algorithm in §A.4. In addition to the given fiber Σ and admissibility level m , we choose the safe mode of the algorithm (see §A.4.3), take $\text{nc}(\Gamma) := (|\sec_\Gamma a_k| \in R_k)$ for the numeric criterion, Γ_{k-1} and G_{k-1} for the base graph and symmetry group, respectively, and

$$(B.1) \quad \mathcal{S}_1 := \{a_k\} \times \text{Sec}_k(\Gamma_{k-1}), \quad \text{Sec}_k(\Gamma) := \prod_{i=1}^N C_*(\sec_\Gamma a_i, b_{ik}),$$

see (A.10), for the initial set of vertices. The result is the plain output, see §A.4.5.

Remark B.2. Since we assume that each graph $\text{sec } a_i$ is discrete, level 1 of the algorithm is “trivial”: we merely test the 1-parameter family $\Gamma(n) := \Sigma \cup \text{sec } a_1$, where $|\text{sec } a_1| = n \leq b_1$, see §A.5.

Likewise, if $b_{12} = b_{21} \leq 1$, level 2 is reduced to testing the 3-parameter family of graphs $\Gamma(n_1, n_2; r) := \Sigma \cup \text{sec } a_1 \cup \text{sec } a_2$, where we let $|\text{sec } a_i| = n_i$, $i = 1, 2$, and $r \leq \min\{n_1, n_2\}$ is the number of sections $s_{1i} \in \text{sec } a_1$ intersecting a section $s_{2j} \in \text{sec } a_2$; when constructing the Gram matrix, we can assume that $s_{1i} \cdot s_{2i} = 1$ for $1 \leq i \leq r$ whereas all other pairs of sections are disjoint.

We can take this observation two steps further: if $\Sigma = \tilde{\mathbf{A}}_m$, $m = 4, 5$, levels 1 to $(m-1)$ reduce to testing an m -parameter family $\Gamma(n_1, \dots, n_{m-1}; r)$. Thus, in fact, we always use at most two “essential” levels.

The output of the N -th level is a complete list of the subgeometric graphs of the form $\Gamma_N := \Sigma \cup \text{sec } \Sigma$ satisfying $|\text{sec } \Sigma| \geq M_\Sigma$. (Note that, at this point, we do *not* require that Γ_N should be a Σ -graph.) In particular, we obtain an upper bound on $|\text{sec } \Sigma|$. To eliminate the overcounting and compute the groups $\text{Aut}(\Gamma_N, \Sigma)$, we apply the explicit sorting (see §A.3).

B.2. Level 0: fiber components. We fix another threshold $M > M_\Sigma + |\Sigma|$ and try to list all m -geometric Σ -graphs $\Gamma \supset \Gamma_N$ satisfying, in addition, the inequality $|\Gamma| \geq M$. For this, we run another instance of the algorithm in §A.4, setting

$$\text{nc}(\Gamma) := (|\Gamma| \geq M) \quad \text{and} \quad \text{NC}(\Gamma) := (\Gamma \text{ is a } \Sigma\text{-graph}) \ \& \ \text{nc}(\Gamma)$$

We take one of the graphs Γ_N and groups $G_N := \text{Aut}(\Gamma_N, \Sigma)$ for the base graph and symmetry group, respectively, and

$$\mathcal{S}_1 := C_*(\text{sec } a_1, b_{10}) \times \dots \times C_*(\text{sec } a_N, b_{N0}),$$

see (A.10), for the initial set of vertices. The preliminary tests are as in Remarks A.6, and we are interested in the saturated output, see §A.4.5.

The choice of the mode of the algorithm depends on the type of Σ . However, in the most common case $\text{rk } \Gamma_N = 19$, we choose the *progressive mode* (see §A.4.4), adding exactly one extra line to obtain (and discard) a graph of rank 20.

Remark B.3 (the validity test). For most fibers Σ , the safe mode of the algorithm (see §A.4.3) is chosen. It runs faster, but the validity of this choice, *i.e.*, the fact that any *sufficiently large* extension $\Gamma \supset \Gamma_N$ is spanned over Γ_N by a collection of *pairwise disjoint* fiber components, needs justification on a case-by-case basis. This is done automatically, using the known list of large combinatorial Σ -pencils (see Appendix C below). Namely, for each $\delta \in \mathbb{N}$, we compute

$$r_{\max}(\delta) := \min\{\alpha(\Pi \setminus \Sigma) \mid \Pi \text{ is a } \Sigma\text{-pencil, } |\Pi \setminus \Sigma| + \delta \geq M\}.$$

Then, if the algorithm starting from a graph $\Gamma_N := \Sigma \cup \text{sec } \Sigma$ does not terminate in $r_{\max}(|\Gamma_N|)$ steps, an error is signaled and Γ_N is returned as unsettled. It is this criterion that dictates our choice of the thresholds M and M_Σ .

B.3. Sections at $\Sigma = \tilde{\mathbf{D}}_4$. We order Σ so that a_1 is the “central” 4-valent vertex. Since any $\tilde{\mathbf{D}}_4$ -graph is pentagon free, *all* sections are pairwise disjoint; thus, in (A.10) we have, for $1 \leq i \neq j \leq 5$,

$$(B.4) \quad b_{i0} = 1, \quad b_{ij} = 0.$$

The other bounds $b_{01} = 6$ and $b_{0k} = 5$ for $k \geq 2$ are not used.

Since $\text{Aut } \tilde{\mathbf{D}}_4 = \mathbb{S}(a_2, \dots, a_5)$ (the full symmetric group), and assuming that a_1 is chosen to have the absolute maximal valency in the graph, we have the following description (assuming $m = 3$, so that the maximal valency of a vertex is 7, see [Lemma 5.1\(2\)](#))

$$(B.5) \quad \pi \in \text{pat}(\Sigma) \quad \text{iff} \quad 6 \geq \pi(a_1) + 3 \geq \pi(a_2) \geq \dots \geq \pi(a_5).$$

Proof of [Lemma 7.3](#). We use the thresholds $M_\Sigma = 11$, $M = 28$, $M_{\text{lines}} = 26$. At level 0, we choose the safe mode and have to add up to four pairwise disjoint extra lines disjoint from Σ , justifying the validity as in [Remark B.3](#). \square

B.4. Sections at $\Sigma = \tilde{\mathbf{A}}_4$. We order Σ cyclically. Since an $\tilde{\mathbf{A}}_4$ -graph is quadrangle free, in [\(A.10\)](#) we have, for $1 \leq i \neq j \leq 5$,

$$(B.6) \quad b_{i0} = 1, \quad b_{ij} = 1 \text{ if } i = j \pm 2 \pmod{5}, \quad b_{ij} = 0 \text{ otherwise.}$$

The other bound $b_{0i} = 6$ (assuming the graph 3-admissible) is not used.

We have $\text{Aut } \tilde{\mathbf{A}}_4 = \mathbb{D}_{10}$ (the dihedral group), and it is not very easy to describe the set $\text{pat}(\Sigma)$; we merely compute it using GAP [\[16\]](#).

Proof of [Lemma 7.6](#). We use the thresholds $M_\Sigma = 14$, $M = 30$, $M_{\text{lines}} = 28$. At level 0, we add up to three pairwise disjoint extra lines in the safe mode. \square

B.5. Sections at $\Sigma = \tilde{\mathbf{A}}_3$. We order Σ cyclically. Any $\tilde{\mathbf{A}}_3$ -graph is triangle free; assuming, in addition, that it is 3-admissible and, hence, bi-quadrangle free, see [Lemma 5.2](#), in [\(A.10\)](#) we have, for $1 \leq i \neq j \leq 4$,

$$(B.7) \quad b_{i0} = 2, \quad b_{ij} = 1 \text{ if } i = j \pm 1 \pmod{4}, \quad b_{ij} = 2 \text{ if } i = j + 2 \pmod{4}.$$

The other bound $b_{0i} = 5$ is not used.

Since $\text{Aut } \tilde{\mathbf{A}}_3 = \mathbb{D}_8$, we have the following relatively simple description of the maximal representatives $\pi \in \text{pat}(\Sigma)$: a pattern $\pi: a_i \mapsto n_i \in \mathbb{N}$, $1 \leq i \leq 4$, belongs to $\text{pat}(\Sigma)$ if and only if

$$(B.8) \quad n_i \leq n_1 \leq 5, \quad n_4 \leq n_2, \quad \text{and} \quad n_4 \leq n_3 \text{ whenever } n_2 = n_1.$$

Proof of [Lemma 7.9](#). We use the thresholds $M_\Sigma = 16$, $M_{\text{lines}} = 30$.

Since many graphs Γ_4 fail the test in [Remark B.3](#), we choose (for all $\tilde{\mathbf{A}}_3$ -graphs) the progressive mode of the algorithm (see [§A.4.4](#)); since this version reliably lists *all* extensions of Γ_4 , the constant M is redundant. In addition to [Remarks A.6](#), for the preliminary tests in [§A.4.2\(3\)](#) we use the fact that the graph must be triangle free to reduce the number of adjacency matrices $\mathbf{m} = [m_{ij}]$. Most notably, we assert that

$$m_{ij} = 0 \quad \text{whenever } v_i \cap v_j \neq \emptyset;$$

besides, for any triple $1 \leq i < j < k \leq |\mathbf{v}|$, at least one of the three entries m_{ij} , m_{ik} , m_{jk} must be 0. With this reduction, the computation remains feasible, in spite of the fact that we may have to add up to five extra vertices. \square

APPENDIX C. LARGE PENCILS

We start with a *fiber* $\Sigma := \{a_1, \dots, a_N\}$ and a combinatorial Σ -pencil $\Pi \supset \Sigma$ (see [§4.3](#)) and set a *goal* $|\Gamma| \geq M$ for a Σ -extension $\Gamma \supset \Pi$. Without loss of generality, we can assume that Π is the *maximal* (with respect to inclusion) pencil in Γ containing Σ ; then we can state the global criterion in terms of sections:

$$\text{NC}(\Gamma) := (\Gamma \text{ is a } \Sigma\text{-graph}) \ \& \ (\text{sec}_\Gamma \emptyset = \Pi \setminus \Sigma) \ \& \ (|\text{sec}_\Gamma \Sigma| \geq M - |\Pi|).$$

We still assume that (1) and (2) in Appendix B hold; besides, we consider *simple* sections only: $v \in \text{sec } a_i$, where $a_i \in \Sigma$ has multiplicity 1, *i.e.*, $n_i = 1$ in (4.1).

The algorithm has up to N level, processing one line $a \in \Sigma$ at a time. The order may differ from that fixed in Appendix B; it is controlled by a certain permutation $\sigma: \{1, \dots, N\} \rightarrow \Sigma$ fixed in advance. We abbreviate $k' := \sigma(k)$ and $a'_k := a_{\sigma(k)}$.

C.1. Level 1 of the algorithm. Let $R_1 := \text{range}_{1'}(\emptyset \hookrightarrow \mathbb{N}, M - |\Pi|)$. We run up to $r_{\max} := \max R_1$ steps of the algorithm in §A.4. In addition to the given fiber Σ and admissibility level m , we choose the safe mode of the algorithm (see §A.4.3), take $\text{nc}(\Gamma) := (|\text{sec}_{\Gamma} a'_1| \in R_1)$ for the local numeric criterion, $\Gamma_0 := \Pi$ for the base graph, and the stabilizer $\text{stab } a'_1 \subset \text{Aut}(\Pi, \Sigma)$ for the group G_0 . The initial set is

$$\mathcal{S}_1 := \{a'_1\} \times \prod_{\Delta \in \Pi_p} C(\Delta, 1) \times \prod_{\Delta \in \Pi_e} C_*(\Delta, 1),$$

where Π_p and Π_e are the sets of, respectively, parabolic and elliptic connected components of Π other than Σ . (Technically, when computing \mathcal{S}_1 , we remove from each component $\Delta \in \Pi_p$ all lines of multiplicity greater than 1. The same should be done for each $\Delta \in \Pi_e$, but the multiplicities are not always known here. In any case, all “wrong” sections are immediately ruled out by the Sylvester test.)

The expected result of level 1 is the plain output, see §A.4.5. Besides, we record

- all intermediate sets $\bar{\mathcal{S}}_r(a'_1) := \bar{\mathcal{S}}_r$ and
- for each graph Γ_1 of rank $\text{rk } \Gamma_1 = 19$ in the output, the set

$$(C.1) \quad \text{Pat}(\Gamma_1, G_0) := \{\max(\pi_{\Gamma} \cdot G_0) \mid \Gamma \in \text{Sat}_m(\Gamma_1) \cup \{\Gamma_1\}\}$$

of the maximal elements of the G_0 -orbits of the section counts (see §A.5) of Γ_1 itself and all its m -geometric saturations.

C.2. Level $k \geq 2$ of the algorithm. Let Γ_{k-1} be one of the graphs in the output of level $(k-1)$. As in §B.1, denote by ρ the restriction of $\pi_{\Gamma_{k-1}}$ to $\{a'_1, \dots, a'_{k-1}\}$ and let $R_k := \text{range}_{k'}(\rho, M_{\Sigma})$. We apply to Γ_{k-1} the algorithm in §A.4, with a few minor modifications explained below. We run up to $r_{\max} := \max R_k$ steps, taking

$$\text{nc}(\Gamma) := (|\text{sec}_{\Gamma} a'_k| \in R_k), \quad G_{k-1} := \text{stab}\{a'_1, \dots, a'_k\} \subset \text{Aut}(\Gamma_{k-1}, \Sigma)$$

(pointwise stabilizer) for the numeric criterion and symmetry group, respectively.

By default, we choose the safe mode (see §A.4.3) and plain output unless either

- $k = N$, so that a'_k is the last point of Σ , or
- $\text{rk } \Gamma_{k-1} = 19$ and

$$(C.2) \quad s_{\min} := \min R_k - \max\{\pi(a'_k) \mid \pi \in \text{Pat}(\Gamma_{k-1}, G_{k-2})\} > 0,$$

in which case the progressive mode (see §A.4.4) and saturated output are used. (Special arrangements can be made for particular types of Σ .) In the case of plain output, we also record the sets $\text{Pat}(\Gamma_k, G_{k-1})$, see (C.1).

To describe the modifications, recall that all simple vertices of Σ constitute a single $(\text{Aut } \Sigma)$ -orbit; hence, we can pick an element $g \in \text{Aut } \sigma$ such that $a'_1 \cdot g = a'_k$ and consider the images $\mathcal{S}(a'_k) := \bar{\mathcal{S}}(a'_1) \cdot g$ (see §C.1; effectively, we merely change the first entry a'_1 of each vertex v to a'_k). Then, we take

$$\mathcal{S}_1 := \mathcal{S}_1(a'_k) \times \text{Sec}_{k'}(\Gamma_{k-1}),$$

see (B.1), for the initial set of vertices and, upon completion of Step 1, define the set $\bar{\mathcal{S}}_1(a'_k)$ as the projection of $\bar{\mathcal{S}}_1$ to the first factor. At each subsequent step r , instead of (A.7), we start from

$$\mathcal{S}_r := (\mathcal{S}_r(a'_k) \cap \bar{\mathcal{S}}_1(a'_k)[r]) \times \text{Sec}_{k'}(\Gamma_{k-1}),$$

thus reusing the results of level 1.

Remark C.3. As yet another performance enhancement, we do not store (just recording their number) the new sections if the algorithm does not improve the rank, *i.e.*, if either $\text{rk } \Gamma_k = \text{rk } \Gamma_{k-1}$ or $\text{rk } \Gamma_k = 20$ for *each* graph Γ_k obtained from a given graph Γ_{k-1} . This convention simplifies the computation on all subsequent levels, as we have a smaller set $\text{sec}_{\Gamma_k} \Sigma$.

C.3. Configurations with large pencils.

Proof of Lemma 7.4. We use the thresholds $M = 28$, $M_{\Pi} = 18$, $M_{\text{lines}} = 26$ and order Σ so that the “central” 4-valent fiber is a'_5 , the last one, so that it is never used: we merely assume that it may have the maximal possible valency 7. \square

Proof of Lemma 7.7. We use the thresholds $M = 30$, $M_{\Pi} = 17$, $M_{\text{lines}} = 28$. \square

Proof of Lemma 7.10. We use the thresholds $M = 32$, $M_{\Pi} = 17$, $M_{\text{lines}} = 30$. For level $r \geq 2$, we choose the progressive mode whenever $\text{rk } \Gamma_{k-1} = 19$; however, if $r \leq 3$ and $s_{\min} \leq 0$ in (C.2), the graph Γ_{k-1} is carried over to level 4. \square

Proof of Lemma 5.5. We let $m = 2$ and choose the thresholds $M = 30$, $M_{\Pi} = 21$, $M_{\text{lines}} = 29$. Besides, we make use of the symmetry $a_2 \leftrightarrow a_3$:

- we do not compute the sets $\text{Pat}(\Gamma_{k-1}, G_{k-2})$ in (C.1), switching to the progressive mode whenever $\text{rk } \Gamma_{k-1} = 19$, and
- we abort the computation if level 2 of the algorithm does not improve rank (see Remark C.3). \square

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