HURWITZ EQUIVALENCE OF BRAID MONODROMIES AND EXTREMAL ELLIPTIC SURFACES

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Abstract. We discuss the equivalence between the categories of certain ribbon graphs and subgroups of the modular group Γ and use this equivalence to construct exponentially large families of not Hurwitz equivalent simple braid monodromy factorizations of the same element. As an application, we also obtain exponentially large families of *topologically* distinct algebraic objects such as extremal elliptic surfaces, real trigonal curves, and real elliptic surfaces.

1. Introduction

Strictly speaking, the principal results of the paper concern extremal elliptic surfaces, see Subsection 1.3. However, we start with discussing a few applications to the braid monodromy, which seems to be a subject of more general interest.

1.1. Braid monodromy. Throughout the paper, we use the notation $[\![\,\cdot\,]\!] = [\![\,\cdot\,]\!]_G$ for the conjugacy class of an element $g \in G$ or a subgroup $H \subset G$ of a group G.

1.1.1. Definition. Given a group G, a (*G-valued*) *monodromy factorization* of length r is a sequence $\bar{\mathfrak{m}} = (\mathfrak{m}_1, \ldots, \mathfrak{m}_r)$ of elements of G. Two monodromy factorizations are *strongly* (*Hurwitz*) *equivalent* if they are related by a finite sequence of *Hurwitz moves*

$$(\ldots, \mathfrak{m}_i, \mathfrak{m}_{i+1}, \ldots) \mapsto (\ldots, \mathfrak{m}_i^{-1}\mathfrak{m}_{i+1}\mathfrak{m}_i, \mathfrak{m}_i, \ldots)$$

and their inverse. Two monodromy factorizations are *weakly equivalent* if they are related by a sequence of Hurwitz moves and their inverse and/or *global conjugation*

$$\bar{\mathfrak{m}} = (\mathfrak{m}_i) \mapsto g^{-1}\bar{\mathfrak{m}}g := (g^{-1}\mathfrak{m}_i g), \quad g \in G.$$

(In the sequel, the weak equivalence is often referred to as just *equivalence*.)

Sometimes it is required that each element $\mathfrak{m}_i$ of a monodromy factorization should belong to the union $\bigcup_j C_j$ of several conjugacy classes C_j fixed in advance. Thus, a $\mathbb{B}_n$-valued monodromy factorization is called *simple* if each $\mathfrak{m}_i$ is conjugate to the Artin generator σ_1, see Definition 5.1.3.

Sometimes, a monodromy factorization is also called a *Hurwitz system.*

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Note that we regard a monodromy as an *anti*-homomorphism, see 1.1.2 below. This convention explains the slightly unusual form of the Hurwitz moves and the fact that the order of multiplication is reversed in 1.1.3(1). However, the precise expressions for the Hurwitz moves are hardly ever used.

In this paper we mainly deal with the first nonabelian braid group $\mathbb{B}_3$ and the closely related groups $\tilde{\Gamma} := SL(2,\mathbb{Z})$ and $\Gamma := PSL(2,\mathbb{Z})$. A $\tilde{\Gamma}$- or Γ-valued monodromy factorization $(\mathfrak{m}_i)$ is called *simple* if each $\mathfrak{m}_i$ belongs to the corresponding conjugacy class $[\![\mathbb{X}\mathbb{Y}]\!]$, see Subsection 2.1 for the notation. The classifications of simple monodromy factorizations (up to weak/strong Hurwitz equivalence) in all three groups coincide, see Proposition 5.1.4.

1.1.2. A G-valued monodromy factorization $\bar{\mathfrak{m}} = (\mathfrak{m}_1, \ldots, \mathfrak{m}_r)$ can be regarded as an anti-homomorphism $\langle\gamma_1, \ldots, \gamma_r\rangle \to G$, $\gamma_i \mapsto \mathfrak{m}_i$, $i = 1, \ldots, r$. In this interpretation, Hurwitz moves generate the canonical action of the braid group $\mathbb{B}_r$ on the free group $\langle\gamma_1, \ldots, \gamma_r\rangle$, and the global conjugation represents the adjoint action of G on itself. Geometrically, anti-homomorphisms as above arise from locally trivial fibrations $X^\sharp \to B^\sharp$ over a punctured disk; then G is the (appropriately defined) mapping class group of the fiber over a fixed point $b \in \partial B^\sharp$ and $\langle\gamma_1, \ldots, \gamma_r\rangle$ is a geometric basis for $\pi_1(B^\sharp, b)$. In this set-up, Hurwitz moves can be interpreted either as basis changes or as automorphisms of $B^\sharp$ fixed on the boundary, see [4], and the topological classification of fibrations reduces to the purely algebraic classification of G-valued monodromy factorizations up to weak Hurwitz equivalence. The best known examples are

- ramified coverings (the fiber is a finite set and $G = \mathbb{S}_n$, see [20]);
- algebraic or, more generally, pseudoholomorphic and Hurwitz curves in $\mathbb{C}^2$ (the fiber is a punctured plane and $G = \mathbb{B}_n$, see [36], [21], [8], [9], [27], [28], [26], [22], [2], [3], [7], [30], [31]);
- (real) elliptic surfaces or, more generally, (real) Lefschetz fibrations of genus one (the fiber is an elliptic curve/topological torus and $G = \tilde{\Gamma}$, see [24], [34], [27], [6], [3], [13], [17], [30], [31], [32]).

The last two subjects are quite popular and the reference lists are far from complete: I tried to cite the founding papers and a few recent results/surveys only.

Usually it is understood that the punctures of $B^\sharp$ correspond to the singular fibers of a fibration $X \to B$ over a disk, the type of each singular fiber F being represented by the conjugacy class of the local monodromy about F. Thus, in the three examples above, simple monodromy factorizations correspond to fibrations with singular fibers which are simplest in the sense that they are not removable by a small local deformation.

1.1.3. The following is a list of the most commonly used weak/strong equivalence invariants of a G-valued monodromy factorization $\bar{\mathfrak{m}}$:

1. the *monodromy at infinity* $\mathfrak{m}_\infty(\bar{\mathfrak{m}}) := \mathfrak{m}_r \ldots \mathfrak{m}_1 \in G$ is a strong invariant; its conjugacy class $[\![\mathfrak{m}_\infty(\bar{\mathfrak{m}})]\!]$ is a weak invariant;
2. the *monodromy group* $\operatorname{Im}(\bar{\mathfrak{m}}) := \langle\mathfrak{m}_1, \ldots, \mathfrak{m}_r\rangle \subset G$ is a strong invariant; its conjugacy class $[\![\operatorname{Im}(\bar{\mathfrak{m}})]\!]$ is a weak invariant;
3. for $G = SL(2,\mathbb{Z})$, the *transcendental lattice* $\mathcal{T}(\bar{\mathfrak{m}})$, see Subsection 7.1 for the definition and generalizations, is a week invariant;
4. for $G = \mathbb{B}_3$, define the (*affine*) *fundamental group* (see [36], [21])

$$\pi_1(\bar{\mathfrak{m}}) := \langle\alpha_1, \alpha_2, \alpha_3 \mid \mathfrak{m}_i(\alpha_j) = \alpha_j \text{ for } i = 1, \ldots, r,\ j = 1, 2, 3\rangle;$$

the homomorphism $\langle \alpha_1, \alpha_2, \alpha_3 \rangle \twoheadrightarrow \pi_1(\bar{\mathfrak{m}})$ is a strong invariant; it depends on $\operatorname{Im}(\bar{\mathfrak{m}})$ only; the isomorphism class of the abstract group $\pi_1(\bar{\mathfrak{m}})$ is a weak invariant; it depends on $[\![\operatorname{Im}(\bar{\mathfrak{m}})]\!]$ only.

Due to Proposition 5.1.4, the invariants (3) and (4) apply equally well to simple $\mathbb{B}_3$-, $\tilde{\Gamma}$-, and Γ-valued monodromy factorizations. Note that often it is the group (4) that is the ultimate goal of computing the monodromy factorization in the first place.

Geometrically, most important is the monodromy at infinity (1); in the set-up of 1.1.2, it corresponds to the monodromy along the boundary ∂B, and the monodromy factorizations $\bar{\mathfrak{m}}$ with a given class $[\![\mathfrak{m}_\infty(\bar{\mathfrak{m}})]\!] \subset G$ enumerate the extensions to B of a given fibration over ∂B. For this reason, a monodromy factorization $\bar{\mathfrak{m}}$ is often regarded as a factorization of a given element $\mathfrak{m}_\infty(\bar{\mathfrak{m}})$ (which explains the term). The geometric importance of the extension problem, a number of partial results, and extensive experimental evidence give rise to the following two long standing questions.

1.1.4. Question. Is the weak/strong equivalence class of a simple $\mathbb{B}_n$-valued monodromy factorization $\bar{\mathfrak{m}}$ determined by the monodromy at infinity $\mathfrak{m}_\infty(\bar{\mathfrak{m}})$? (Note that the length of $\bar{\mathfrak{m}}$ is determined by $\mathfrak{m}_\infty(\bar{\mathfrak{m}})$, see 5.1.5.)

1.1.5. Question. If two simple $\mathbb{B}_n$-valued monodromy factorizations $\bar{\mathfrak{m}}_1$, $\bar{\mathfrak{m}}_2$ have the same monodromy at infinity and are weakly equivalent, are they also strongly equivalent? In other words, if a simple monodromy factorization $\bar{\mathfrak{m}}$ is conjugated by an element of G commuting with $\mathfrak{m}_\infty(\bar{\mathfrak{m}})$, is the result strongly equivalent to $\bar{\mathfrak{m}}$?

The answer to Question 1.1.4 is in the affirmative if $n = 3$ and $\mathfrak{m}_\infty(\bar{\mathfrak{m}})$ is a central (see [27]) or, more generally, positive (with respect to the Artin basis, see [31]) element of $\mathbb{B}_3$. Furthermore, for any n, two monodromy factorizations sharing the same monodromy at infinity are known to be *stably equivalent*, see [22] or [26] for details. An example of two nonequivalent simple $\mathbb{B}_4$-valued monodromy factorizations of length six was recently constructed in [25]. The corresponding Hurwitz curves differ by the number of components (one is irreducible and one is not); hence the monodromy factorizations differ by the fundamental group.

The condition that $\bar{\mathfrak{m}}$ should be simple in Question 1.1.4 is crucial: in general, a monodromy factorization is not unique. The first example was essentially found in [36], and a great deal of other examples have been discovered since then. A few new examples are discussed in Subsections 5.5 and 5.6. In particular, we give a very simple, not computer aided, proof of the non-equivalence of the two monodromy factorizations considered in [3].

1.2. Principal results. We answer Questions 1.1.4 and 1.1.5 in the negative for the braid group $\mathbb{B}_3$ (and related groups Γ and $\tilde{\Gamma}$, see Proposition 5.1.4). The inclusion $\mathbb{B}_3 \hookrightarrow \mathbb{B}_n$ implies a negative answer for the other braid groups as well, at least concerning the strong equivalence, see 5.1.7.

Let $T(k)$ be the number of isotopy classes of trees $\Xi \subset S^2$ with k trivalent vertices and $(k+2)$ monovalent vertices (and no other vertices), see Section 4 and Corollary 4.2.2. Let $C(k) = \binom{2k}{k}/(k+1)$ be the k-th Catalan number, and let $\tilde{T}(k) = (5k+4)C(k)/(k+2)$, see Subsection 4.2 and Corollary 4.2.2. Note that each of the three series grows faster that a^k for any $a < 4$. The first few values of $T(k)$ and $\tilde{T}(k)$ are listed in Table 1.

Table 1. A few values of $T(k)$ and $\tilde{T}(k)$

k	0	1	2	3	4	5	6	7	...	10	...	15
$T(k)$	1	1	1	1	4	6	19	49	...	1424	...	570285
$\tilde{T}(k)$	2	3	7	19	56	174	561	1859	...	75582	...	45052515

1.2.1. Theorem. *For each integer $k \geqslant 0$, there is a set $\{\bar{\mathfrak{m}}_i\}$, $i = 1, \ldots, \tilde{T}(k)$, of simple Γ-valued monodromy factorizations of length $(k+2)$ that share the same*

- *monodromy at infinity $\mathfrak{m}_\infty(\bar{\mathfrak{m}}_i) = (\mathbb{XY})^{-5k-4}$,*
- *transcendental lattice $\mathcal{T}(\bar{\mathfrak{m}}_i)$, see Example 7.2.3, and*
- *fundamental group $\pi_1(\bar{\mathfrak{m}}_i)$ (which is $\mathbb{Z}$ for $k \geqslant 2$)*

but are not strongly equivalent: the monodromy groups $\operatorname{Im}(\bar{\mathfrak{m}}_i) \subset \Gamma$ are pairwise distinct subgroups of index $6(k+1)$.

Recall once again that, due to Proposition 5.1.4 below, both the transcendental lattice and the fundamental group are well defined for a *simple* Γ-valued monodromy factorization, as it lifts to a unique simple $\tilde{\Gamma}$- (respectively, $\mathbb{B}_3$-) valued one.

1.2.2. Theorem. *For each k, the monodromy factorizations $\bar{\mathfrak{m}}_i$ in Theorem 1.2.1 form $T(k)$ distinct weak equivalence classes: they are distinguished by the conjugacy classes $[\![\operatorname{Im}(\bar{\mathfrak{m}}_i)]\!]$ of the monodromy groups.*

Since $T(k) < \tilde{T}(k)$ for all $k \geqslant 0$, one has the following corollary.

1.2.3. Corollary. *For each integer $k \geqslant 0$, there is a pair of conjugate simple Γ-valued monodromy factorizations of length $(k+2)$ that share the same monodromy at infinity $(\mathbb{XY})^{-5k-4}$ but are not strongly equivalent.* $\square$

Theorems 1.2.1 and 1.2.2 are proved in Subsection 5.2; the monodromy factorizations in question are given by (5.2.2), and their $\mathbb{B}_3$-valued counterparts are given by (5.3.1). The first example of weakly but not strongly equivalent $\mathbb{B}_3$-valued monodromy factorizations given by Corollary 1.2.3 has length two; it is as simple as

$$\bar{\mathfrak{m}}' = (\sigma_1^2\sigma_2\sigma_1^{-2}, \sigma_2), \qquad \bar{\mathfrak{m}}'' = (\sigma_1\sigma_2\sigma_1^{-1}, \sigma_1^{-1}\sigma_2\sigma_1),$$

see Example 5.3.3. The first example of non-equivalent monodromy factorizations given by Theorem 1.2.2 has length six, see Example 5.3.2. In Subsection 5.4 we construct another example of not weakly equivalent monodromy factorizations of length two; they also differ by the monodromy groups, which are of infinite index. A few other examples (not necessarily simple) are considered in Subsections 5.5 and 5.6.

1.3. Elliptic surfaces. Recall that an *extremal elliptic surface* can be defined as a Jacobian elliptic surface X of maximal Picard number, $\operatorname{rk} NS(X) = h^{1,1}(X)$, and minimal Mordell-Weil rank, $\operatorname{rk} MW(X) = 0$. (For an alternative description, in terms of singular fibers, see 2.2.3. Yet another characterization is the following: a Jacobian elliptic surface is extremal if and only if its transcendental lattice is positive definite, see [16].) Extremal elliptic surfaces are rigid (any small fiberwise equisingular deformation of such a surface X is isomorphic to X); they are defined over algebraic number fields.

In this paper, we mainly deal with elliptic surfaces with singular fibers of Kodaira types I_p and I_p^*. To shorten the statements, we call singular fibers of all other types, *i.e.*, Kodaira's II, III, IV and II*, III*, IV*, *exceptional*. (These types are related to the exceptional simple singularities/Dynkin diagrams $\mathbf{E}_6$, $\mathbf{E}_7$, $\mathbf{E}_8$.)

Given two elliptic surfaces X_1, X_2, a fiberwise homeomorphism $\varphi\colon X_1 \to X_2$ is said to be 2-*orientation preserving* (*reversing*) if it preserves (respectively, reverses) the complex orientation of the bases and the fibers of the two elliptic fibrations.

1.3.1. Theorem. *Two extremal elliptic surfaces without exceptional fibers are isomorphic if and only if they are related by a* 2*-orientation preserving fiberwise homeomorphism.*

Theorem 1.3.1 is not proved separately, as it is an immediate consequence of Theorem 2.5.3 below: the topological invariant distinguishing the surfaces is the conjugacy class in $\tilde{\Gamma}$ of the monodromy group of the homological invariant $\tilde{\mathfrak{h}}_X$, see 2.2.2. In fact, we show that appropriate subgroups of $\tilde{\Gamma}$ classify extremal elliptic surfaces without exceptional fibers, both analytically and topologically.

Two extensions of Theorem 1.3.1 to somewhat wider classes of surfaces are proved in Subsections 3.3 (see Remark 3.3.4) and 3.4.

As a by-product, we obtain exponentially large collections of non-homeomorphic elliptic surfaces sharing the same combinatorial type of singular fibers.

1.3.2. Theorem. *For each integer $k \geqslant 0$, there is a collection of $T(k)$ extremal elliptic surfaces that share the same combinatorial type of singular fibers, which is*

- $(k+2)\mathrm{I}_1 \oplus \mathrm{I}^*_{5k+4}$ *if k is even, or*
- $(k+2)\mathrm{I}_1 \oplus \mathrm{I}_{5k+4}$ *if k is odd,*

but are not related by a 2*-orientation preserving fiberwise homeomorphism.*

This theorem is proved in Subsection 4.3, and a few generalizations are discussed in Subsection 4.5. In fact, the surfaces were constructed in [12]. In [16], it is shown that they share as well such topological invariants as the transcendental lattice, see Example 7.2.3, and the fundamental group of the branch locus.

The proof of Theorems 1.3.1 and 2.5.3 is based on an explicit computation of the monodromy group $\operatorname{Im}\tilde{\mathfrak{h}}_X$ of an extremal elliptic surface X in terms of its skeleton Sk_X, see 2.2.5. In a sense, we show that Sk_X *is* $\operatorname{Im}\tilde{\mathfrak{h}}_X$ (assuming that X has no type II* singular fibers). As another consequence, we obtain an algebraic description of the reduced monodromy groups of such surfaces, see Subsection 3.5.

The principal tool in the proofs is a relation between subgroups of the modular group Γ and certain ribbon graphs, see Subsection 2.3. As yet another consequence of this construction, we obtain a few results (which may be known to the experts) on the subgroups of Γ. To me, the most interesting seem Corollaries 3.2.5 and 3.6.2 describing the structure of subgroups and Proposition 4.4.1 characterizing the monodromy groups of simple monodromy factorizations (see also Remarks 4.4.2 and 4.4.3).

1.4. Real trigonal curves and real elliptic surfaces. We consider a few other applications of the relation between ribbon graphs and subgroups of Γ, primarily to illustrate that some classification problems are wilder than they may seem.

Recall that the *Hirzebruch surface* is the geometrically ruled surface $\Sigma_k \to \mathbb{P}^1$, $k > 0$, with an exceptional section E of self-intersection $-k$. Up to isomorphism, there is a unique real structure (*i.e.*, anti-holomorphic involution) $\mathrm{conj}\colon \Sigma_k \to \Sigma_k$ with nonempty real part $(\Sigma_k)_\mathbb{R} := \operatorname{Fix}\mathrm{conj}$. A curve $C \subset \Sigma_k$ is *real* if it is invariant under conj. A *trigonal curve* is a curve $C \subset \Sigma_k$ disjoint from E and intersecting each fiber of the ruling at three points (counted with multiplicity). Such a curve is *generic* if all its singular fibers are of type I_1 (simple tangency of the curve and a fiber of the ruling). A generic curve is necessarily nonsingular.

(Often, an abstract trigonal curve is defined as a curve with a linear system of degree three. Any such curve admits an embedding to a Hirzebruch surface, and by a sequence of elementary transformation the image can be made disjoint from the exceptional section, although possibly singular. We adhere to the definition given in the previous paragraph as it is commonly accepted in the literature on topology of real algebraic varieties.)

1.4.1. Theorem. *For each integer $k \geqslant 0$, there is a collection of $T(k)$ generic real trigonal curves $C_i \subset \Sigma_{2k+2}$ such that all real parts $(C_i)_\mathbb{R} \subset (\Sigma_{2k+2})_\mathbb{R}$ are isotopic but the curves are not related by an equivariant 2-orientation preserving fiberwise auto-homeomorphism of Σ_{2k+2} preserving the orientation of the real part $\mathbb{P}^1_\mathbb{R}$ of the base of the ruling.*

Theorem 1.4.1 is proved in Subsection 6.2, and a generalization is discussed in Subsection 6.3. The real part of each curve C_i in Theorem 1.4.1 consists of a 'long' component L isotopic to $E_\mathbb{R}$ (see 6.1.3) and $(5k+4)$ ovals, necessarily unnested, all in the same connected component of $(\Sigma_{2k+2})_\mathbb{R} \smallsetminus (L \cup E_\mathbb{R})$.

For each curve C_i as in Theorem 1.4.1, the double covering $X_i \to \Sigma_{2k+2}$ ramified at $C_i \cup E$ is a real Jacobian elliptic surface. Since the curves C_i are distinguished by the braid monodromy, one has the following corollary.

1.4.2. Corollary. *For each integer $k \geqslant 0$, there are two collections of $T(k)$ real Jacobian elliptic surfaces $X_i \to \mathbb{P}^1$ such that all real parts $(X_i)_\mathbb{R}$ are fiberwise homeomorphic but the surfaces are not related by an equivariant 2-orientation preserving fiberwise homeomorphism of Σ_{2k+2} preserving the orientation of the real part $\mathbb{P}^1_\mathbb{R}$ of the base of the elliptic pencil.* $\square$

In other words, each of the two collections consists of $T(k)$ pairwise non-isomorphic directed real Lefschetz fibrations of genus 1 in the sense of [32]. The real parts $(X_i)_\mathbb{R}$ can be described in terms of the *necklace diagrams*, see [32]: they are chains of $(5k+4)$ copies of the same stone, which is either –○– or –□–.

1.5. Contents of the paper. In Section 2, we introduce the basic objects and prove principal technical results relating extremal elliptic surfaces, 3-regular ribbon graphs, and geometric subgroups of Γ. Section 3 deals with a few generalizations of these results to wider classes of ribbon graphs/subgroups. In Section 4, we introduce *pseudo-trees*, which are ribbon graphs constructed from oriented rooted binary trees. It is this relation that is responsible for the exponential growth in most examples. Theorem 1.3.2 is proved here. In Sections 5 and 6, we prove the results concerning, respectively, simple monodromy factorizations and real trigonal curves. Finally, in Section 7 we introduce the notion of transcendental lattice of a monodromy factorization and consider a few examples.

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2. Elliptic surfaces

In this section, we introduce some basic notions and prove the principal technical results: Corollary 2.3.6 and Theorem 2.4.5, establishing a connection between 3-regular ribbon graphs and geometric subgroups of Γ, and Theorems 2.5.2 and 2.5.3, relating extremal elliptic surfaces, their skeletons, and monodromy groups.

2.1. The modular group. Let $\mathcal{H} = \mathbb{Z}a \oplus \mathbb{Z}b$ be a rank 2 free abelian group with the skew-symmetric bilinear form $\bigwedge^2 \mathcal{H} \to \mathbb{Z}$ given by $a \cdot b = 1$. We fix the notation $\mathcal{H}$, a, b throughout the paper and *define* $\tilde{\Gamma} := SL(2, \mathbb{Z})$ as the group $\operatorname{Sp}\mathcal{H}$ of symplectic automorphisms of $\mathcal{H}$; it is generated by the operators $\mathbb{X}, \mathbb{Y} \colon \mathcal{H} \to \mathcal{H}$ given (in the basis $\{a, b\}$ above) by the matrices

$$\mathbb{X} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}, \qquad \mathbb{Y} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

One has $\mathbb{X}^3 = \mathrm{id}$ and $\mathbb{Y}^2 = -\mathrm{id}$. If $c = -a - b \in \mathcal{H}$, then $\mathbb{X}$ acts *via*

$$(a, b) \overset{\mathbb{X}}{\longmapsto} (c, a) \overset{\mathbb{X}}{\longmapsto} (b, c) \overset{\mathbb{X}}{\longmapsto} (a, b).$$

The *modular group* $\Gamma := PSL(2, \mathbb{Z})$ is the quotient $\tilde{\Gamma}/\pm\mathrm{id}$. We retain the notation $\mathbb{X}$, $\mathbb{Y}$ for the generators of Γ. One has

$$\Gamma = \langle \mathbb{X} \,|\, \mathbb{X}^3 = 1 \rangle * \langle \mathbb{Y} \,|\, \mathbb{Y}^2 = 1 \rangle \cong \mathbb{Z}_3 * \mathbb{Z}_2.$$

A subgroup $H \subset \Gamma$ is called *geometric* if it is torsion free and of finite index. Since $\Gamma = \mathbb{Z}_3 * \mathbb{Z}_2$, the factors generated by $\mathbb{X}$ and $\mathbb{Y}$, a subgroup $H \subset \Gamma$ is torsion free if and only if it is disjoint from the conjugacy classes $[\![\mathbb{X}]\!]$ and $[\![\mathbb{Y}]\!]$, or, equivalently, if both $\mathbb{X}$ and $\mathbb{Y}$ act freely on the quotient Γ/H.

Similarly, a subgroup $\tilde{H} \subset \tilde{\Gamma}$ is called *geometric* if it is torsion free and of finite index. A subgroup $\tilde{H} \subset \tilde{\Gamma}$ is torsion free if and only if $-\mathrm{id} \notin \tilde{H}$ and the image of $\tilde{H}$ in Γ is torsion free.

2.2. Extremal elliptic surfaces. In this subsection, we recall a few well known facts concerning Jacobian elliptic surfaces. The principal references are [18] or the original paper [24]. For more details concerning skeletons, we refer to [12].

A *Jacobian elliptic surface* is a compact complex surface X equipped with an elliptic fibration $\mathrm{pr} \colon X \to B$ (*i.e.*, a fibration with all but finitely many fibers nonsingular elliptic curves) and a distinguished section $E \subset X$ of pr. (From the existence of a section it follows that X has no multiple fibers.) Throughout the paper we assume that surfaces are *relatively minimal*, *i.e.*, that fibers of the elliptic pencil contain no (-1)-curves.

2.2.1. Each nonsingular fiber of a Jacobian elliptic surface $\mathrm{pr} \colon X \to B$ is an abelian group, and the multiplication by (-1) extends through the singular fibers of X. The quotient $X/\pm 1$ blows down to a geometrically ruled surface $\Sigma \to B$ over the same base B, and the double covering $X \to \Sigma$ is ramified over the exceptional section E of Σ and a certain curve $C \subset \Sigma$ disjoint from E and intersecting each generic fiber of the ruling at three points.

2.2.2. Denote by $B^\sharp \subset B$ the set of regular values of pr, and define the (*functional*) *j-invariant* $j_X\colon B \to \mathbb{P}^1$ as the analytic continuation of the function $B^\sharp \to \mathbb{C}^1$ sending each nonsingular fiber of pr to its classical j-invariant (divided by 12^3). The surface X is called *isotrivial* if $j_X = \text{const}$.

The monodromy $\tilde{\mathfrak{h}}_X\colon \pi_1(B^\sharp, b) \to \tilde{\Gamma} = \operatorname{Sp} H_1(\mathrm{pr}^{-1}(b))$, $b \in B^\sharp$, of the locally trivial fibration $\mathrm{pr}^{-1}(B^\sharp) \to B^\sharp$ is called the *homological invariant* of X. Its reduction $\mathfrak{h}_X\colon \pi_1(B^\sharp) \to \Gamma$ is called the *reduced monodromy*; it is determined by the j-invariant. Together, j_X and $\tilde{\mathfrak{h}}_X$ determine X up to isomorphism, and any pair $(j, \tilde{\mathfrak{h}})$ that agrees in the sense just described gives rise to a unique isomorphism class of Jacobian elliptic surfaces.

2.2.3. According to [29], a Jacobian elliptic surface X is extremal if and only if it satisfies the following conditions:

(1) j_X has no critical values other than 0, 1, and ∞;
(2) each point in $j_X^{-1}(0)$ has ramification index at most 3, and each point in $j_X^{-1}(1)$ has ramification index at most 2;
(3) X has no singular fibers of types I_0^*, II, III, or IV.

2.2.4. Recall that a *ribbon graph* is a graph with a distinguished cyclic order of edges at each vertex. A *left turn path* in a ribbon graph is a combinatorial path (a sequence of adjacent vertices) $v_0, \ldots, v_n$ with the property that, for each $i = 1, \ldots, n-1$, the edge $[v_i, v_{i+1}]$ is the immediate predecessor of $[v_i, v_{i-1}]$ with respect to the cyclic order at v_i. A *region* is a minimal left turn cycle.

Each graph embedded to an oriented surface inherits a natural ribbon graph structure. Conversely, patching each region of a connected ribbon graph with an oriented disk, one obtains a minimal oriented surface supporting the graph.

The *genus* of a connected ribbon graph is defined as the genus of its minimal supporting surface. Explicitly, the genus g is given by

$$2 - 2g = \#\{\text{vertices}\} - \#\{\text{edges}\} + \#\{\text{regions}\}.$$

2.2.5. The *skeleton* of a non-isotrivial elliptic surface $\mathrm{pr}\colon X \to B$ (not necessarily extremal) is the embedded bipartite graph $\mathrm{Sk}_X := j_X^{-1}[0,1] \subset B$. The pull-backs of 0 and 1 are called $\bullet$- and $\circ$-vertices of Sk_X, respectively. (Thus, Sk_X is the *dessin d'enfants* of j_X in the sense of Grothendieck; however, we reserve the word 'dessin' for the more complicated graphs describing arbitrary, not necessarily extremal, surfaces, *cf.* Subsection 6.1.) *A priori*, j_X may have critical values in the open interval $(0,1)$, hence the edges of Sk_X may meet at points other than $\bullet$- or $\circ$-vertices. However, by a small fiberwise equisingular deformation of X the skeleton Sk_X can be made *generic* in the sense that the edges of Sk_X meet only at $\bullet$- or $\circ$-vertices and the valency of each $\bullet$- (respectively, $\circ$-) vertex is $\leqslant 3$ (respectively, $\leqslant 2$).

The skeleton Sk_X of an extremal elliptic surface X is always generic. In addition, each region of Sk_X (*i.e.*, component of $B \smallsetminus \mathrm{Sk}_X$) is a topological disk; in particular, Sk_X is connected. Furthermore, each region contains a single critical point of j_X, the critical value being ∞. Thus, in this case Sk_X can be regarded as an abstract ribbon graph, and B is its minimal supporting surface. Extending the projection $\mathrm{Sk}_X \to [0,1]$ to B (with a single critical point inside each region), one recovers the ramified covering $j_X\colon B \to \mathbb{P}^1$; then, the analytic structure on B is given by the Riemann existence theorem. It follows that the skeleton Sk_X of an extremal elliptic surface X determines its j-invariant $j_X\colon B \to \mathbb{P}^1$ (as an analytic function); hence the pair $(\mathrm{Sk}_X, \tilde{\mathfrak{h}}_X)$ determines X.

2.2.6. The exceptional singular fibers of an elliptic surface X are in a one-to-one correspondence with the $\bullet$-vertices of Sk_X of valency $\ne 0 \bmod 3$ and its $\circ$-vertices of valency $\ne 0 \bmod 2$. Hence, if X is extremal and without exceptional fibers, all $\bullet$- and $\circ$-vertices of Sk_X are of valency 3 and 2, respectively. Since Sk_X is a bipartite graph, its $\circ$-vertices can be ignored, assuming that such a vertex is to be inserted at the middle of each edge connecting two $\bullet$-vertices. Under this convention, the skeleton of an extremal elliptic surface without exceptional fibers is a 3-regular ribbon graph. As explained above, each region of Sk_X is a disk containing a single singular fiber of X. Hence Sk_X is a strict deformation retract of $B^\sharp$, and the homological invariant can be regarded as an anti-homomorphism $\tilde{\mathfrak{h}}_X : \pi_1(\mathrm{Sk}_X) \to \tilde{\Gamma}$. It is explained in [16] (see also Remark 2.5.6 below) that $\tilde{\mathfrak{h}}_X$ can be encoded in terms of an orientation of Sk_X.

2.3. Skeletons: another point of view. Following [16], we start with redefining a 3-regular ribbon graph combinatorially as a set of ends of its edges. However, in the further exposition we will make no distinction between a graph in the sense of Definition 2.3.1 below and its geometric realization, defined in the obvious way. To justify this dual approach, we point out that the combinatorial definition establishes a relation between ribbon graphs and subgroups of the modular group Γ, whereas the topological point of view makes this relation useful, as it lets one appeal to one's geometric intuition when studying subgroups.

We will also redefine a few notions related to graphs (like connectedness, paths, *etc.*); each time, unless it is immediately obvious, we will try to explain the relation between a new notion and its conventional topological counterpart defined in terms of the geometric realization.

2.3.1. Definition. A *3-regular ribbon graph* is a collection $\mathrm{Sk} = (\mathcal{E}, \mathrm{op}, \mathrm{nx})$, where $\mathcal{E} = \mathcal{E}_{\mathrm{Sk}}$ is a finite set, $\mathrm{op}\colon \mathcal{E} \to \mathcal{E}$ is a free involution, and $\mathrm{nx}\colon \mathcal{E} \to \mathcal{E}$ is a free automorphism of order three. The orbits of op are called the *edges* of Sk, the orbits of nx are called its *vertices*, and the orbits of $\mathrm{nx}^{-1}\,\mathrm{op}$ are called its *faces* or *regions*.

A *based* 3*-regular ribbon graph* is a pair $(\mathrm{Sk}, \mathrm{e})$, where $\mathrm{e} \in \mathcal{E}_{\mathrm{Sk}}$.

With a certain abuse of the language, we will often refer to the elements of $\mathcal{E}_{\mathrm{Sk}}$ as the *points* of Sk. Usually not leading to a confusion (as combinatorial graphs do not have points), this convention will let us avoid excessive awkward terminology and keep intact such well established terms as 'base point', 'starting point', *etc.* However, when the geometric realization is involved in the discussion in an essential way, we will use the term *end* (of a given edge at a given vertex). For the further ease of reading, we use a distinctive notation e for the elements of $\mathcal{E}_{\mathrm{Sk}}$.

2.3.2. Remark. As explained above, $\mathcal{E}$ is the set of ends of edges of the geometric realization of the graph. In this geometric language, op assigns to an end the other end of the same edge, and nx assigns the next end at the same vertex with respect to the cyclic order at the vertex (constituting the ribbon graph structure).

2.3.3. Remark. Alternatively, one can consider $\mathcal{E}_{\mathrm{Sk}}$ as the set of edges of Sk regarded as a bipartite ribbon graph, see 2.2.6. Then the orbits of op and nx represent, respectively, the $\circ$- and $\bullet$-vertices of Sk. Considering bipartite ribbon graph with the valency of $\bullet$- and $\circ$-vertices equal to (respectively, dividing) two given integers p and q, one can extend, almost literally, the material of this and next subsections (respectively, the generalizations found in Section 3) to the subgroups of the group $\langle x, y \mid x^p = y^q = 1\rangle$. However, I do not know any interesting geometric applications of this group.

2.3.4. Given a 3-regular ribbon graph Sk, the set $\mathcal{E}_{\mathrm{Sk}}$ admits a canonical left Γ-action. To be precise, we define a homomorphism $\Gamma \to \mathbb{S}(\mathcal{E}_{\mathrm{Sk}})$ to the group $\mathbb{S}(\mathcal{E}_{\mathrm{Sk}})$ of permutations of $\mathcal{E}_{\mathrm{Sk}}$ *via* $\mathbb{X} \mapsto \mathrm{nx}^{-1}$, $\mathbb{Y} \mapsto \mathrm{op}$. According to this convention, the vertices, edges, and regions of Sk are the orbits of $\mathbb{X}$, $\mathbb{Y}$, and $\mathbb{XY}$, respectively. The graph Sk is *connected* if and only if the canonical Γ-action is transitive. A connected 3-regular ribbon graph is called a 3*-skeleton*.

Given an element $\mathrm{e} \in \mathcal{E}_{\mathrm{Sk}}$, we denote by $\mathrm{Stab(e)} \subset \Gamma$ its stabilizer. Stabilizers of all points of a 3-skeleton form a whole conjugacy class of subgroups of Γ; it is denoted by $[\![\mathrm{Stab\,Sk}]\!]$ and is called the *stabilizer* of Sk.

A *morphism* of 3-skeletons $\mathrm{Sk}' = (\mathcal{E}', \mathrm{op}', \mathrm{nx}')$ and $\mathrm{Sk}'' = (\mathcal{E}'', \mathrm{op}'', \mathrm{nx}'')$ is defined as a map $\varphi\colon \mathcal{E}' \to \mathcal{E}''$ commuting with the Γ-action, *i.e.*, such that $\varphi \circ \mathrm{op}' = \mathrm{op}'' \circ \varphi$ and $\varphi \circ \mathrm{nx}' = \mathrm{nx}'' \circ \varphi$. In other words, φ is a morphism of Γ-sets. A morphism of based 3-skeletons $(\mathrm{Sk}', \mathrm{e}')$ and $(\mathrm{Sk}'', \mathrm{e}'')$ is required, in addition, to take e' to e''. The group of automorphisms of a 3-skeleton Sk is denoted Aut Sk; we regard it as a subgroup of the symmetric group $\mathbb{S}(\mathcal{E}_{\mathrm{Sk}})$.

The following two statements, although crucial for the sequel, are immediate consequences of the definitions.

2.3.5. Theorem. *The functors* $(\mathrm{Sk}, \mathrm{e}) \mapsto \mathrm{Stab(e)}$, $H \mapsto (\Gamma/H, H/H)$ *establish an equivalence of the categories of*

- *based* 3*-skeletons and morphisms and*
- *geometric subgroups* $H \subset \Gamma$ *and inclusions.* ☐

It follows that any morphism of 3-skeletons is a topological covering of their geometric realizations.

2.3.6. Corollary. *The maps* $\mathrm{Sk} \mapsto [\![\mathrm{Stab\,Sk}]\!]$, $[\![H]\!] \mapsto \Gamma/H$ *establish a canonical one-to-one correspondence between the sets of*

- *isomorphism classes of* 3*-skeletons and*
- *conjugacy classes of geometric subgroups* $H \subset \Gamma$. ☐

If a 3-skeleton Sk is fixed, the isomorphism classes of based 3-skeletons $(\mathrm{Sk}, \mathrm{e})$ are naturally enumerated by the orbits of Aut Sk. Hence one has the following corollary, concerning properties of geometric subgroups.

2.3.7. Corollary. *The conjugacy class* $[\![H]\!]$ *of a geometric subgroup* $H \subset \Gamma$ *is in a one-to-one correspondence with the set of orbits of* $\mathrm{Aut}(\Gamma/H)$. *Furthermore, there is an anti-isomorphism* $\mathrm{Aut}(\Gamma/H) = N(H)/H$, *where* $N(H)$ *is the normalizer of* H (*acting on* Γ/H *by the right multiplication*). ☐

2.3.8. Remark. Theorem 2.3.5, as well as its generalizations 3.2.1, 3.6.1 below, relating subgroups of Γ and ribbon graphs resemble the results of [5]. However, the two constructions differ: in [5], finite index subgroups of the congruence subgroup $\Gamma(2)$ are encoded using bipartite ribbon graphs with vertices of arbitrary valency. Our approach is closer to that of [6], where the modular j-function on a modular curve B (see [34] and Remark 2.5.5) is described in terms of a special triangulation of B. Theorem 2.4.5 below and its generalizations in Section 3 make the geometric relation between ribbon graphs and subgroups of Γ even more transparent.

2.4. Paths (chains) in a 3-skeleton. The treatment of paths found in [16] is not quite satisfactory for our purposes; we choose a slightly different approach here. To avoid confusion, we will use the term 'chain'; the geometric background behind the formal combinatorial definition is explained in Remark 2.4.2 below.

2.4.1. Definition. A *chain* in a 3-skeleton $\mathrm{Sk} = (\mathcal{E}, \mathrm{op}, \mathrm{nx})$ is a pair $\gamma = (\mathrm{e}, w)$, where $\mathrm{e} \in \mathcal{E}_{\mathrm{Sk}}$ and w is a word in the alphabet $\{\mathrm{op}, \mathrm{nx}, \mathrm{nx}^{-1}\}$. The *evaluation map* val sends a chain $\gamma = (\mathrm{e}, w)$ to the element $\mathrm{val}\,\gamma \in \Gamma$ obtained by replacing $\mathrm{op} \mapsto \mathbb{Y}$, $\mathrm{nx}^{\pm 1} \mapsto \mathbb{X}^{\pm 1}$ in w and multiplying in Γ. The *starting* and *ending* points of γ are, respectively, $\gamma_0 := \mathrm{e} \in \mathcal{E}_{\mathrm{Sk}}$ and $\gamma_1 := (\mathrm{val}\,\gamma)^{-1}\mathrm{e} \in \mathcal{E}_{\mathrm{Sk}}$. A chain γ is a *loop* if $\gamma_0 = \gamma_1$. The *product* of two chains $\gamma' = (\mathrm{e}', w')$ and $\gamma'' = (\mathrm{e}'', w'')$ is defined whenever $\gamma''_0 = \gamma'_1$; it is $\gamma' \cdot \gamma'' := (\mathrm{e}', w'w'')$, where $w'w''$ is the concatenation.

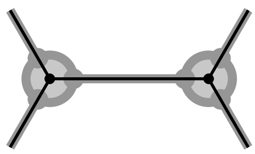


FIGURE 1. A 3-skeleton Sk (black), auxiliary graph Sk° (bold grey), and space $\mathrm{Sk}^\bullet$ deformation equivalent to Sk (bold and light grey)

2.4.2. Remark. Intuitively, our definition of chain represents the fact that, at each point $\mathrm{e} \in \mathcal{E}_{\mathrm{Sk}}$, one can choose among three directions: following the edge or walking around the vertex preserving or reversing the cyclic order. The inverse in the definition of γ_1 is due to the fact that the action of Γ is left rather than right, hence the order of the elements of w should be reversed. (This is also one of the reasons why $\mathbb{X}$ is defined to act *via* nx^{-1}.) Strictly speaking, what is defined is a combinatorial path (a chain of consecutive edges) in the auxiliary graph Sk° obtained from Sk by shortening each edge and replacing each vertex with a small circle (shown in bold grey lines in Figure 1). The vertices of Sk° are in a natural one-to-one correspondence with the elements of $\mathcal{E}_{\mathrm{Sk}}$. When speaking about path homotopies, fundamental groups, *etc.*, we replace Sk° with the topological space $\mathrm{Sk}^\bullet$ obtained from Sk° by patching each circle with a disk (light grey in the figure) and consider the homomorphisms induced by the inclusion $\mathrm{Sk}^\circ \hookrightarrow \mathrm{Sk}^\bullet$ and the strict deformation retraction $\mathrm{Sk}^\bullet \twoheadrightarrow \mathrm{Sk}$.

Clearly, each chain (e, w) gives rise to a path (in the conventional topological sense) in the geometric realization: for example, one can divide the unit segment into $|w|$ equal pieces and map each piece constantly onto the corresponding vertex (for each instance of $\mathrm{nx}^{\pm 1}$ in w) or linearly onto the corresponding edge (for each instance of op). The resulting path connects two vertices of Sk and is equipped with a distinguished marking (edge end) at each of its endpoints. Similarly, (e, w) gives rise to a topological path in the auxiliary skeleton Sk°.

The following two observations are also straightforward.

2.4.3. Lemma. *A chain γ is a loop if and only* $\mathrm{val}\,\gamma \in \mathrm{Stab}\,\gamma_0$. *Conversely, given* $\mathrm{e} \in \mathcal{E}_{\mathrm{Sk}}$, *any element of* $\mathrm{Stab}(\mathrm{e})$ *has the form* $\mathrm{val}\,\gamma$ *for some loop* $\gamma = (\mathrm{e}, w)$. □

2.4.4. Lemma. *Evaluation is multiplicative*: $\mathrm{val}(\gamma_1 \cdot \gamma_2) = \mathrm{val}\,\gamma_1\,\mathrm{val}\,\gamma_2$. □

2.4.5. Theorem. *Given a based* 3*-skeleton* (Sk, e)*, the evaluation map restricts to a well defined isomorphism* $\mathrm{val}\colon \pi_1(\mathrm{Sk}, \mathrm{e}) \to \mathrm{Stab}(\mathrm{e})$.

Proof. Due to Lemmas 2.4.3 and 2.4.4, it suffices to show that val is well defined (*i.e.*, it takes equal values on homotopic loops) and $\mathrm{Ker}\,\mathrm{val} = \{1\}$. Both statements follow from comparing the cancellations in $\pi_1(\mathrm{Sk}, \mathrm{e})$ and in Γ.

Since $\Gamma = \mathbb{Z}_3 * \mathbb{Z}_2$ is a free product, two words in $\{\mathbb{Y}, \mathbb{X}, \mathbb{X}^{-1}\}$ represent the same element of Γ if and only if they are obtained from each other by a sequence of cancellations of subwords of the form $\mathbb{Y}\mathbb{Y}$, $\mathbb{X}\mathbb{X}^{-1}$, $\mathbb{X}^{-1}\mathbb{X}$, $\mathbb{X}\mathbb{X}\mathbb{X}$, or $\mathbb{X}^{-1}\mathbb{X}^{-1}\mathbb{X}^{-1}$. The first three cancellations constitute the combinatorial definition of path homotopy in the auxiliary graph Sk°, see Remark 2.4.2: they correspond to cancelling an edge immediately followed by its inverse. The last two cancellations normally generate the kernel of the inclusion homomorphism $\pi_1(\mathrm{Sk}^\circ, \mathrm{e}) \to \pi_1(\mathrm{Sk}^\bullet, \mathrm{e})$: they correspond to contracting circles in $\mathrm{Sk}^\circ \subset \mathrm{Sk}^\bullet$ to vertices of the original 3-skeleton Sk.

An alternative proof of the fact that val is well defined is given by Lemma 2.5.1 below, which provides an invariant geometric description of this map. □

2.4.6. Corollary. *Any geometric subgroup* $H \subset \Gamma$ *(respectively, any geometric subgroup* $\tilde{H} \subset \tilde{\Gamma}$*) has index divisible by six,* $[\Gamma : H] = 6k$ *(respectively, divisible by twelve,* $[\tilde{\Gamma} : \tilde{H}] = 12k$*) and is isomorphic to a free group on* $(k+1)$ *generators.*

Proof. Let $\mathrm{Sk} = \Gamma/H$, see Theorem 2.3.5. Then $[\Gamma : H] = |\mathcal{E}_{\mathrm{Sk}}|$. On the other hand, since Sk is a 3-regular graph, one has $|\mathcal{E}_{\mathrm{Sk}}| = 6k$ and Sk has $2k$ vertices and $3k$ edges. Then $\chi(\mathrm{Sk}) = -k$ and $\pi_1(\mathrm{Sk})$ is a free group on $(k+1)$ generators.

If $\tilde{H} \subset \tilde{\Gamma}$ is a geometric subgroup, then $\tilde{H} \not\ni -\mathrm{id}$ and the projection $\tilde{H} \to \Gamma$ is an isomorphism onto its image, which is a geometric subgroup of Γ. □

2.4.7. Remark. The universal covering of a 3-skeleton Sk is a 3-regular tree; hence it is the Farey tree. The automorphism group $\mathrm{Aut}\,F$ of the Farey tree F can be identified with Γ: it is generated by the rotations about a vertex or the center of an edge. Thus, geometrically, $\mathrm{Sk} = F/H$ for a finite index subgroup $H \subset \mathrm{Aut}\,F$ acting freely on F, and Theorem 2.4.5 becomes a well known property of topological coverings. If the action of H on F is not free, one needs to consider the orbifold fundamental group $\pi_1^{\mathrm{orb}}(F/H)$, see Subsection 3.2 below. If $[\Gamma : H] = \infty$, the quotient F/H is an infinite graph, see Subsections 3.1 and 3.6.

2.5. The homological invariant. Fix a Jacobian elliptic surface $\mathrm{pr}\colon X \to B$ without exceptional fibers and let $\mathrm{Sk} = \mathrm{Sk}_X$ be the skeleton of X. Assume that Sk is generic, hence 3-regular. Below, we treat Sk as its geometric realization, thus using the term (edge) ends for the elements of $\mathcal{E}_{\mathrm{Sk}}$.

Consider the double covering $X \to \Sigma$ ramified at $C \cup E$, see 2.2.1. Pick a vertex v of Sk, let F_v be the fiber of X over v, and let $\bar{F}_v$ be its projection to Σ. Then, F_v is the double covering of $\bar{F}_v$ ramified at $\bar{F}_v \cap (C \cup E)$ (the three black points in Figure 2 and ∞).

Recall that the three points of intersection $\bar{F}_v \cap C$ are in a canonical one-to-one correspondence with the three edge ends at v, see [12]. Choose one of the ends (a *marking* at v in the terminology of [12]) and let $\{\alpha_1, \alpha_2, \alpha_3\}$ be the canonical basis for the group $\pi_1(\bar{F}_v \smallsetminus (C \cup E))$ defined by this end (see [12] and Figure 2; unlike [12], we take for the reference point the zero section of Σ, which is well defined in the presence of C; this choice removes the ambiguity in the definition of canonical basis). Then $H_1(F_v) = \pi_1(F_v)$ is generated by the lifts $a = \alpha_2\alpha_1$ and

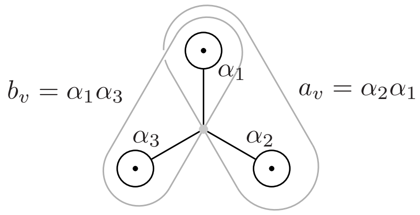


FIGURE 2. The basis in $H_1(F_v)$

$b = \alpha_1\alpha_3$ (the two grey cycles in the figure). To be precise, one needs to choose one of the two pull-backs of the zero section and take it for the reference point for $\pi_1(F_v)$ (the grey point at the center of the figure). Thus, a choice of an end at v gives rise to an isometry $H_1(F_v) = \mathcal{H}$, which is canonical up to $\pm\,\mathrm{id}$.

Now, consider a copy F_{e} of F_v for each end e at v and identify its homology with $\mathcal{H}$ using e as the marker. (Alternatively, one can assume that a separate fiber is chosen over each vertex of the auxiliary graph Sk°, see Remark 2.4.2.) Under this identification, the monodromy $\tilde{\mathfrak{h}}_\gamma\colon H_1(F_{\gamma_0}) \to H_1(F_{\gamma_1})$ of the locally trivial fibration $\mathrm{pr}^{-1}(\mathrm{Sk}) \to \mathrm{Sk}$ along the path defined by a chain γ in Sk reduces to a well defined element $\mathfrak{h}_\gamma \in \Gamma$.

2.5.1. Lemma. *In the notation above, one has* $\mathfrak{h}_\gamma = (\mathrm{val}\,\gamma)^{-1}$.

Proof. Since both maps $\gamma \mapsto \mathfrak{h}_\gamma$ and $\gamma \mapsto (\mathrm{val}\,\gamma)^{-1}$ reverse products, it suffices to prove the assertion for a chain $\gamma = (\mathrm{e}, w)$ with $w = \mathrm{op}$ or $\mathrm{nx}^{\pm 1}$, *i.e.*, for a single edge of Sk°.

Circumventing a vertex of the original skeleton Sk in the positive direction is the change of basis induced by a change of the marker (rotation through $-2\pi/3$ about the center in Figure 2); its transition matrix is $\mathbb{X}^{-1} = (\mathrm{val}\ \mathrm{nx})^{-1}$. Following an edge of Sk is a lift of the monodromy $m_{1,1}$ in [12]: during the monodromy, the black ramification point surrounded by α_1 crosses the segment connecting the ramification points surrounded by α_2 and α_3; modulo $\pm\,\mathrm{id}$, the corresponding linear operator is given by $\mathbb{Y} = (\mathrm{val}\ \mathrm{op})^{-1}$. ☐

Let v be a vertex of Sk and let $\mathrm{e} \in v$. We will use the notation $\pi_1(B^\sharp, \mathrm{e})$ for the group $\pi_1(B^\sharp, v)$, meaning that the fiber F_v is identified with $\mathcal{H}$ using e as a marker. Thus, we will speak about the reduced monodromy $\mathfrak{h}_X\colon \pi_1(B^\sharp, \mathrm{e}) \to \Gamma$.

2.5.2. Theorem. *Let X be an extremal elliptic surface without exceptional fibers, and let* e *be a representative of a vertex of* Sk_X. *Then the reduced monodromy* $\mathfrak{h}_X\colon \pi_1(B^\sharp, \mathrm{e}) \to \Gamma$ *takes values in* $\mathrm{Stab}(\mathrm{e})$, *both maps in the diagram*

$$\pi_1(\mathrm{Sk}_X, \mathrm{e}) \xrightarrow{\mathrm{in}_*} \pi_1(B^\sharp, \mathrm{e}) \xrightarrow{\mathfrak{h}_X} \mathrm{Stab}(\mathrm{e}) \subset \Gamma$$

are (anti-)isomorphisms, and the composed map is given by $\gamma \mapsto (\mathrm{val}\,\gamma)^{-1}$.

Proof. Since Sk_X is a strict deformation retract of $B^\sharp$, see 2.2.6, the inclusion homomorphism $\mathrm{in}_*\colon \pi_1(\mathrm{Sk}_X) \to \pi_1(B^\sharp)$ is an isomorphism. The rest follows from Lemma 2.5.1 and Theorem 2.4.5. ☐

2.5.3. Theorem. *The map $X \to [\![\operatorname{Im}\tilde{\mathfrak{h}}_X]\!]$ establishes a bijection between the set of isomorphism classes of extremal elliptic surfaces without exceptional fibers and the set of conjugacy classes of geometric subgroups of $\tilde{\Gamma}$.*

Proof. It suffices to show that a subgroup $\tilde{H} \subset \tilde{\Gamma}$ defines a unique extremal elliptic surface. Since $\tilde{H}$ is geometric, in particular $-\operatorname{id} \notin \tilde{H}$, the projection $\tilde{\Gamma} \to \Gamma$ induces an isomorphism of $\tilde{H}$ to a geometric subgroup $H \subset \Gamma$. The latter determines a skeleton $\operatorname{Sk} \subset B$, hence a j-invariant $j_X : B \to \mathbb{P}^1$ and corresponding reduced monodromy $\mathfrak{h}_X : \pi_1(B^\sharp) \to H$. Then, the inverse isomorphism $H \to \tilde{H}$ is merely a lift of $\mathfrak{h}_X$ to a homological invariant $\tilde{\mathfrak{h}}_X$; together with j_X, it defines a unique isomorphism class of Jacobian elliptic surfaces, which are necessarily extremal due to [29], see 2.2.3. □

Since the conjugacy class of the monodromy group of a fibration is obviously invariant under fiberwise homeomorphisms, Theorem 2.5.3 implies Theorem 1.3.1 in the introduction.

2.5.4. Remark. One can easily see that two extremal elliptic surfaces without exceptional singular fibers are anti-isomorphic if and only if their monodromy subgroups are conjugated by an element of $GL(2,\mathbb{Z}) \smallsetminus \tilde{\Gamma}$. (This conjugation results in a homeomorphism of the skeletons reversing the cyclic order at each vertex.) In other words, surfaces are anti-isomorphic if and only if they are related by a 2-orientation reversing homeomorphism.

2.5.5. Remark. The inverse map sending a geometric subgroup $H \subset \tilde{\Gamma}$ to an extremal elliptic surface in Theorem 2.5.3 is equivalent to Shioda's construction [34] of modular elliptic surfaces, where the base B of the elliptic fibration is obtained as the quotient $\{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}/H$ and the j-invariant j_X is the descent of the modular j-invariant. A generalization of the results of this section to arbitrary finite index subgroups of Γ is considered in Subsections 3.2 and 3.3, see Remark 3.3.4; such subgroups correspond to skeletons with monovalent $\bullet$- and $\circ$-vertices allowed. For a further generalization to arbitrary subgroups, see Subsections 3.1 and 3.6; finitely generated subgroups can still be encoded by finite ribbon graphs.

2.5.6. Remark. In [16] it is shown that, for an extremal elliptic surface X without exceptional singular fibers, the homological invariant $\tilde{\mathfrak{h}}_X$ admits a simple geometric description in terms of an orientation of Sk_X: one defines the value $\tilde{\mathfrak{h}}_X(\gamma)$ on a loop γ in Sk_X to be $\pm(\operatorname{val}\gamma)^{-1} \in \tilde{\Gamma}$, depending on the parity of the number of edges travelled by γ in the opposite direction. This correspondence is not one-to-one, as distinct orientations may give rise to the same homological invariant.

3. Digression: a few generalizations

In this section, we generalize some results of Section 2 to arbitrary subgroups of Γ: finitely generated subgroups can still be encoded by finite graphs. Proofs are merely sketched, as they repeat, almost literally, those in Section 2. The material of this section is not used in the proofs of the principal results of the paper stated in the introduction. However, Subsection 3.1 is used in the construction of nonequivalent monodromy factorization of length two, see Subsection 5.4.

3.1. Infinite skeletons. In order to study subgroups of Γ of infinite index, we modify Definition 2.3.1 and define a *generalized 3-regular ribbon graph* as a triple

$\mathrm{Sk} = (\mathcal{E}_{\mathrm{Sk}}, \mathrm{op}, \mathrm{nx})$, where $\mathcal{E}_{\mathrm{Sk}}$ is a set (not necessarily finite) and op and nx are free automorphisms of $\mathcal{E}_{\mathrm{Sk}}$ of order 2 and 3, respectively. A *generalized* 3-*skeleton* is a connected generalized 3-regular ribbon graph.

All notions introduced in Subsections 2.3 and 2.4 and most statements proved there extend to the general case with obvious changes. We restate Theorems 2.3.5 and 2.4.5.

3.1.1. Theorem. *The functors* $(\mathrm{Sk}, \mathrm{e}) \mapsto \mathrm{Stab}(\mathrm{e})$*,* $H \mapsto (\Gamma/H, H/H)$ *establish an equivalence of the categories of*

- *based generalized* 3*-skeletons and morphisms and*
- *torsion free subgroups* $H \subset \Gamma$ *and inclusions.* □

3.1.2. Theorem. *Given a based generalized* 3*-skeleton* $(\mathrm{Sk}, \mathrm{e})$*, the evaluation map restricts to a well defined isomorphism* $\mathrm{val}\colon \pi_1(\mathrm{Sk}, \mathrm{e}) \to \mathrm{Stab}(\mathrm{e})$*.* □

A generalized 3-skeleton Sk is called *almost contractible* if the group $\pi_1(\mathrm{Sk})$ is finitely generated. Under Theorem 3.1.1, almost contractible skeletons correspond to finitely generated torsion free subgroups.

3.1.3. Proposition. *There is a one-to-one correspondence between the sets of*

1. *conjugacy classes of proper finitely generated torsion free subgroups* $H \subset \Gamma$*,*
2. *almost contractible* 3*-skeletons with at least one cycle, and*
3. *connected finite ribbon graphs with all vertices of valency* 3 *or* 1 *and such that distinct monovalent vertices are adjacent to distinct trivalent vertices.*

Under this correspondence $H \leftrightarrow \mathrm{Sk} \leftrightarrow \mathrm{Sk}^c$ *one has (anti-)isomorphisms* $N(H)/H = \mathrm{Aut}\,\mathrm{Sk} = \mathrm{Aut}\,\mathrm{Sk}^c$ *and* $H = \pi_1(\mathrm{Sk}) = \pi_1(\mathrm{Sk}^c)$*; in fact,* Sk^c *is embedded to* Sk *as an induced subgraph and a strict deformation retract.*

The finite ribbon graph Sk^c corresponding to an almost contractible 3-skeleton Sk under Proposition 3.1.3 is called the *compact part* of Sk. In the drawings, the monovalent vertices of Sk^c (those that are to be extended to 'half' Farey trees) are represented by triangles $\vartriangle$, *cf.* Figure 8 in Subsection 5.4. The last condition in 3.1.3(3) is the requirement that Sk^c should admit no further contraction to a subgraph with all vertices of valency 3 or 1. This condition makes Sk^c canonical.

Proof. Each almost contractible 3-skeleton Sk contains an induced subgraph Sk' such that $\mathrm{Sk} \smallsetminus \mathrm{Sk}'$ is a forest: one can pick a finite collection of loops representing a basis for $\pi_1(\mathrm{Sk})$ and take for Sk' the induced subgraph generated by all vertices contained in at least one of the loops. (The notation $\mathrm{Sk} \smallsetminus \mathrm{Sk}'$ stands for the induced subgraph generated by the vertices of Sk that are not in Sk'.) The complement $\mathrm{Sk} \smallsetminus \mathrm{Sk}'$ is a finite disjoint union of *infinite branches*, each infinite branch being a tree with one bivalent vertex and all other vertices trivalent. Unless Sk is the Farey tree itself (corresponding to the trivial subgroup of Γ), each infinite branch is contained in a unique maximal one. The maximal infinite branches are pairwise disjoint, and contracting each such branch to its only bivalent vertex produces the compact part Sk^c as in the statement, the monovalent vertices of Sk^c corresponding to the maximal infinite branches contracted. (The last condition in 3.1.3(3) is due to the fact that, if two monovalent vertices u_1, u_2 were adjacent to the same vertex v then, together with v, the two infinite branches represented by u_1 and u_2 would form a larger infinite branch.)

Since the construction is canonical, any automorphism of Sk preserves Sk^c and hence restricts to an automorphism of Sk^c. Conversely, any automorphism of Sk^c

extends to a unique automorphism of Sk: the uniqueness is due to the fact that *ribbon* graphs are considered; once an automorphism of such a graph fixes a vertex v and an edge adjacent to v, it is the identity. □

3.2. Skeletons with monovalent vertices. As another generalization, we lift the requirement that op and nx should be free and define a $(3,1)$-*ribbon graph* as a triple $\mathrm{Sk} = (\mathcal{E}_{\mathrm{Sk}}, \mathrm{op}, \mathrm{nx})$, where $\mathcal{E}_{\mathrm{Sk}}$ is a finite set and op and nx are automorphisms of $\mathcal{E}_{\mathrm{Sk}}$ of order 2 and 3, respectively. A $(3,1)$-*skeleton* is a connected $(3,1)$-ribbon graph. Thus, a $(3,1)$-skeleton is allowed to have monovalent •-vertices (which are the one element orbits of nx) and 'hanging edges' (one element orbits of op); the latter are represented in the figures by monovalent ◦-vertices attached to these edges, *cf.* Figure 3 below.

As above, all notions introduced in Subsections 2.3 and 2.4 extend to the case of $(3,1)$-skeletons. Theorem 2.3.5 takes the following form.

3.2.1. Theorem. *The functors* $(\mathrm{Sk}, \mathrm{e}) \mapsto \mathrm{Stab}(\mathrm{e})$, $H \mapsto (\Gamma/H, H/H)$ *establish an equivalence of the categories of*

- *based* $(3,1)$-*skeletons and morphisms and*
- *finite index subgroups* $H \subset \Gamma$ *and inclusions.* □

3.2.2. Denote by $D^2_1 \cong D^2$, $D^2_2 \cong \mathbb{P}^1_{\mathbb{R}}$, and D^2_3 the *CW*-complexes obtained by attaching a single 2-cell D^2 to a circle S^1 *via* a map $\partial D^2 \to S^1$ of degree 1, 2, or 3, respectively. Given $\mathrm{e} \in \mathcal{E}_{\mathrm{Sk}}$, define the *orbifold fundamental group* $\pi_1^{\mathrm{orb}}(\mathrm{Sk}, \mathrm{e})$ as the fundamental group $\pi_1(\mathrm{Sk}^\bullet, \mathrm{e})$, where the space $\mathrm{Sk}^\bullet$ is obtained from Sk by replacing a neighborhood of each trivalent •-vertex, monovalent ◦-vertex, or monovalent •-vertex with a copy of D^2_1, D^2_2, or D^2_3, respectively, *cf.* Figure 3. (Note that $\pi_1^{\mathrm{orb}}(\mathrm{Sk}, \mathrm{e})$ is indeed the orbifold fundamental group, with the orbifold structure given by declaring each monovalent ◦- or •-vertex a ramification point of ramification index 2 or 3, respectively. With this convention, the universal covering of Sk is again the Farey tree, *cf.* Remark 2.4.7.) Contracting a maximal tree not containing a monovalent vertex, one establishes a homotopy equivalence between $\mathrm{Sk}^\bullet$ and a wedge of circles and copies of D^2_2 and D^2_3. Hence, $\pi_1^{\mathrm{orb}}(\mathrm{Sk}, \mathrm{e})$ is a free product

$$\pi_1^{\mathrm{orb}}(\mathrm{Sk}, \mathrm{e}) = \circledast_{n_0}\mathbb{Z} * \circledast_{n_2}\mathbb{Z}_2 * \circledast_{n_3}\mathbb{Z}_3, \tag{3.2.3}$$

where n_2 and n_3 are the numbers of monovalent ◦- and •-vertices, respectively, and $n_0 = 1 - \chi(\mathrm{Sk}) = 1 - \chi(\mathrm{Sk}^\bullet)$. Observe that $|\mathcal{E}_{\mathrm{Sk}}| = 6n_0 + 3n_2 + 4n_3 - 6$ (a simple combinatorial computation of the Euler characteristic).

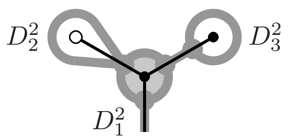


Figure 3. A $(3,1)$-skeleton Sk (black), auxiliary graph Sk° (bold grey), and space $\mathrm{Sk}^\bullet$ (bold and light grey)

Definition 2.4.1 of chains, loops, and the evaluation map extends literally to the case of $(3,1)$-skeletons. Thus, we are speaking about combinatorial paths in the auxiliary graph Sk° obtained by fattening the vertices of Sk as shown in Figure 3. (Note though that we disregard the direction of a path along the single edge replacing a $\circ$-vertex and the adjacent edge of Sk.) It is straightforward that $\pi_1^{\mathrm{orb}}(\mathrm{Sk})$ can be defined as the group of loops modulo an appropriate equivalence relation. Next statement is proved similar to Theorem 2.4.5.

3.2.4. Theorem. *Given a based* $(3,1)$*-skeleton* (Sk,e), *the evaluation map* val *factors through a well defined isomorphism* $\mathrm{val}\colon \pi_1^{\mathrm{orb}}(\mathrm{Sk},\mathrm{e}) \to \mathrm{Stab}(\mathrm{e})$. □

3.2.5. Corollary. *Any finite index subgroup* $H \subset \Gamma$ *is a free product* (3.2.3), *and one has* $[\Gamma : H] = 6n_0 + 3n_2 + 4n_3 - 6$. □

3.3. Extremal elliptic surfaces without type II^* **fibers.** Using the concept of $(3,1)$-skeleton introduced in the previous section and the description of the braid monodromy of the branch locus found in [12] (the monodromy $l_1(2) \mapsto \mathbb{Y}\mathbb{X}^{-1}\mathbb{Y}$ and $l_1(3) \mapsto \mathbb{Y}$ for monovalent $\bullet$- and $\circ$-vertices, respectively; as in Subsection 2.5, the homomorphism $\mathbb{B}_3 \to \Gamma$ is given by (5.1.1) below), one arrives at the following generalization of Theorem 2.5.2.

3.3.1. Theorem. *Let* X *be an extremal elliptic surface without type* II^* *fibers, and let* $\mathrm{e} \in \mathcal{E}$ *be a representative of a vertex of the skeleton* Sk_X. *Then the reduced monodromy* $\mathfrak{h}_X\colon \pi_1(B^\sharp,\mathrm{e}) \to \Gamma$ *factors as follows*:

$$\pi_1(B^\sharp,\mathrm{e}) \twoheadrightarrow \pi_1^{\mathrm{orb}}(\mathrm{Sk}_X,\mathrm{e}) \xrightarrow{\cong} \mathrm{Stab}(\mathrm{e}) \subset \Gamma$$

where the rightmost anti-isomorphism is the map $\gamma \mapsto (\mathrm{val}\,\gamma)^{-1}$. □

3.3.2. Remark. In the presence of monovalent vertices, Sk_X is no longer a subspace of $B^\sharp$. The first arrow in Theorem 3.3.1 is the composition of the homomorphisms induced by the strict deformation retraction $B^\sharp \to \mathrm{Sk}'$ and the inclusion $\mathrm{Sk}' \hookrightarrow \mathrm{Sk}^\bullet$, where Sk' is obtained from Sk°, see Figure 3, by patching with disks the circles surrounding the *trivalent* $\bullet$-vertices only.

3.3.3. Corollary. *The map* $X \to [\![\mathrm{Im}\,\tilde{\mathfrak{h}}_X]\!]$ *establishes a bijection between the set of isomorphism classes of extremal elliptic surfaces without type* II^* *or* III^* *fibers and the set of conjugacy classes of finite index subgroups* $\tilde{H} \subset \tilde{\Gamma}$ *such that* $-\mathrm{id} \notin \tilde{H}$.

Proof. Let X be a surface as in the statement, let $\tilde{H} = \mathrm{Im}\,\tilde{\mathfrak{h}}_X \subset \tilde{\Gamma}$ (with respect to some base point in $B^\sharp$), and let $H = \mathrm{Im}\,\mathfrak{h}_X \subset \Gamma$ be the projection of $\tilde{H}$ to Γ. Under the assumptions, Sk_X has no $\circ$-vertices and hence $\pi_1^{\mathrm{orb}}(\mathrm{Sk}_X) = H$ is a free product of copies of $\mathbb{Z}$ and $\mathbb{Z}_3$ only. Furthermore, each order 3 generator of H represents the monodromy about a type IV^* singular fiber of X, see 2.2.3(3), and hence lifts to an order 3 element of $\tilde{H}$. Thus, the projection $\tilde{H} \to H$ admits a section and hence is an isomorphism. The rest of the proof follows that of Theorem 2.5.3. □

3.3.4. Remark. Corollary 3.3.3 covers Shioda's construction [34] to full extent and generalizes Theorem 1.3.1 to surfaces with type IV^* fibers allowed. Apparently, considering the homological invariant itself rather than just its image, one can further generalize Theorem 1.3.1 to type III^* singular fibers. The special case of rational base is considered in Theorem 3.4.1 below.

3.3.5. Remark. Surprisingly, type II^* singular fibers do not fit into the approach of this paper at all, as they are represented by bivalent •-vertices of the skeleton, *i.e.*, orbits of $\underset{\sim}{\mathrm{nx}}$ of length two. Possibly, such skeletons can be treated as homogeneous spaces of $\tilde\Gamma$ rather than Γ, but the precise statements are not quite clear at the moment. An attempt of considering such more general skeletons is made in [16].

3.4. The case of rational base. In this subsection, we assume that the base B of an elliptic fibration $X \to B$ is rational, $B \cong \mathbb{P}^1$. In this case, the homological invariant $\tilde{\mathfrak{h}}_X$ (lifting a given reduced monodromy $\mathfrak{h}_X$) can be defined in terms of a *type specification* of X, *i.e.*, a choice of one of the two possible types (whose local monodromies differ by $-\mathrm{id}$) of each singular fiber. Moreover, the types of all but one singular fibers can be chosen arbitrary, whereas the type of the remaining fiber is determined by the requirement that the total multiplicity of all singular fibers, which equals the topological Euler characteristic $\chi(X)$, should be divisible by 12. (The multiplicities of the two lifts of a given element of Γ differ by 6, *cf.* 5.1.2.)

If X is extremal and has no type II^* singular fibers, its type specification can be described in terms of the reduced monodromy group $H = \operatorname{Im}\mathfrak{h}_X$. Indeed, in view of condition 2.2.3(3), the types of the exceptional fibers of X are fixed. The non-exceptional singular fibers are in a one-to-one correspondence with the regions of Sk_X, equivalently, with the orbits of $\mathbb{XY}$, equivalently, with the H-conjugacy classes of maximal unipotent subgroups of H, and a type specification consists in assigning a lift $\langle \pm g^{-1}(\mathbb{XY})^n g\rangle \subset \tilde\Gamma$ to each such conjugacy class $[\![\langle g^{-1}(\mathbb{XY})^n g\rangle]\!]_H$.

3.4.1. Theorem. *Two extremal elliptic surfaces X_1, X_2 over the rational base $B = \mathbb{P}^1$ and without type II^* singular fibers are isomorphic if and only if they are related by a 2-orientation preserving fiberwise homeomorphism.*

Proof. The 'only if' part is obvious. For the 'if' part, it suffices to notice that a 2-orientation preserving homeomorphism $X_1 \to X_2$ induces an orientation preserving homeomorphism $B_1 \to B_2$ taking punctures to punctures, commuting with the homological invariants $\pi_1(B_1^\sharp) \to \Gamma \leftarrow \pi_1(B_2^\sharp)$ (and hence taking H_1 to H_2) and preserving the type specification (as distinct types of singular elliptic fibers differ topologically, for example by the local monodromy). Hence, X_1 and X_2 are isomorphic. □

3.4.2. Remark. The extremality condition in Theorem 3.4.1 can be relaxed by replacing 2.2.3(3) by the requirement that the surface should have no singular fibers of type I_0^*, II^*, or IV. In this case, a type specification would also choose a lift $\langle \pm g^{-1}\mathbb{X}g\rangle$ for each conjugacy class $[\![\langle g^{-1}\mathbb{X}g\rangle]\!]_H$ of order 3 subgroups of H (monovalent •-vertices) and a lift $\langle \pm g^{-1}\mathbb{Y}g\rangle$ for each conjugacy class $[\![\langle g^{-1}\mathbb{Y}g\rangle]\!]_H$ of order 2 subgroups of H (monovalent ∘-vertices).

3.4.3. Remark. The combinatorial type of singular fibers of an extremal (or more general as in Remark 3.4.2) elliptic surface X is determined by its type specification and the following combinatorial information about its skeleton Sk_X: the numbers of monovalent •- and ∘-vertices and the shapes of the regions of Sk_X. Each monovalent •- (respectively, ∘-) vertex gives rise to a singular fiber of type II or IV^* (respectively, III or III^*), and each n-gonal region gives rise to a singular fiber of type I_n or I_n^*. There are large numbers of skeletons sharing these data; some examples are considered in Subsections 4.3, 4.5, and 5.6 below.

3.5. The monodromy group of an elliptic surface. For an elliptic surface X, introduce the following fiber counts:

- $n_{\rm II}$ is the number of fibers of type II or IV*;
- $n_{\rm III}$ is the number of fibers of type III or III*;
- $n_{\rm IV}$ is the number of fibers of type IV or II*;
- t is the number of fibers of type $\rm I_p^*$, $p \geqslant 0$, II*, III*, or IV*.

Let, further, $\chi(X)$ be the topological Euler characteristic of X.

3.5.1. Theorem. *Let X be an extremal elliptic surface without type* II* *singular fibers. Then the reduced monodromy group* $\operatorname{Im}\mathfrak{h}_X \subset \Gamma$ *is a subgroup of index* $\chi(X) - 6t - 2n_{\rm II} - 3n_{\rm III}$ *isomorphic to the free product*

$$\circledast_n \mathbb{Z} * \circledast_{n_{\rm III}} \mathbb{Z}_2 * \circledast_{n_{\rm II}} \mathbb{Z}_3,$$

where $n = \frac{1}{6}\chi(X) - t - n_{\rm II} - n_{\rm III} + 1$.

Proof. The statement follows from Theorem 3.3.1, Corollary 3.2.5, and the fact that

$$\chi(X) = |\mathcal{E}_{\rm Sk}| + 6t + 2n_{\rm II} + 3n_{\rm III} + 4n_{\rm IV}, \tag{3.5.2}$$

where $\rm Sk = Sk_X$. (Here, we admit skeletons with bivalent $\bullet$-vertices as well.) For the latter, observe that $\chi(X)$ equals the total multiplicity of the singular fibers of X. Exceptional singular fibers are accounted for by the mono- and bivalent $\bullet$-vertices and monovalent $\circ$-vertices of Sk. Besides, there is one fiber of type $\rm I_p$ or $\rm I_p^*$ inside each p-gonal region of Sk. The sum of all indices p is the total number of corners of all regions of Sk, *i.e.*, $|\mathcal{E}_{\rm Sk}|$. Finally, each $*$-type fiber increases the total multiplicity by 6. □

3.5.3. Theorem. *Let X be a non-isotrivial elliptic surface without type* II* *or* IV *singular fibers. Then the index of the reduced monodromy group* $\operatorname{Im}\mathfrak{h}_X \subset \Gamma$ *of X divides* $\chi(X) - 6t - 2n_{\rm II} - 3n_{\rm III}$. *In particular, it is finite.*

Proof. Let Sk be the skeleton of X. After a fiberwise equisingular deformation of X, not necessarily small, one can assume that Sk is generic and connected. (For the modifications of skeletons resulting in deformations of surfaces, see [12] or [17].) Hence Sk is a $(3,1)$-skeleton. This time, each region of Sk may contain several singular fibers of X. Hence, instead of Theorem 3.3.1, one has a diagram

$$\pi_1(B^\sharp, \mathrm{e}) \longleftrightarrow \pi_1(\mathrm{Sk}', \mathrm{e}) \twoheadrightarrow \pi_1^{\rm orb}(\mathrm{Sk}_X, \mathrm{e}) \xrightarrow{\cong} \mathrm{Stab}(\mathrm{e}) \subset \Gamma$$

(where Sk$'$ is the auxiliary space introduced in Remark 3.3.2) and an inclusion $\mathrm{Stab(e)} \subset \operatorname{Im}\mathfrak{h}_X$. It remains to observe that $[\Gamma : \mathrm{Stab(e)}] = |\mathcal{E}_{\rm Sk}|$ and that (3.5.2) holds for any non-isotrivial surface X. □

3.5.4. Remark. The reduced monodromy group $\operatorname{Im}\mathfrak{h}_X$ of an isotrivial elliptic surface X is either trivial or conjugate to the subgroup generated by $\mathbb{X}$ or $\mathbb{Y}$. In particular, $[\Gamma : \operatorname{Im}\mathfrak{h}_X] = \infty$. At present, I do not know whether the index of $\operatorname{Im}\mathfrak{h}_X$ is necessarily finite if X is a non-isotrivial surface with type II* or IV singular fibers.

3.6. Further generalizations. Combined, the constructions of Subsections 3.1 and 3.2 give rise to the notion of *generalized* (*i.e.*, possibly infinite) $(3,1)$-*skeleton.* Theorems 3.1.1 and 3.2.1 would combine to the following statement.

3.6.1. Theorem. *The functors* $(\mathrm{Sk}, \mathrm{e}) \mapsto \mathrm{Stab}(\mathrm{e})$, $H \mapsto (\Gamma/H, H/H)$ *establish an equivalence of the categories of*

- *based generalized* $(3,1)$*-skeletons and morphisms and*
- *subgroups* $H \subset \Gamma$ *and inclusions.* □

The orbifold fundamental group $\pi_1^{\mathrm{orb}}(\mathrm{Sk}, \mathrm{e})$ of a generalized $(3,1)$-skeleton Sk is defined as in 3.2.2, and Theorem 3.2.4 extends to this case literally. Since $\mathrm{Sk}^\bullet$ is still homotopy equivalent to a wedge of circles and copies of D_2^2 and D_3^2, one obtains the following corollary.

3.6.2. Corollary. *Any subgroup of* Γ *is a free product* (*possibly infinite*) *of copies of cyclic groups* $\mathbb{Z}$, $\mathbb{Z}_2$, *and* $\mathbb{Z}_3$. □

3.6.3. Under Theorem 3.6.1, finitely generated subgroups correspond to *almost contractible* $(3,1)$*-skeletons*, which are defined as those with finitely generated group $\pi_1^{\mathrm{orb}}(\mathrm{Sk})$. Following the proof of Proposition 3.1.3, one can easily show that any almost contractible $(3,1)$-skeleton Sk representing a finitely generated subgroup $H \subset \Gamma$, $H \neq \{1\}$ (so that Sk is not the Farey tree), admits a strict deformation retraction to a canonically defined finite induced subgraph $\mathrm{Sk}^c \subset \mathrm{Sk}$, called the *compact part* of Sk, with the following properties:

(1) all vertices of Sk^c are of valency 3 or 1;
(2) the monovalent vertices of Sk^c are divided into three types: ∘, •, or ▵ (the latter representing maximal infinite branches of Sk);
(3) distinct ▵-vertices are adjacent to distinct trivalent vertices.

Under this correspondence $H \leftrightarrow \mathrm{Sk} \leftrightarrow \mathrm{Sk}^c$ one has (anti-)isomorphisms $N(H)/H = \operatorname{Aut}\mathrm{Sk} = \operatorname{Aut}\mathrm{Sk}^c$ and $H = \pi_1^{\mathrm{orb}}(\mathrm{Sk}) = \pi_1^{\mathrm{orb}}(\mathrm{Sk}^c)$, where $\pi_1^{\mathrm{orb}}(\mathrm{Sk}^c)$ is defined similar to $\pi_1^{\mathrm{orb}}(\mathrm{Sk})$, as the fundamental group of the space $(\mathrm{Sk}^c)^\bullet$ obtained from Sk^c by replacing each monovalent ∘- or •-vertex with a copy of D_2^2 or D_3^2, respectively.

4. Pseudo-trees

Here, we introduce and count admissible trees and related 3-regular ribbon graphs, called pseudo-trees; they are the principal source of most exponentially large examples stated in the introduction.

4.1. Admissible trees and pseudo-trees. An embedded tree $\Xi \subset S^2$ is called *admissible* if all its vertices have valency 3 (*nodes*) or 1 (*leaves*). Two such trees are called *isomorphic* if they are related by an orientation preserving auto-homeomorphism of S^2. Each admissible tree Ξ gives rise to its *associated* 3*-skeleton* Sk_Ξ: its embedded geometric realization is obtained by attaching a small loop to each leaf of Ξ, see Figure 4, left, and the ribbon graph structure is induced from the embedding. A 3-skeleton obtained in this way is called a *pseudo-tree*. Clearly, each pseudo-tree is a skeleton of genus 0; two pseudo-trees $\mathrm{Sk}_{\Xi'}$ and $\mathrm{Sk}_{\Xi''}$ are isomorphic as ribbon graphs if and only if the trees Ξ' and Ξ'' are isomorphic.

An admissible tree has a certain number $k \geqslant 0$ of nodes and $(k+2)$ leaves. The number of isomorphism classes of admissible trees with k nodes is denoted by $T(k)$; it equals to the number of isomorphism classes of pseudo-trees with $(2k+2)$ vertices.

4.1.1. Remark. Certainly, instead of fixing a particular embedding $\Xi \subset S^2$, one can merely consider Ξ as a ribbon graph, see 2.2.4; we always assume that Ξ is equipped with the ribbon graph structure induced from S^2. Then Sk_Ξ is obtained

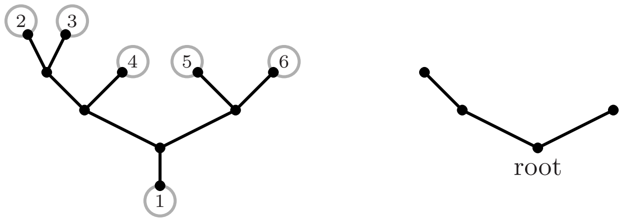


FIGURE 4. An admissible tree Ξ (black) and associated 3-skeleton Sk_Ξ (left); the related oriented rooted binary tree (right)

from Ξ by attaching a loop at each monovalent vertex and extending the ribbon graph structure; the latter extension is obviously unique up to isomorphism. The sole reason for considering embedded rather than abstract ribbon graphs is an attempt to make the exposition more geometric: drawing ribbon graphs of genus 0 in the plane and assuming the 'blackboard thickening'.

4.1.2. A *marking* of an admissible tree Ξ is a choice of one of its leaves v_1. Given a marking, one can number all leaves of Ξ consecutively, starting from v_1 and moving in the clockwise direction, *i.e.*, following left turn paths (see Figure 4, where the indices of the leaves are shown inside the loops). Declaring the node adjacent to v_1 the root and removing all leaves, one obtains an oriented rooted binary tree B with k vertices (see, *e.g.*, [23] for the related terminology). This procedure, intuitively clear from Figure 4, can be formally described as follows:

- orient the edges of Ξ upwards from v_1: an edge $[u', u'']$ is directed from u' to u'' if u' is closer to v_1 in Ξ;
- with this convention, each node v of Ξ has exactly one incoming edge e_1 and two outgoing edges e_2, e_3; if (e_1, e_2, e_3) is the cyclic order at v, declare e_2 and e_3 the right and left edges at v, respectively (and their other ends, the right and left children of v, respectively);
- let B be the induced subgraph of Ξ spanned by its nodes, retaining the orientation of the edges (the parent/child relation) and the left/right labels (the binary tree orientation of B); the only parentless node of B is its root.

Conversely, an oriented rooted binary tree B gives rise to a marked admissible tree:

- extend B to a proper binary tree by inserting all missing children (left and/ or right) at *each* vertex of B (node or leaf);
- attach an extra leaf v_1 at the root of B, directing the new edge from v_1 to the root;
- at each node (trivalent vertex) of the resulting tree Ξ, define the cyclic order of the edges as $(\{\text{incoming}\}, \{\text{right}\}, \{\text{left}\})$.

As a consequence, the number of isomorphism classes of marked admissible trees with k nodes is given by the Catalan number $C(k)$, see, *e.g.*, [11].

4.1.3. The *vertex distance* m_i between two consecutive leaves v_i, v_{i+1} of a marked admissible tree Ξ is the vertex length of the shortest left turn path in Ξ from v_i to v_{i+1}. For example, in Figure 4 one has $(m_1, m_2, m_3, m_4, m_5) = (5, 3, 4, 5, 3)$; for another example, see Figure 7 in Subsection 5.3. The vertex distance between two leaves v_i, v_j, $j > i$, is defined to be $\sum_{s=i}^{j-1} m_s$; it is the vertex length of the shortest left turn path connecting v_i to v_j *in the associated* 3*-skeleton* Sk_Ξ.

The vertex distance between two consecutive vertices is equal to the shortest distance in the tree; however, this is not true in general.

One can extend the sequence $(m_1, \ldots, m_{k+1})$ by appending the vertex distance m_{k+2} from v_{k+2} to v_1; then one has $m_1 + \ldots + m_{k+2} = 5k + 4$ (the number of edges in the boundary of the outer region: each of the $(2k + 1)$ edges of Ξ contributes to this number twice, and each of the $(k+2)$ loops, once). Two marked trees are isomorphic if and only if their sequences $(m_1, \ldots, m_{k+1})$ are equal. Two unmarked trees are isomorphic if and only if the corresponding extended sequences $(m_1, \ldots, m_{k+1}, m_{k+2})$ differ by a cyclic permutation. Note that *not* any sequence $(m_1, \ldots, m_{k+1})$ gives rise to a marked admissible tree, see [16] for a criterion.

4.2. Counts. As above, let $T(k)$ be the number of isomorphism classes of pseudo-trees with $(2k + 2)$ vertices. Let, further, $T_i(k)$, $i \geqslant 0$, be the number of classes of pseudo-trees Sk with $|\operatorname{Aut}\mathrm{Sk}| = i$.

For a pseudo-tree Sk with $(2k + 2)$ vertices, denote by $\mathfrak{O}_{\mathrm{Sk}}$ the orbit of $\mathbb{X}\mathbb{Y}$ corresponding to the outer $(5k+4)$-gonal region of Sk. The number of isomorphism classes of based 3-skeletons $(\mathrm{Sk}, \mathrm{e})$, where Sk is a pseudo-tree with $(2k+2)$ vertices and $\mathrm{e} \in \mathfrak{O}_{\mathrm{Sk}}$, is denoted by $\tilde{T}(k)$.

4.2.1. Lemma. *For a pseudo-tree* $\mathrm{Sk} = \mathrm{Sk}_\Xi$ *one has* $|\operatorname{Aut}\mathrm{Sk}| \leqslant 3$, i.e., $T_i(k) = 0$ *for* $i > 3$. *The numbers* $T_1(k)$, $T_2(k)$, $T_3(k)$ *are subject to the relations*

$$\sum_{i=1}^{3} \frac{T_i(k)}{i} = \frac{C(k)}{k+2},$$

$$T_2(k) = \begin{cases} C(k'), & \text{if } k = 2k', \\ 0, & \text{otherwise,} \end{cases} \qquad T_3(k) = \begin{cases} C(k'), & \text{if } k = 3k' + 1, \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, the group $\operatorname{Aut}\mathrm{Sk} = \operatorname{Aut}\Xi$ *acts freely on the set of leaves of the original tree* Ξ *and on the set* $\mathcal{E}_{\mathrm{Sk}}$ *of edge ends of* Sk.

Proof. Obviously, one has $\operatorname{Aut}\mathrm{Sk}_\Xi = \operatorname{Aut}\Xi$. Any combinatorial automorphism of Ξ is represented by a piecewise linear auto-homeomorphism $\varphi\colon \Xi \to \Xi$. Since Ξ is contractible, φ has a fixed point p, which is necessarily isolated (assuming that $\varphi \neq \mathrm{id}$), as an automorphism of a connected ribbon graph fixing an edge is the identity. If p is at the center of an edge of Ξ (respectively, p is a vertex of Ξ), then φ^2 (respectively, φ^3) fixes a whole edge of Ξ and thus is the identity.

FIGURE 5. An automorphism of an admissible tree

A tree Ξ with an automorphism φ is shown in Figure 5. It is clear that such a tree admits no automorphisms other than powers of φ: the fixed point q of such an automorphism would belong to one of the grey areas and the vertices of Ξ would be distributed unevenly about q. Let k' be the number of nodes of the subtree Ξ'

shown in the figure. In Figure 5, left ($|\operatorname{Aut}\Xi| = 2$), one has $k = 2k'$; in Figure 5, right ($|\operatorname{Aut}\Xi| = 3$), one has $k = 3k' + 1$. In each case, the trees Ξ admitting such an automorphism φ can be parameterized by the marked subtrees Ξ', distinguished being the leaf extending towards the fixed point of φ. Their number is $C(k')$, which proves the expressions for $T_2(k)$ and $T_3(k)$.

It is also clear from Figure 5 that a non-trivial automorphism does not fix a leaf of Ξ or an edge end of Sk. Then the first relation in the statement is the usual orbit count: a tree Ξ with $|\operatorname{Aut}\Xi| = i$ admits $(k+2)/i$ essentially distinct markings, and the total number of marked trees is $C(k)$. □

4.2.2. Corollary. *For each integer $k \geqslant 0$, one has*

$$T(k) = \frac{C(k)}{k+2} + \frac{T_2(k)}{2} + \frac{2T_3(k)}{3}, \qquad \tilde{T}(k) = \frac{5k+4}{k+2}C(k),$$

where $T_2(k)$ and $T_3(k)$ are given by Lemma 4.2.1.

Proof. Since $T_i(k) = 0$ for $i > 3$, the expression for $T(k) = T_1(k) + T_2(k) + T_3(k)$ follows directly from Lemma 4.2.1.

For each pseudo-tree Sk, one has $|\mathfrak{O}_{\mathrm{Sk}}| = 5k + 4$ and $\operatorname{Aut}\mathrm{Sk}$ acts freely on $\mathfrak{O}_{\mathrm{Sk}}$. Hence $\tilde{T}(k) = (5k+4)\sum_{i=1}^{3} T_i(k)/i = (5k+4)C(k)/(k+2)$ due to the first relation in Lemma 4.2.1. □

4.3. Proof of Theorem 1.3.2. The surfaces in question were constructed in [12]. Each surface X corresponds to a pseudo-tree Sk with $(2k + 2)$ vertices, with the type specification (see Subsection 3.4 and Remark 3.4.3) chosen so that the singular fiber of X inside each monogonal region of Sk should be of type I_1. The type of the singular fiber inside the remaining $(5k + 4)$-gonal region (the outer region in Figure 4, left) is then determined by the parity of k: it is of type I_{5k+4} if k is odd or I^*_{5k+4} if k is even.

The $T(k)$ distinct pseudo-trees with $(2k + 2)$ vertices give rise to $T(k)$ pairwise non-isomorphic extremal elliptic surfaces; Theorem 1.3.1 implies that they are not related by a 2-orientation preserving fiberwise homeomorphism. □

4.4. Digression: generalized pseudo-trees. The construction of Subsection 4.1 producing a 3-skeleton from a tree can be generalized. A function ℓ defined on the set of leaves of an admissible tree Ξ and taking values in $\{0, \circ, \bullet, \vartriangle\}$ is called *admissible* if no two leaves v_1, v_2 with $\ell(v_1) = \ell(v_2) = \vartriangle$ are adjacent to the same node. An *admissible pair* is a pair (Ξ, ℓ), where Ξ is an admissible tree and ℓ is an admissible function on the set of leaves of Ξ. Each admissible pair (Ξ, ℓ) gives rise to an (almost contractible) $(3, 1)$-skeleton $\mathrm{Sk}_{(\Xi,\ell)}$, whose compact part Sk^c is obtained from Ξ by attaching a small loop to each leaf v with $\ell(v) = 0$ and replacing each other leaf v with a monovalent vertex of type $\ell(v)$, *cf.* Figures 8 and 9 in Section 5. Thus, one has $\mathrm{Sk}_\Xi = \mathrm{Sk}_{(\Xi,0)}$. A generalized $(3, 1)$-skeleton obtained in this way is called a *generalized pseudo-tree.*

Clearly, two generalized pseudo-trees $\mathrm{Sk}_{(\Xi',\ell')}$ and $\mathrm{Sk}_{(\Xi'',\ell'')}$ are isomorphic if and only if so are pairs (Ξ', ℓ') and (Ξ'', ℓ''), *i.e.*, if there exists an isomorphism $\varphi\colon \Xi' \to \Xi''$ such that $\ell' = \ell'' \circ \varphi$.

For a generalized pseudo-tree $\mathrm{Sk} = \mathrm{Sk}_{(\Xi,\ell)}$, we denote by $n_*(\mathrm{Sk})$, $* \in \{\circ, \bullet, \vartriangle\}$, the number of *monovalent* $*$-vertices of the compact part Sk^c. Thus, $n_*(\mathrm{Sk}) = |\ell^{-1}(*)|$.

4.4.1. Proposition. *Let $H \subset \Gamma$ be a proper finitely generated subgroup. Then H is generated by $H \cap [\![\mathbb{X}\mathbb{Y}]\!]_\Gamma$ if and only if Γ/H is a generalized pseudo-tree without monovalent vertices* (i.e., *a skeleton* $\mathrm{Sk}_{(\Xi,\ell)}$ *with* ℓ *taking values in* $\{0, \vartriangle\}$). *If this is the case, H admits a free basis consisting of elements conjugate to* $\mathbb{X}\mathbb{Y}$.

Proof. Let $\mathrm{Sk} = \Gamma/H$. It is an almost contractible $(3,1)$-skeleton, see 3.6.3. Since H is proper, Sk has a well defined compact part Sk^c, which is *not* isomorphic to the skeleton $\bullet\!\!-\!\!\circ$ representing Γ itself. Hence, each monogonal region of Sk (orbit of $\mathbb{X}\mathbb{Y}$ of length one) is bounded by an edge with both ends attached to a trivalent $\bullet$-vertex. (The only exceptional monogonal region is the 'outer' region in the skeleton $\bullet\!\!-\!\!\circ$ representing Γ.) It follows that the edge bounding a monogonal region cannot belong to any subtree of Sk^c.

Let Ξ be a maximal tree in Sk^c not containing a monovalent $\circ$- or $\bullet$-vertex. Contracting Ξ establishes a homotopy equivalence of the space $(\mathrm{Sk}^c)^\bullet$ computing $\pi_1^{\mathrm{orb}}(\mathrm{Sk}^c) = H$, see 3.6.3, to a wedge W of circles and copies of D^2_2 and D^2_3. Each monogonal region of Sk produces a separate circle in W, and the H-conjugacy classes of loops represented by these circles constitute the intersection $H \cap [\![\mathbb{X}\mathbb{Y}]\!]$. Thus, H is generated by $H \cap [\![\mathbb{X}\mathbb{Y}]\!]$ if and only if W has no other circles or copies of D^2_2 or D^2_3, *i.e.*, Sk^c consists of several monogonal regions attached to the (unique) maximal subtree $\Xi \subset \mathrm{Sk}^c$. □

4.4.2. Remark. Proposition 4.4.1 gives a geometric characterization of the proper subgroups $H \subset \Gamma$ that can appear as the monodromy group of a simple Γ-valued monodromy factorization, see Definition 5.1.3. Note that Γ itself can also appear in this way (it is generated by the images $\mathbb{X}\mathbb{Y}$ and $\mathbb{X}^2\mathbb{Y}\mathbb{X}^{-1}$ of σ_1 and σ_2, respectively, see (5.1.1) below); it is the only monodromy group that is not free.

4.4.3. Remark. According to Proposition 4.4.1, the study of simple Γ-valued monodromy factorizations is often reduced to that of monodromy factorizations with the values in a free group, which may be easier. For example, in some cases (if $\mathfrak{m}_\infty$ is positive with respect to an appropriate basis in the image), one can mimic Artin's proof of his Theorem 16 in [4] to establish the uniqueness of a monodromy factorization of a given element $\mathfrak{m}_\infty$ *with a given monodromy group.*

4.5. Digression: more examples of elliptic surfaces. Let $\mathrm{Sk} = \mathrm{Sk}_{(\Xi,\ell)}$ be a finite generalized pseudo-tree (thus, we assume that $n_\vartriangle(\mathrm{Sk}) = 0$) obtained from an admissible tree Ξ with k nodes. Let $n_* = n_*(\mathrm{Sk})$. For the type specification (see Subsection 3.4 and Remark 3.4.3), assign type I_1 to each monogonal region of Sk and types IV^* and III^* to the monovalent $\bullet$- and $\circ$-vertices, respectively. Then the fiber inside the remaining outer region of Sk is of type I_s if $k + n_\bullet + n_\circ$ is odd or I^*_s otherwise, where $s = 5k + 4 - n_\bullet - 2n_\circ$. (For even more examples, one could also vary the types I_1 or I^*_1 of the fibers in the monogonal regions, adjusting the type of the remaining fiber accordingly.)

The skeleton Sk and the type specification described above define an extremal elliptic surface X with the combinatorial type of singular fibers

$$(k + 2 - n_\bullet - n_\circ)\mathrm{I}_1 \oplus n_\bullet \mathrm{IV}^* \oplus n_\circ \mathrm{III}^* \oplus \{\mathrm{I}_s \text{ or } \mathrm{I}^*_s\}.$$

The surfaces corresponding to non-isomorphic pairs (Ξ, ℓ) are neither analytically isomorphic nor related by a 2-orientation preserving fiberwise homeomorphism, as they have non-conjugate reduced monodromy groups.

5. Monodromy factorizations

This section deals with monodromy factorizations. We prove Theorems 1.2.1 and 1.2.2 and discuss a few sporadic examples arising from generalized pseudo-trees and from maximizing plane sextics.

5.1. Preliminaries. The *braid group* $\mathbb{B}_3$ is the group

$$\mathbb{B}_3 = \langle \sigma_1, \sigma_2 \,|\, \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle = \langle u, v \,|\, u^3 = v^2 \rangle,$$

where $u = \sigma_2\sigma_1$ and $v = \sigma_2\sigma_1^2$. The center $Z(\mathbb{B}_3)$ is the infinite cyclic group generated by $u^3 = v^2$, and the quotient $\mathbb{B}_3/Z(\mathbb{B}_3)$ is isomorphic to Γ. In order to be consistent with Subsection 2.5, we define the epimorphism $\mathbb{B}_3 \twoheadrightarrow \tilde{\Gamma}$ (and further to Γ) *via*

$$\sigma_1 \mapsto \mathbb{X}\mathbb{Y}, \quad \sigma_2 \mapsto \mathbb{X}^2\mathbb{Y}\mathbb{X}^{-1}. \tag{5.1.1}$$

(Then $u \mapsto -\mathbb{X}^{-1}$ and $v \mapsto -\mathbb{Y}$.)

5.1.2. The abelianization $\mathbb{B}_3/[\mathbb{B}_3, \mathbb{B}_3]$ is the cyclic group $\mathbb{Z}$. The image of a braid $\beta \in \mathbb{B}_3$ in the abelianization $\mathbb{B}_3/[\mathbb{B}_3, \mathbb{B}_3] = \mathbb{Z}$ is called its *degree* $\deg \beta$. (By convention, $\deg \sigma_1 = 1$.) A braid $\beta \in \mathbb{B}_3$ is uniquely recovered from its image $\bar{\beta} \in \Gamma$ and its degree $\deg \beta$; the latter is determined by $\bar{\beta}$ up to a multiple of 6. (The degree of an element of Γ or $\tilde{\Gamma}$ is defined, respectively, modulo 6 or 12.)

5.1.3. Definition. A $\mathbb{B}_3$- (respectively, Γ- or $\tilde{\Gamma}$-) valued monodromy factorization $(\mathfrak{m}_i)$, $i = 1, \ldots, r$, is called *simple* if each entry $\mathfrak{m}_i$ belongs to the conjugacy class $[\![\sigma_1]\!]$ (respectively, $[\![\mathbb{X}\mathbb{Y}]\!]_\Gamma$ or $[\![\mathbb{X}\mathbb{Y}]\!]_{\tilde{\Gamma}}$).

5.1.4. Proposition. *For each $r \geqslant 1$, the epimorphisms $\mathbb{B}_3 \twoheadrightarrow \tilde{\Gamma} \twoheadrightarrow \Gamma$ establish bijections between the sets of simple $\mathbb{B}_3$-, $\tilde{\Gamma}$-, and Γ-valued monodromy factorizations of length r; these bijections preserve the weak/strong equivalence classes.*

Proof. Each element $x \in [\![\mathbb{X}\mathbb{Y}]\!] \subset \Gamma$ lifts to a unique element $x' \in [\![\sigma_1]\!] \subset \mathbb{B}_3$ and to a unique element $x'' \in [\![\mathbb{X}\mathbb{Y}]\!] \subset \tilde{\Gamma}$ (characterized by the requirement that $\deg x' = 1$ and $\deg x'' = 1 \bmod 12$), establishing a one-to-one correspondence between the sets of monodromy factorizations. The weak and strong Hurwitz equivalences are preserved due to the fact that both $\mathbb{B}_3 \twoheadrightarrow \Gamma$ and $\tilde{\Gamma} \twoheadrightarrow \Gamma$ are central extensions. □

5.1.5. The advantage of considering the braid group $\mathbb{B}_3$ rather than the modular group Γ is the fact that, in $\mathbb{B}_3$, the length r of a simple monodromy factorization of an element $\mathfrak{m}_\infty \in \mathbb{B}_3$ is uniquely determined by $\mathfrak{m}_\infty$: one has $r = \deg \mathfrak{m}_\infty$. Hence, for $\mathbb{B}_3$, the problem of uniqueness of a simple monodromy factorization of a given element can be restated in the language of *factorization semigroup*, see [22] and [31].

5.1.6. Definition. The *factorization semigroup* is the semigroup $\mathfrak{B}_n$ (with the binary operation denoted by $\cdot$) generated by the elements $\beta \in [\![\sigma_1]\!]_{\mathbb{B}_n}$ subject to the *Hurwitz relations* $\beta_1 \cdot \beta_2 = \beta_1^{-1}\beta_2\beta_1 \cdot \beta_1 = \beta_2 \cdot \beta_2\beta_1\beta_2^{-1}$. The *evaluation anti-homomorphism* $v\colon \mathfrak{B}_n \to \mathbb{B}_n$ is defined *via* $v\colon \beta_1 \cdot \beta_2 \cdot \ldots \cdot \beta_r \mapsto \beta_r \ldots \beta_2\beta_1$.

5.1.7. It is clear that an element $\bar{\mathfrak{m}} \in \mathfrak{B}_n$ represents a strong Hurwitz equivalence class of simple $\mathbb{B}_n$-valued monodromy factorizations (of length $\deg v(\bar{\mathfrak{m}})$) and the value $v(\bar{\mathfrak{m}})$ is merely the monodromy at infinity $\mathfrak{m}_\infty(\bar{\mathfrak{m}})$. Our Theorem 1.2.1 states that, for $n = 3$, the evaluation map v is not injective; moreover, the cardinality of the pull-back $v^{-1}(\beta)$, $\beta \in \mathbb{B}_3$, may grow exponentially in the degree $\deg \beta$. Using the canonical inclusion $\mathbb{B}_3 \hookrightarrow \mathbb{B}_n$, one can easily conclude that the same assertion holds for any integer $n \geqslant 3$: the cardinality of the pull-back $v^{-1}(\beta)$, $\beta \in \mathbb{B}_n$, may grow exponentially in the degree $\deg \beta$.

According to [31], the fact that v is not injective implies that $\mathfrak{B}_n$ does *not* have the cancellation property, *i.e.*, an equality $\alpha_1 \cdot \beta = \alpha_2 \cdot \beta$ or $\beta \cdot \alpha_1 = \beta \cdot \alpha_1$ in $\mathfrak{B}_n$ does not necessarily imply that $\alpha_1 = \alpha_2$.

5.2. Proof of Theorems 1.2.1 and 1.2.2. Consider a marked admissible tree (Ξ, v_1) with k nodes and $(k+2)$ leaves and let $\mathrm{Sk} = \mathrm{Sk}_\Xi$ be the associated pseudo-tree, see Subsection 4.1. Let $(m_1, \ldots, m_{k+1})$ be the sequence of consecutive vertex distances, see 4.1.3, and consider the distances

$$n_i = m_i + \ldots + m_{k+1}, \; i = 1, \ldots, k+1, \quad n_{k+2} = 0 \tag{5.2.1}$$

from v_i to v_{k+2} in Sk.

Let $\mathrm{e} \in \mathcal{E}_{\mathrm{Sk}}$ be the edge end at v_{k+2} that belongs to the original tree, see the grey dot in Figure 6, and consider the basis $\{\gamma_1, \ldots, \gamma_{k+2}\}$ for $\pi_1(\mathrm{Sk}, \mathrm{e})$, where γ_i is the class represented by the loop of Sk attached at v_i which is connected to e by the shortest left turn path in Sk (the grey loop in Figure 6).

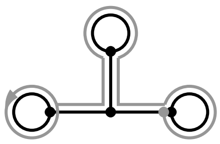


FIGURE 6. A loop γ_i (grey)

In terms of Definition 2.4.1, the loop representing a basis element γ_i is (e, w_i), where

$$w_i = (\mathrm{nx}\;\mathrm{op})^{n_i}(\mathrm{nx}\;\mathrm{op}\;\mathrm{nx}^{-1}\,\mathrm{nx}^{-1})(\mathrm{op}\;\mathrm{nx}^{-1})^{n_i}.$$

The product $\gamma_1 \ldots \gamma_{k+1}$ is homotopic to the boundary of the outer $(5k+4)$-gonal region of Sk; after cancellation, $\gamma_1 \ldots \gamma_{k+2} \sim (\mathrm{e}, (\mathrm{nx}\;\mathrm{op})^{5k+4})$.

Define the Γ-valued monodromy factorization $\bar{\mathfrak{m}} = \bar{\mathfrak{m}}(\mathrm{Sk}, \mathrm{e}) = (\mathfrak{m}_1, \ldots, \mathfrak{m}_{k+2})$ *via*

$$\mathfrak{m}_i = (\operatorname{val} \gamma_i)^{-1} = (\mathbb{X}\mathbb{Y})^{n_i}(\mathbb{X}^2\mathbb{Y}\mathbb{X}^{-1})(\mathbb{X}\mathbb{Y})^{-n_i}. \tag{5.2.2}$$

By construction, one has $\mathfrak{m}_\infty(\bar{\mathfrak{m}}) = (\mathbb{X}\mathbb{Y})^{-5k-4}$, see Lemma 2.4.4, and $\operatorname{Im}(\bar{\mathfrak{m}}) = \pi_1(\mathrm{Sk}, \mathrm{e}) = \operatorname{Stab}(\mathrm{e})$, see Theorem 2.5.3. Regarding each $\mathfrak{m}_i$ in (5.2.2) as an element of $\tilde{\Gamma}$ and adjusting degree modulo 12, one obtains $\mathfrak{m}_\infty(\bar{\mathfrak{m}}) = -(-\mathbb{X}\mathbb{Y})^{-5k-4} \in \tilde{\Gamma}$.

5.2.3. Remark. Note that the particular choice of a basis $\{\gamma_i\}$ used above is not very important: by Artin's theorem [4], any other basis $\{\gamma'_i\}$ with the property that each γ'_i is conjugate to some γ_j and $\gamma'_1 \ldots \gamma'_{k+2} = \gamma_1 \ldots \gamma_{k+2}$ is obtained from $\{\gamma_i\}$ by a sequence of Hurwitz moves; hence the resulting monodromy factorization $\bar{\mathfrak{m}}'$ would be strongly equivalent to $\bar{\mathfrak{m}}$.

Now, observe that e belongs to the orbit $\mathfrak{O}_{\mathrm{Sk}}$ introduced in 4.2. Let $\mathrm{e}' \in \mathfrak{O}_{\mathrm{Sk}}$ be another element of this orbit, $\mathrm{e}' = (\mathbb{X}\mathbb{Y})^s\mathrm{e}$, and consider the monodromy factorization $\bar{\mathfrak{m}}' = \bar{\mathfrak{m}}(\mathrm{Sk}, \mathrm{e}') := (\mathbb{X}\mathbb{Y})^s\bar{\mathfrak{m}}(\mathrm{Sk}, \mathrm{e})(\mathbb{X}\mathbb{Y})^{-s}$. Clearly, one has $\mathfrak{m}_\infty(\bar{\mathfrak{m}}') =$

$(\mathbb{XY})^{-5k-4}$ and $\operatorname{Im}(\bar{\mathfrak{m}}') = \pi_1(\mathrm{Sk}, \mathrm{e}') = \operatorname{Stab}(\mathrm{e}')$. As above, the strong equivalence class of $\bar{\mathfrak{m}}(\mathrm{Sk}, \mathrm{e}')$ does not depend on the particular choice of a basis for $\pi_1(\mathrm{Sk}, \mathrm{e}')$; for this reason, we omit the reference to the marking of the original tree Ξ in the notation.

Considering all $\tilde{T}(k)$ pairwise non-isomorphic pairs $(\mathrm{Sk}, \mathrm{e})$, $\mathrm{e} \in \mathfrak{O}_{\mathrm{Sk}}$, see Subsection 4.2 and Corollary 4.2.2, one obtains $\tilde{T}(k)$ distinct monodromy factorizations $\bar{\mathfrak{m}}(\mathrm{Sk}, \mathrm{e})$; they differ by the monodromy groups $\operatorname{Im}(\mathfrak{m}(\mathrm{Sk}, \mathrm{e})) = \operatorname{Stab}(\mathrm{e})$, see Theorem 2.3.5. Disregarding the base points e, one arrives at $T(k)$ weak equivalence classes, which differ by the conjugacy class $[\![\operatorname{Im}(\mathfrak{m}(\mathrm{Sk}, \mathrm{e}))]\!] = [\![\operatorname{Stab}\mathrm{Sk}]\!]$, see Corollary 2.3.6.

The transcendental lattices and fundamental groups of the monodromy factorizations constructed above are computed in [16]; for the former, see Example 7.2.3. □

5.2.4. Remark. The monodromy factorizations (5.2.2) represent the reduced homological invariants of the extremal elliptic surfaces constructed in Subsection 4.3.

5.3. Examples. Thus, the $T(k)$ weak equivalence classes of monodromy factorizations given by Theorem 1.2.2 are numbered by the isomorphism classes of admissible trees with k nodes. They are given by (5.2.2), where the sequence $(n_1, \ldots, n_{k+2})$ is obtained from the vertex distances $(m_1, \ldots, m_{k+1})$ of the tree, see 4.1.3. The lifts to simple $\mathbb{B}_3$-valued monodromy factorizations are

$$\mathfrak{m}_i = \sigma_1^{n_i} \sigma_2 \sigma_1^{-n_i}, \ i = 1, \ldots, k+2, \qquad \mathfrak{m}_\infty = (\sigma_1\sigma_2)^{3(k+1)} \sigma_1^{-5k-4}. \tag{5.3.1}$$

(For $\mathfrak{m}_\infty$, we multiply σ_1^{-5k-4} by a power of the central element $(\sigma_1\sigma_2)^3$ in order to match the degree.)

5.3.2. Example. The simplest example of non-equivalent monodromy factorizations given by Theorem 1.2.2 is obtained when $k = 4$. The four admissible trees with four nodes and their vertex distances are shown in Figure 7. The fact that the resulting monodromy factorizations are not equivalent can be proved directly, using `GAP` [19]. Let $\bar{\mathfrak{m}}$ be one of the monodromy factorizations, let $H = \operatorname{Im}(\bar{\mathfrak{m}})$ be its monodromy group, and let N be the normalizer of H in Γ. Then, as Corollary 2.3.7 predicts, the index $[N : H]$ equals 1, 2, and 3 for the trees in Figure 7, left, middle, and right, respectively. In particular, the four groups belong to at least three distinct conjugacy classes. The two groups corresponding to the two trees in the middle (which are related by an orientation reversing diffeomorphism of the sphere) are conjugate in $PGL(2, \mathbb{Z})$ but not in Γ.

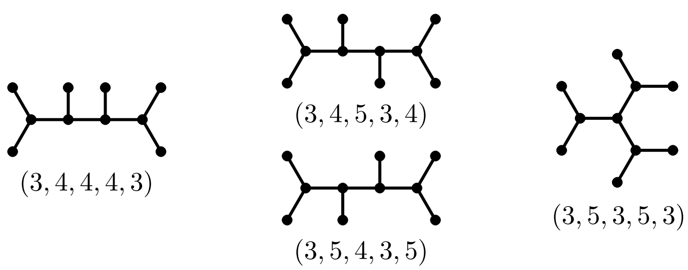


FIGURE 7. Admissible trees with four nodes

5.3.3. Example. The simplest example of weakly but not strongly equivalent monodromy factorizations with the same monodromy at infinity is given by Theorem 1.2.1 with $k = 0$. The only admissible tree without nodes (two leaves connected by an edge) gives rise to two monodromy factorizations:

$$\bar{\mathfrak{m}}' = (\sigma_1^2\sigma_2\sigma_1^{-2}, \sigma_2), \qquad \bar{\mathfrak{m}}'' = (\sigma_1\sigma_2\sigma_1^{-1}, \sigma_1^{-1}\sigma_2\sigma_1).$$

Let $H', H'' \subset \Gamma$ be their monodromy groups (reduced to Γ). Using `GAP` [19], one can see that $[\Gamma : H'] = [\Gamma : H''] = 6$ whereas $[\Gamma : H' \cap H''] = 24$. Hence $H' \neq H''$.

5.4. Non-equivalent monodromy factorizations of length two. Consider the almost contractible generalized pseudo-trees represented by the two ribbon graphs shown in Figure 8. (Recall that each $\vartriangle$-vertex is to be extended to a maximal infinite branch, which is a 'half' of the Farey three, see Subsection 3.1.) They are obviously not isomorphic; hence their stabilizers are not conjugate.

FIGURE 8. Almost contractible pseudo-trees with two loops

In each skeleton Sk, let $\mathrm{e} \in \mathcal{E}_{\mathrm{Sk}}$ be the edge end represented by a grey dot in the figure, and pick a basis $\{\gamma_1, \gamma_2\}$ for $\pi_1(\mathrm{Sk}, \mathrm{e})$ so that each γ_i, $i = 1, 2$, is conjugate to the boundary of a monogonal region of Sk and $\gamma_1\gamma_2$ is homotopic to a circle encompassing the compact part Sk^c of Sk. (The particular choice of bases is not important, see Remark 5.2.3.) Let $\bar{\mathfrak{m}}(\mathrm{Sk}) = ((\operatorname{val}\gamma_1)^{-1}, (\operatorname{val}\gamma_2)^{-1})$. For example, the bases can be chosen so that

$$\bar{\mathfrak{m}}(\mathrm{Sk}_{\mathrm{left}}) = ((\mathbb{X}\mathbb{Y})(\mathbb{X}^2\mathbb{Y}\mathbb{X}^{-1})(\mathbb{X}\mathbb{Y})^{-1}, (\mathbb{Y}\mathbb{X}\mathbb{Y})(\mathbb{X}^2\mathbb{Y}\mathbb{X}^{-1})(\mathbb{Y}\mathbb{X}\mathbb{Y})^{-1}),$$
$$\bar{\mathfrak{m}}(\mathrm{Sk}_{\mathrm{right}}) = (\mathbb{X}^2\mathbb{Y}\mathbb{X}^{-1}, (\mathbb{Y}\mathbb{X}\mathbb{Y}\mathbb{X}^2\mathbb{Y})(\mathbb{X}^2\mathbb{Y}\mathbb{X}^{-1})(\mathbb{Y}\mathbb{X}\mathbb{Y}\mathbb{X}^2\mathbb{Y})^{-1}).$$

The $\mathbb{B}_3$-valued simple lifts of the two factorizations are

$$\bar{\mathfrak{m}}(\mathrm{Sk}_{\mathrm{left}}) = (\sigma_1\sigma_2\sigma_1^{-1}, \sigma_2\sigma_1^3\sigma_2\sigma_1^{-3}\sigma_2^{-1}), \quad \bar{\mathfrak{m}}(\mathrm{Sk}_{\mathrm{right}}) = (\sigma_2, \beta\sigma_2\beta^{-1}),$$

where $\beta = \sigma_2\sigma_1^2\sigma_2^{-1}\sigma_1$. One has

$$\mathfrak{m}_\infty(\bar{\mathfrak{m}}(\mathrm{Sk}_{\mathrm{left}})) = \mathfrak{m}_\infty(\bar{\mathfrak{m}}(\mathrm{Sk}_{\mathrm{right}})) = \mathbb{Y}\mathbb{X}(\mathbb{X}\mathbb{Y})^{-3}\mathbb{Y}\mathbb{X}(\mathbb{X}\mathbb{Y})^{-3}$$

(or, respectively, $\mathfrak{m}_\infty = (\sigma_2\sigma_1^3\sigma_2\sigma_1^{-1})^2 \in \mathbb{B}_3$). On the other hand, the monodromy groups $[\![\operatorname{Im}(\bar{\mathfrak{m}}(\mathrm{Sk}))]\!] = [\![\operatorname{Stab}\mathrm{Sk}]\!]$ are not conjugate in Γ (although they are conjugate in $PGL(2, \mathbb{Z})$); hence the two monodromy factorizations are not weakly equivalent.

5.4.1. Remark. The two pseudo-trees differ by an orientation reversing autohomeomorphism of the sphere. This fact implies that the corresponding Hurwitz curves and Lefschetz fibrations are anti-isomorphic. Hence, the two monodromy factorizations have isomorphic fundamental groups and transcendental lattices, see 1.1.3.

5.5. Digression: non-simple monodromy factorizations. Let $\mathrm{Sk} = \mathrm{Sk}_{(\Xi,\ell)}$ be a generalized pseudo-tree obtained from an admissible tree Ξ with k nodes, see Subsection 4.4. Denote $n_* = n_*(\mathrm{Sk})$ for $* \in \{\bullet, \circ, \vartriangle\}$.

Consider an embedding $\mathrm{Sk}^c \subset S^2$, patch each monogonal region of Sk^c with a disk, and let B be a regular neighborhood of the result. Denote by $B^\sharp$ the punctured disk obtained from B by removing a point inside each monogonal region of Sk and all monovalent $\bullet$- and $\circ$-vertices of Sk, see the shaded area in Figure 9. There is an epimorphism $\rho\colon \pi_1(B^\sharp) \twoheadrightarrow \pi_1^{\mathrm{orb}}(\mathrm{Sk})$, *cf.* Theorem 3.3.1.

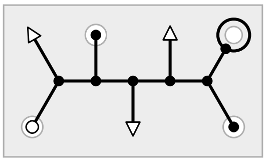


FIGURE 9. A generalized pseudo-tree Sk^c and punctured disk $B^\sharp$

Fix a point $b \in \partial B$ and pick a geometric basis $\{\gamma_1, \ldots, \gamma_s\}$ for $\pi_1(B, b)$ such that $\gamma_1 \ldots \gamma_s = [\partial B]$. (The precise choice is not important as different bases would produce weakly equivalent monodromy factorizations, *cf.* Remark 5.2.3.) Define the monodromy factorization $\bar{\mathfrak{m}}(\mathrm{Sk}) = (\mathfrak{m}_1, \ldots, \mathfrak{m}_s)$ of length $s = k + 2 - n_\vartriangle$ *via* $\mathfrak{m}_i = (\operatorname{val}\rho(\gamma_i))^{-1}$, $i = 1, \ldots, s$. It has $n_\bullet$ elements in $[\![\mathbb{X}]\!]$, $n_\circ$ elements in $[\![\mathbb{Y}]\!]$, and $k+2-n_\bullet-n_\circ-n_\vartriangle$ elements in $[\![\mathbb{X}\mathbb{Y}]\!]$. Thus, $\bar{\mathfrak{m}}$ is simple if and only if $n_\bullet = n_\circ = 0$.

If $n_\vartriangle = 0$, the conjugacy class of the monodromy at infinity $\mathfrak{m}_\infty(\bar{\mathfrak{m}}(\mathrm{Sk}))$ equals $[\![(\mathbb{X}\mathbb{Y})^{-n}]\!]$, where $n = 5k + 4 - n_\bullet - 2n_\circ$, and $\bar{\mathfrak{m}}(\mathrm{Sk})$ represents the reduced homological invariant of an extremal elliptic surface constructed in Subsection 4.5. In general, the monodromy at infinity can be found as follows. Let $(m_1, \ldots, m_{n_\vartriangle})$ be the sequence of vertex distances in Sk^c between consecutive $\vartriangle$-vertices, each distance being the length of the shortest left turn path connecting two $\vartriangle$-vertices, with only $\bullet$-vertices counted. (For example, for the graph shown in Figure 9, starting from the upper left corner, one has $(m_1, m_2, m_3) = (6, 9, 4)$; in Figure 8, for both graphs one has $(m_1, m_2) = (5, 5)$.) Then, the conjugacy class of the monodromy at infinity $\mathfrak{m}_\infty(\bar{\mathfrak{m}}(\mathrm{Sk}))$ is represented by the *right to left* product

$$\prod_{i=1}^{n_\vartriangle} (\mathbb{X}\mathbb{Y})^{m_i-1}\mathbb{X} = \ldots (\mathbb{X}\mathbb{Y})^{m_2-1}\mathbb{X}\ (\mathbb{X}\mathbb{Y})^{m_1-1}\mathbb{X}. \tag{5.5.1}$$

Note that $\sum_{i=1}^{n_\vartriangle} m_i = 5k + 4 - n_\bullet - 2n_\circ - 2n_\vartriangle$.

5.5.2. Lemma. *Given two generalized pseudo-trees* Sk', Sk'', *the monodromies at infinity* $\mathfrak{m}_\infty(\bar{\mathfrak{m}}(\mathrm{Sk}'))$ *and* $\mathfrak{m}_\infty(\bar{\mathfrak{m}}(\mathrm{Sk}''))$ *are conjugate in* Γ *if and only if the corresponding sequences* (m'_i) *and* (m''_j) *differ by a cyclic permutation.*

Proof. The 'if' part is obvious. For the converse, observe that the admissibility condition in Subsection 4.4 implies that each entry m'_i, m''_j is at least 2. Then the cyclic word w given by (5.5.1) admits no cancellations and the distances m_i can be recovered from the distances in w between consecutive occurrences of $\mathbb{X}^2$. $\square$

5.6. Digression: maximizing plane sextics. We conclude this section with a few examples arising from maximizing plane sextics.

Consider a plane sextic $C \subset \mathbb{P}^2$ with simple singularities only and with a distinguished type $\mathbf{E}$ singular point P. Let L_∞ be the (only) tangent to C at P. Assume that L_∞ is not a component of C and let $C^{\mathrm{a}} \subset \mathbb{C}^2 = \mathbb{P}^2 \smallsetminus L_\infty$ be the affine part of C. It is a *horizontal curve* in the sense of [3] (or *Hurwitz curve* in the sense of [26]) of degree 3 with respect to the pencil $\mathcal{P} = \{L_t\}$, $t \in \mathbb{C}^1$, of lines through P; in other words, the projection $C^{\mathrm{a}} \to \mathbb{C}^1$ defined by $\mathcal{P}$ is a proper map. Hence, using $\mathcal{P}$ and an appropriately chosen section of the projection, one can define the braid monodromy $\mu_C \colon \pi_1(B^\sharp) \to \mathbb{B}_3$, where $B^\sharp$ is the base $\mathbb{C}^1$ of the pencil with the singular fibers removed. Then, choosing a geometric basis for $\pi_1(B^\sharp)$, one can represent μ_C by a monodromy factorization $\bar{\mathfrak{m}}_C$, which is well defined up to weak Hurwitz equivalence.

The minimal resolution of singularities X of the double plane ramified at a sextic C as above is a $K3$-surface, and the pencil $\mathcal{P}$ lifts to an elliptic pencil $X \to \mathbb{P}^1$ with a distinguished section. One can easily show (see, *e.g.*, [13]) that X is extremal if and only if C is *maximizing*, *i.e.*, if its total Milnor number takes its maximal possible value 19. When this is the case, the combinatorial type of singular fibers of X is determined by the combinatorial type of singularities of C as follows:

– the distinguished singular point P of type $\mathbf{E}_6$, $\mathbf{E}_7$, or $\mathbf{E}_8$ gives rise to a singular fiber of type I_6, I_2^*, or III^*, respectively,
– each other singular point gives rise to a singular fiber of the following type: $\mathbf{A}_p \mapsto \mathrm{I}_{p+1}$, $p \geqslant 1$, $\mathbf{D}_q \mapsto \mathrm{I}_{q-4}^*$, $q \geqslant 4$, $\mathbf{E}_6 \mapsto \mathrm{IV}^*$, $\mathbf{E}_7 \mapsto \mathrm{III}^*$, $\mathbf{E}_8 \mapsto \mathrm{II}^*$,
– a number of type I_1 fibers are added to make the total multiplicity 24.

Furthermore, the $\tilde{\Gamma}$-valued reduction of the braid monodromy μ_C is the homological invariant $\tilde{\mathfrak{h}}_X$.

In [3], the authors construct a pair of reducible maximizing sextics C_1, C_2 with the set of singularities $\mathbf{E}_6 \oplus \mathbf{A}_7 \oplus \mathbf{A}_3 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$ and, using the fact that both curves and all their singular fibers can be chosen real, compute their monodromy factorizations $\bar{\mathfrak{m}}_1$, $\bar{\mathfrak{m}}_2$. Then, reducing $\bar{\mathfrak{m}}_1$ and $\bar{\mathfrak{m}}_2$ to the finite group $SL(2, \mathbb{Z}_{32})$ and using `GAP` [19], they compute their Hurwitz orbits and show that they are disjoint, concluding that $\bar{\mathfrak{m}}_1$ and $\bar{\mathfrak{m}}_2$ are not weakly equivalent and thus distinguishing the curves. (Both orbits are of length 15360.) In [15], the same pair of sextics is constructed using trigonal curves or, equivalently, extremal elliptic $K3$-surfaces; their skeletons are as shown in Figure 10, with the distinguished fiber L_∞ corresponding to the outer region. Since the skeletons are obviously not isomorphic, Theorem 2.5.3 implies that $[\![\mathrm{Im}(\bar{\mathfrak{m}}_1)]\!] \neq [\![\mathrm{Im}(\bar{\mathfrak{m}}_2)]\!]$.

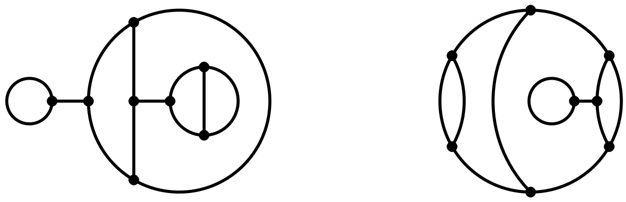


FIGURE 10. The set of singularities $\mathbf{E}_6 \oplus \mathbf{A}_7 \oplus \mathbf{A}_3 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$

5.6.1. Remark. Strictly speaking, constructed in [15] is merely a pair of not deformation equivalent sextics with the set of singularities $\mathbf{E}_6 \oplus \mathbf{A}_7 \oplus \mathbf{A}_3 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$.

However, it follows from [33] that this set of singularities is realized by exactly two equisingular deformation families. Hence, the pairs found in [3] and [15] coincide.

A number of other examples is found in [14] and [15]. Listed in Table 2 are all sets of singularities realized by a pair C_1, C_2 of *irreducible* maximizing plane sextics with a distinguished type $\mathbf{E}$ singular point and with essentially different skeletons. (More precisely, we ignore pairs of anti-isomorphic curves.) For each such pair, Theorem 2.5.3 implies that the corresponding monodromy factorizations $\bar{\mathfrak{m}}_1$, $\bar{\mathfrak{m}}_2$ are not weakly equivalent, as their monodromy groups are not conjugate. For the sets of singularities marked with a *, the corresponding monodromy factorizations differ by their transcendental lattices, see Example 7.2.2 below.

Table 2. Irreducible maximizing sextics with a type $\mathbf{E}$ singular point

$^*\mathbf{E}_8 \oplus \mathbf{A}_{10} \oplus \mathbf{A}_1$	$\mathbf{E}_6 \oplus \mathbf{D}_5 \oplus \mathbf{A}_8$	$(\mathbf{E}_6 \oplus \mathbf{A}_8 \oplus \mathbf{A}_2) \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$
$\mathbf{E}_8 \oplus \mathbf{A}_8 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$	$^*\mathbf{E}_6 \oplus \mathbf{D}_5 \oplus \mathbf{A}_6 \oplus \mathbf{A}_2$	$^*\mathbf{E}_6 \oplus \mathbf{A}_7 \oplus \mathbf{A}_4 \oplus \mathbf{A}_2$
$^*\mathbf{E}_8 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_1$	$^*\mathbf{E}_6 \oplus \mathbf{A}_{10} \oplus \mathbf{A}_3$	$^*\mathbf{E}_6 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$
$\mathbf{E}_8 \oplus \mathbf{A}_5 \oplus \mathbf{A}_4 \oplus \mathbf{A}_2$	$^*\mathbf{E}_6 \oplus \mathbf{A}_{10} \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$	$\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus 2\mathbf{A}_4$
$(2\mathbf{E}_6 \oplus \mathbf{A}_5) \oplus \mathbf{A}_2$	$\mathbf{E}_6 \oplus \mathbf{A}_9 \oplus \mathbf{A}_4$	$(\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus 2\mathbf{A}_2) \oplus \mathbf{A}_4$
$2\mathbf{E}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_3$	$\mathbf{E}_6 \oplus \mathbf{A}_8 \oplus \mathbf{A}_4 \oplus \mathbf{A}_1$	

5.6.2. Remark. It is worth mentioning that there also are three pairs C_1, C_2 of irreducible maximizing sextics, those with the sets of singularities

$$\mathbf{E}_7 \oplus \mathbf{E}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_2, \quad \mathbf{E}_7 \oplus \mathbf{A}_{10} \oplus \mathbf{A}_2, \quad \mathbf{E}_7 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_2$$

(the distinguished point P being that of type $\mathbf{E}_7$), such that, within each pair, the curves are not deformation equivalent but are represented by isomorphic skeletons, hence have equivalent monodromy factorizations. It follows that the affine parts C_1^{a}, C_2^{a} are isotopic in the class of Hurwitz curves, see [22]. In fact, the curves constituting each pair are related by a quadratic birational transformation biholomorphic in the affine part $\mathbb{P}^2 \smallsetminus L_\infty$.

6. Real trigonal curves

Here, we give a brief introduction to theory of real trigonal curves (see [17] for more details), prove Theorem 1.4.1, and consider a few generalizations.

6.1. Dessins. Recall that a *real structure* on a complex analytic variety X is an anti-holomorphic involution $\operatorname{conj}\colon X \to X$. A map, subvariety, *etc.* is called *real* if it commutes with/is preserved by conj.

For each Hirzebruch surface $\Sigma_k \to B \cong \mathbb{P}^1$, $k \geqslant 1$, fix a (unique up to automorphism) real structure $\operatorname{conj}\colon \Sigma_k \to \Sigma_k$ with nonempty real part, see [10] or [35]. Recall that the ruling of Σ_k restricts to an S^1-fibration $(\Sigma_k)_\mathbb{R} \to B_\mathbb{R} \cong \mathbb{P}^1_\mathbb{R} \cong S^1$ of the real parts, which is orientable if and only if k is even. The real part $E_\mathbb{R}$ of the exceptional section $E \subset \Sigma_k$ is a section of this fibration.

In what follows, we fix an orientation of $B_\mathbb{R}$ and denote by B_+ the closure of the connected component of $B \smallsetminus B_\mathbb{R}$ whose complex orientation agrees with the chosen orientation of the boundary $\partial B_+ = B_\mathbb{R}$.

6.1.1. Given a trigonal curve $C \subset \Sigma_k$, one can define the *j-invariant* $j_C : B \to \mathbb{P}^1$ by sending a nonsingular fiber $\bar{F}$ to the j-invariant of the elliptic curve F covering $\bar{F}$ and ramified at $\bar{F} \cap (C \cup E)$. (Here, the target is the standard Riemann sphere $\mathbb{C} \cup \{\infty\}$.) Following [30] (see also [17] for more details), define the *dessin* of C as the graph $j_C^{-1}(\mathbb{P}^1_{\mathbb{R}}) \subset B$ with the following extra decoration:

- the pull-backs of 0, 1, and ∞ are $\bullet$-, $\circ$-, and $\times$-vertices, respectively;
- the pull-backs of $[0, 1]$, $[1, \infty]$, and $[-\infty, 0]$ are bold, dotted, and solid edges, respectively.

(Thus, the skeleton introduced in 2.2.5 is obtained from the dessin by removing all $\times$-vertices and solid and dotted edges.) The dessin of a real curve is invariant under the complex conjugation in B; for this reason, we only draw the part contained in the closed disk B_+. Vertices and edges of the dessin that belong to the boundary ∂B_+ are called *real*.

6.1.2. Remark. Note that the j-invariant of a real curve may have real critical values other than 0, 1, or ∞ not removable by a small equivariant deformation. For this reason, a generic symmetric dessin may have non-removable *monochrome* vertices in the boundary ∂B_+, *cf.* Figure 11.

According to [30] and [17], a dessin in the *topological* disk B_+ determines a real trigonal curve C, which is well defined up to equivariant fiberwise deformation. (The converse is not true: a deformation of C may result in a non-trivial modification of its dessin, see [17] for details. We do not use this fact here.)

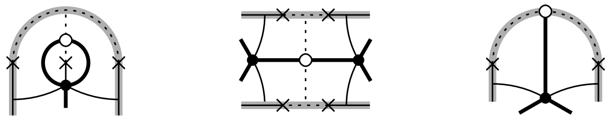
6.1.3. From now on, we assume all curves nonsingular and generic, *i.e.*, we assume that all singular fibers are of Kodaira type I_1.

The real part $C_{\mathbb{R}} = \operatorname{Fix} \operatorname{conj}|_C$ of a real trigonal curve $C \subset \Sigma_k$ consists of a *long component* L isotopic to $E_{\mathbb{R}}$ and a number of *ovals*, *i.e.*, components contractible in $(\Sigma_k)_{\mathbb{R}}$. By Bézout's theorem, ovals of a trigonal curve are never nested. The critical values of the restriction $p : C_{\mathbb{R}} \to B_{\mathbb{R}}$ of the ruling are the real $\times$-vertices of the dessin of C. Pairs of such vertices bound *maximal dotted segments* in ∂B_+, each segment containing a number of monochrome vertices and, possibly, a number of real $\circ$-vertices. The projection p is three-to-one over the interior of each dotted segment, and it is one-to-one outside the dotted segments. A maximal dotted segment containing an even number of $\circ$-vertices is the projection of an oval, *cf.* Figure 11(a) and (b); a segment containing an odd number of $\circ$-vertices is the projection of a *zigzag* in L, *cf.* Figure 11(c). (For the sake of brevity, we merely *define* a zigzag as the pull-back of a said maximal dotted segment. Intuitively, it is a Z-shaped fragment of the long component of the curve.) For further details concerning recovering the topology of a curve from its dessin, see [17] and [30].

The real $\circ$-vertices of the dessin are the points where $C_{\mathbb{R}}$ crosses the zero section of Σ_k. It follows that, if k is even, two ovals of $C_{\mathbb{R}}$ belong to the same connected component of the complement $(\Sigma_k)_{\mathbb{R}} \smallsetminus (L \cup E_{\mathbb{R}})$ if and only if they are separated by an even number of real $\circ$-vertices.

6.2. Proof of Theorem 1.4.1. To construct a curve C_i as in the statement, consider one of the $T(k)$ pseudo-trees Sk_i with k nodes, see Subsection 4.1, and extend it to a dessin as shown in Figure 11(a) and (b).

More precisely, embed Sk_i to the sphere S^2 (which is *not* the base of the elliptic pencil being constructed), patch each loop of Sk_i with the disk bounded by this loop, and take for B_+ a regular neighborhood of the result in S^2. In other words, B_+ is the space $\mathrm{Sk}_i \cup \bigcup_j D_j \cup_\varphi (\partial R \times I)$, where D_j are the disks attached, ∂R is the boundary of the outer region of Sk_i, represented as a union of copies of edges of Sk_i, and the attaching map $\varphi\colon \partial R \times \{0\} \to \mathrm{Sk}_i$ sends each such copy onto the corresponding edge linearly. (Each loop contributes once to ∂R, and each other edge contributes twice.) The real part ∂B_+ is identified with $\partial R \times \{1\}$.



(a) (b) (c)

FIGURE 11. Extending a skeleton Sk (bold) to a dessin

Now, place a $\circ$-vertex at the center of each edge (*cf.* the convention in 2.2.6), place a $\times$-vertex v_j at the center of each disk D_j, and connect v_j to the $\bullet$- and $\circ$-vertices in ∂D_j by the radii of D_j (solid and dotted, respectively), *cf.* Figure 11(a). The dessin structure in $\partial R \times \{0\}$ copies that of Sk_i (the $\bullet$- and $\circ$-vertices). Declare each segment $\bullet \times I$ a solid edge, and each segment $\circ \times I$ a dotted edge, obtaining corresponding monochrome vertices in $\partial R \times \{1\}$. Finally, in $\partial R \times \{1\}$, place a $\times$-vertex between each pair of consecutive monochrome vertices, and connect this $\times$-vertex to its monochrome neighbors by appropriate edges (solid or dotted). In this manner, each (copy of an) edge in the base $\partial R \times \{0\}$ gives rise to an oval type fragment, see 6.1.3, in the boundary $\partial R \times \{1\} = \partial B_+$.

Each loop of Sk_i gives rise to an oval in ∂B_+, see Figure 11(a), and each edge of the original tree Ξ_i gives rise to two ovals, see Figure 11(b). Thus, we obtain the dessin of a real trigonal curve in Σ_{2k+2} with $(5k+4)$ ovals.

All curves obtained are topologically distinct: they differ by the monodromy group $[\![\mathrm{Stab}\,\mathrm{Sk}_i]\!]$ of the monodromy $\pi_1(B_+^\sharp) \to \Gamma$, where $B_+^\sharp$ is the interior of B_+ with the inner $\times$-vertices removed. $\square$

6.2.1. Remark. Note that the curves constructed in the proof have no zigzags (Figure 11(c) is not used). Moreover, the dessins have no real $\circ$-vertices, hence the real parts of the curves do not cross the zero section, see 6.1.3. It follows that all ovals of the real part are to the same side from the long component.

6.3. Digression: more ribbon curves. The real trigonal curves constructed in Subsection 6.2 are *ribbon curves* in the sense of [17]. This construction can be generalized. Let $\mathrm{Sk} = \mathrm{Sk}_{(\Xi,\ell)}$ be the generalized pseudo-tree obtained from an admissible tree Ξ with k nodes and a function ℓ taking values in $\{0, \bullet\}$, see Subsection 4.4. Let $z = n_\bullet(\mathrm{Sk})$. Extend Sk do a dessin as shown in Figure 11. The new element here is Figure 11(c): the edge adjacent to a monovalent $\bullet$-vertex v is extended towards ∂B_+ and v is replaced with a $\circ$-vertex (which is bivalent in the complete dessin in B), giving rise to a zigzag rather than an oval. The result is the dessin of a real trigonal curve $C \subset \Sigma := \Sigma_{2k+2-z}$ with $(5k+4-z)$ ovals and z zigzags.

6.3.1. To distinguish the curves topologically, consider the region $B_+^\sharp$ obtained from the interior of B_+ by adding small regular neighborhoods of the zigzags and

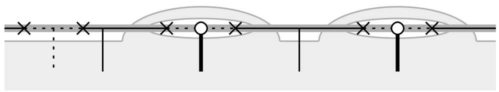


FIGURE 12. The region $B_+^\sharp$

removing the zigzags themselves and all inner ×-vertices, see Figure 12. Since zigzags are clearly distinguishable topologically, the monodromy $\pi_1(B_+^\sharp) \to \Gamma$ is a topological invariant of the curve. On the other hand, at least topologically, a pair of ×-vertices constituting a zigzag can collapse to a single type II singular fiber; hence the Γ-valued monodromy about a whole zigzag equals that about a monovalent •-vertex. Thus, the image of the monodromy $\pi_1(B_+^\sharp) \to \Gamma$ equals $[\![\operatorname{Stab}\operatorname{Sk}]\!]$, and distinct skeletons produce non-isotopic curves.

6.3.2. Let $(m_1, \ldots, m_z)$ be the sequence of vertex distances between consecutive monovalent •-vertices of Sk (*cf.* Subsection 5.5; the monovalent •-vertices themselves are also included into the count, so that each $m_i \geqslant 3$). Then the topology of pair $(\Sigma_\mathbb{R}, C_\mathbb{R})$ is uniquely determined by the following two properties:

- $C_\mathbb{R}$ does not intersect the zero section except once inside each zigzag;
- the pair of zigzags of $C_\mathbb{R}$ corresponding to a pair of consecutive monovalent •-vertices at a distance m is separated by $(m-3)$ ovals.

Similar to Lemma 5.5.2, one can easily see that the curves C', C'' obtained from two skeletons Sk', Sk'' as above have fiberwise isotopic real parts if and only if the corresponding sequences (m_i'), (m_j'') differ by a cyclic permutation.

6.3.3. If $z = n_\bullet(\operatorname{Sk})$ is even, the double covering X of Σ ramified at C and E is a generic Jacobian real elliptic surface. The surfaces obtained from distinct skeletons Sk or distinct (not related by an automorphism of Sk) lifts of the real structure are neither deformation equivalent nor isomorphic in the class of directed real Lefschetz fibrations, as they differ by the homological invariants, *cf.* 6.3.1. The necklace diagram of X, see [32], can be recovered from the sequence $(m_1, \ldots, m_z)$ introduced in 6.3.2: reading from m_z down to m_1, each pair m_{2i}, m_{2i-1} gives rise to a copy of —>—, followed by $(m_{2i}-3)$ copies of —○—, a copy of —<—, and $(m_{2i-1}-3)$ copies of —□—. Two sequences produce isomorphic necklace diagrams if and only if they differ by an *even* cyclic permutation. (Thus, the lift of the real structure is encoded in the choice of a marked monovalent •-vertex of Sk.)

6.3.4. Remark. In the terminology of [17], the curves constructed in this section are *ribbon curves* with all blocks of type I_1 or II_3. Conversely, any such curve C over the rational base is obtained by the above construction, and the ribbon curve structure of C is encoded by the original skeleton Sk. It follows that both the fiberwise deformation type and the fiberwise isotopy type of C determine its ribbon curve structure. In [17], a similar assertion is stated for ribbon curves with all blocks of type I_2 or II_3.

6.3.5. Remark. It is worth emphasizing that the analytic and topological classifications of the curves constructed above coincide. This fact substantiates the conjecture that real trigonal curves are *quasi-simple*, *i.e.*, the fiberwise equisingular deformation type of such a curve $C \subset \Sigma_k$ is determined by the topological type of the quadruple $(\Sigma_k, C; \operatorname{pr}, \operatorname{conj})$, where $\operatorname{pr}\colon \Sigma_k \to \mathbb{P}^1$ is the ruling.

7. The transcendental lattice

In this section, we give a formal definition of a new invariant of monodromy factorizations, which we call the *transcendental lattice*, and discuss a few open questions.

7.1. The construction. Fix a commutative ring R, two R-modules $\mathcal{L}$, $\mathcal{V}$, and a skew-symmetric bilinear form $\bigwedge^2\mathcal{L}\to\mathcal{V}$, $x\wedge y\mapsto x\cdot y$. (In case $\mathcal{V}$ has a 2-torsion, we assume, in addition, that $x\cdot x=0$ for all $x\in\mathcal{L}$.) Fix, further, a symplectic (with respect to the chosen form) representation $G\to\operatorname{Sp}\mathcal{L}$.

7.1.1. Definition. Given a G-valued monodromy factorization $\bar{\mathfrak{m}}=(\mathfrak{m}_1,\ldots,\mathfrak{m}_r)$, define the following objects:

(1) the R-module $\mathcal{L}\otimes\bar{\mathfrak{m}}:=\bigoplus_{i=1}^r\mathcal{L}$;
(2) the R-linear map $\chi\colon\mathcal{L}\otimes\bar{\mathfrak{m}}\to\mathcal{L}$, $\bigoplus_i x_i\mapsto\sum_i(\mathfrak{m}_i-1)x_i$;
(3) the R-quadratic map $q\colon\mathcal{L}\otimes\bar{\mathfrak{m}}\to\mathcal{V}$,

$$\bigoplus x_i\mapsto-\sum_{i=1}^r x_i\cdot\mathfrak{m}_i x_i+\sum_{1\leqslant i<j\leqslant r}(\mathfrak{m}_i-1)x_i\cdot(\mathfrak{m}_j-1)x_j.$$

(Here, q is R-quadratic in the sense that $q(rx)=r^2q(x)$ for all $x\in\mathcal{L}\otimes\bar{\mathfrak{m}}$, $r\in R$ and $(x,y)\mapsto q(x+y)-q(x)-q(y)\in\mathcal{V}$ is a $\mathcal{V}$-valued bilinear form.)

Let $\mathcal{L}_{\bar{\mathfrak{m}}}=\operatorname{Ker}\chi$, and define $\mathcal{L}_{\bar{\mathfrak{m}}}^\perp=\{x\in\mathcal{L}_{\bar{\mathfrak{m}}}\,|\,q(y+x)=q(y)\text{ for all }y\in\mathcal{L}_{\bar{\mathfrak{m}}}\}$. Then, $\mathcal{L}_{\bar{\mathfrak{m}}}^\perp\subset\mathcal{L}_{\bar{\mathfrak{m}}}$ is an R-submodule and the quotient $\mathcal{T}(\bar{\mathfrak{m}}):=\mathcal{L}_{\bar{\mathfrak{m}}}/\mathcal{L}_{\bar{\mathfrak{m}}}^\perp$ inherits a quadratic map $q\colon\mathcal{T}(\bar{\mathfrak{m}})\to\mathcal{V}$. It is called the *transcendental lattice* of $\bar{\mathfrak{m}}$ (defined by the representation $G\to\operatorname{Sp}\mathcal{L}$).

7.1.2. Lemma. *One has* $q(x+y)-q(x)-q(y)=\chi(x)\cdot\chi(y)\bmod 2\mathcal{V}$ *for any pair* $x,y\in\mathcal{L}\otimes\bar{\mathfrak{m}}$.

Proof. The proof is a simple computation taking into account the fact that each $\mathfrak{m}_i$ is a symplectic automorphism of $\mathcal{L}$, so that $\mathfrak{m}_i x_i\cdot\mathfrak{m}_i y_i+x_i\cdot y_i=2(x_i\cdot y_i)=0\bmod 2\mathcal{V}$. □

7.1.3. Corollary. *If $\mathcal{V}$ is free of 2-torsion, the quadratic form $q\colon\mathcal{L}_{\bar{\mathfrak{m}}}\to\mathcal{V}$ extends to a symmetric bilinear form $\mathcal{L}_{\bar{\mathfrak{m}}}\otimes\mathcal{L}_{\bar{\mathfrak{m}}}\to\mathcal{V}$.* □

The symmetric bilinear extension of q is also denoted by q. Its kernel equals the submodule $\mathcal{L}_{\bar{\mathfrak{m}}}^\perp$ defined above, and q factors to a nondegenerate symmetric bilinear form $q\colon\mathcal{T}(\bar{\mathfrak{m}})\otimes\mathcal{T}(\bar{\mathfrak{m}})\to\mathcal{V}$. The pair $(\mathcal{T}(\bar{\mathfrak{m}}),q)$ is still called the *transcendental lattice* of $\bar{\mathfrak{m}}$.

7.1.4. Remark. Assume that $\mathcal{L}=H_1(F)$ for a punctured oriented surface F and that the map $G\to\operatorname{Sp}\mathcal{L}$ is induced by a certain representation of G in the mapping class group of F. In these settings, a weak Hurwitz equivalence class of a G-valued monodromy factorization $\bar{\mathfrak{m}}$ of length r represents an F-bundle $X\to B^\sharp$ over a disk $B^\sharp$ with r punctures, see 1.1.2, one has $\mathcal{L}_{\bar{\mathfrak{m}}}=H_2(X)$, and the symmetric bilinear form $q\colon\mathcal{L}_{\bar{\mathfrak{m}}}\otimes\mathcal{L}_{\bar{\mathfrak{m}}}\to\mathbb{Z}$ is given by the intersection index, $q\colon x\otimes y\mapsto x\circ y$. Indeed, X is homotopy equivalent to an F-bundle over a wedge of circles in $B^\sharp$, and $H_2(X)$ can be computed by applying the Mayer–Vietoris exact sequence to the union of $F\times\{\text{basepoint}\}$ and a number of cylinders $F\times I$, the map χ serving essentially as the boundary homomorphism. Then, the self-intersection of a 2-cycle can be found by shifting the wedge to another copy, transversal to the original one, see [16] and [1] for details. (In [16], a similar approach is also used to compute the intersection form of an F-bundle over any skeleton.) Definition 7.1.1 is thus a mere generalization of this simple algorithm.

The group $\mathcal{L}\otimes\bar{\mathfrak{m}}$ can be interpreted as $H_2(X,F_b)$, where F_b is the fiber over a point $b\in\partial B^\sharp$, but the quadratic form $q\colon\mathcal{L}\otimes\bar{\mathfrak{m}}\to\mathbb{Z}$ does not seem to have a simple geometric meaning. Examples show that the associated bilinear form does not need to be divisible by 2, see [16] or Example 7.2.6 below.

7.1.5. Definition. A (*weak*) *isomorphism* between two triples $(\mathcal{M}_1;\chi_1,q_1)$ and $(\mathcal{M}_2;\chi_2,q_2)$, where $\mathcal{M}_i$ is an R-module, $\chi_i\colon\mathcal{M}_i\to\mathcal{L}$ is an R-linear map, and $q_i\colon\mathcal{M}_i\to\mathcal{V}$ is an R-quadratic map, is an R-isomorphism $\varphi\colon\mathcal{M}_1\to\mathcal{M}_2$ such that $q_1=q_2\circ\varphi$ and $\chi_1=\chi_2\circ\varphi$ (respectively, $\chi_1=g\circ\chi_2\circ\varphi$ for some $g\in G$).

7.1.6. Proposition. *The triples* $(\mathcal{L}\otimes\bar{\mathfrak{m}};\chi,q)$ *and* $(\mathcal{L}\otimes\bar{\mathfrak{m}}';\chi',q')$ *corresponding to two strongly* (*respectively, weakly*) *equivalent monodromy factorizations* $\bar{\mathfrak{m}}$ *and* $\bar{\mathfrak{m}}'$ *are isomorphic* (*respectively, weakly isomorphic*). *In particular, the transcendental lattice* $q\colon\mathcal{T}(\bar{\mathfrak{m}})\to\mathcal{V}$ *is a weak equivalence invariant of* $\bar{\mathfrak{m}}$.

Proof. If $\bar{\mathfrak{m}}'$ is obtained from $\bar{\mathfrak{m}}$ by a global conjugation, $\mathfrak{m}'_i=g^{-1}\mathfrak{m}_i g$, $g\in G$, the weak isomorphism $\mathcal{L}\otimes\bar{\mathfrak{m}}'\to\mathcal{L}\otimes\bar{\mathfrak{m}}$ is $\varphi\colon\bigoplus x'_i\mapsto\bigoplus gx'_i$; then $\chi'=g^{-1}\circ\chi\circ\varphi$.

Assume that $\bar{\mathfrak{m}}'$ is obtained from $\bar{\mathfrak{m}}$ by one inverse Hurwitz move,

$$\mathfrak{m}'_i=\mathfrak{m}_{i+1},\quad \mathfrak{m}'_{i+1}=\mathfrak{m}_{i+1}\mathfrak{m}_i\mathfrak{m}_{i+1}^{-1},\quad \mathfrak{m}'_j=\mathfrak{m}_j,\ j\ne i,i+1.$$

Then the isomorphism $\varphi\colon\bigoplus x'_i\mapsto\bigoplus x_i$ is given by

$$x_i=\mathfrak{m}_{i+1}^{-1}x'_{i+1},\quad x_{i+1}=x'_i+(\mathfrak{m}_i-1)\mathfrak{m}_{i+1}^{-1}x'_{i+1},\quad x_j=x'_j,\ j\ne i,i+1.$$

It is straightforward that

$$(\mathfrak{m}_i-1)x_i+(\mathfrak{m}_{i+1}-1)x_{i+1}=(\mathfrak{m}'_i-1)x'_i+(\mathfrak{m}'_{i+1}-1)x'_{i+1};\tag{7.1.7}$$

hence $\chi'=\chi\circ\varphi$. Furthermore, due to (7.1.7), the essentially different terms in the expressions for q and q' are

$$-x_i\cdot\mathfrak{m}_i x_i-x_{i+1}\cdot\mathfrak{m}_{i+1}x_{i+1}+(\mathfrak{m}_i-1)x_i\cdot(\mathfrak{m}_{i+1}-1)x_{i+1}$$

(and the corresponding primed terms). Rewrite the latter sum in the form

$$-x_i\cdot\mathfrak{m}_i x_i+[(\mathfrak{m}_i-1)x_i-x_{i+1}]\cdot(\mathfrak{m}_{i+1}-1)x_{i+1}$$

(using $x_{i+1}\cdot x_{i+1}=0$) and observe that $(\mathfrak{m}_i-1)x_i-x_{i+1}=-x'_i$ and

$$(\mathfrak{m}_{i+1}-1)x_{i+1}=(\mathfrak{m}'_i-1)x'_i+(\mathfrak{m}'_{i+1}-1)x'_{i+1}-\mathfrak{m}_{i+1}^{-1}(\mathfrak{m}'_{i+1}-1)x'_{i+1}.$$

Multiplying out and using the fact that $\cdot$ is skew-symmetric and $\mathfrak{m}_{i+1}=\mathfrak{m}'_i$ is a symplectic automorphism, one obtains $q'=q\circ\varphi$. $\square$

7.2. Examples and open questions. The transcendental lattice $q\colon\mathcal{T}(\bar{\mathfrak{m}})\to\mathcal{V}$ is a relatively new invariant (regarded as an invariant of a monodromy factorization) and I do not know how powerful it is. In particular, I do not know if it can be expressed in terms of other known invariants.

7.2.1. Problem. Is there a relation between $\mathcal{T}(\bar{\mathfrak{m}})$ and other known invariants, for example $[\![\operatorname{Im}(\bar{\mathfrak{m}})]\!]$ and $[\![\mathfrak{m}_\infty(\bar{\mathfrak{m}})]\!]$?

Most known examples of computation of $\mathcal{T}(\bar{\mathfrak{m}})$ use the identity representation $\tilde{\Gamma} = \operatorname{Sp}\mathcal{H}$, see 2.1, and deal with a monodromy factorization representing the homological invariant of an extremal elliptic surfaces X. In this case, $\mathcal{T}$ is indeed the transcendental lattice of X, *i.e.*, the orthogonal complement $NS(X)^\perp \subset H_2(X)$, with the form induced by the intersection index; this relation explains the terminology, and it is the computation in [16] that inspired Definition 7.1.1.

7.2.2. Example. The $\tilde{\Gamma}$-valued reductions of the (non-simple) monodromy factorizations arising from the pairs of plane sextics with the sets of singularities marked with a * in Table 2, see Subsection 5.6, differ by their transcendental lattices. An easy way to prove this fact is to compare the geometric classification of curves found in [14], [15] and their arithmetic classification found in [33]. The same argument shows that the other pairs in Table 2 have isomorphic transcendental lattices.

7.2.3. Example. For each $k \geqslant 0$, the simple $\tilde{\Gamma}$-valued monodromy factorizations given by Theorem 1.2.1 have isomorphic transcendental lattices, see [16]. If k is even, one has $\mathcal{T} \cong \mathbf{D}_k$ (with the usual convention $\mathbf{D}_0 = 0$ and $\mathbf{D}_2 = 2\mathbf{A}_1$); if $k = 2s-1$ is odd, then $\mathcal{T}$ is the orthogonal complement $(3\mathbf{v}_1 + \ldots + 3\mathbf{v}_s + \mathbf{v}_{s+1} + \ldots + \mathbf{v}_{2s-1})^\perp$ in the orthogonal direct sum $\bigoplus_{i=1}^{2s-1}\mathbb{Z}\mathbf{v}_i$, $\mathbf{v}_1^2 = 1$.

7.2.4. A *coloring* of length r is a function $\ell\colon \{1,\ldots,r\} \to \{\pm 1\}$. Given a simple Γ-valued monodromy factorization $\bar{\mathfrak{m}} = (\mathfrak{m}_1,\ldots,\mathfrak{m}_r)$ and a coloring ℓ of length r, define $\mathcal{T}(\bar{\mathfrak{m}},\ell)$ as the transcendental lattice of the $\tilde{\Gamma}$-valued lift of $\bar{\mathfrak{m}}$ obtained as follows: an entry $\mathfrak{m}_i = g_i^{-1}\mathbb{X}\mathbb{Y}g_i$, $i = 1,\ldots,r$, $g_i \in \Gamma$, lifts to $g_1^{-1}\ell(i)\mathbb{X}\mathbb{Y}g_i \subset \tilde{\Gamma}$. Alternatively, this lift can be described as the one with the eigenvalues of sign $\ell(i)$; in this form, the concept can be extended to a wider class of monodromy factorizations, for example, to those with unipotent entries, which arise from elliptic surfaces/trigonal curves over the rational base and without exceptional singular fibers. The following statement is immediate.

7.2.5. Proposition. *Assume that two simple* Γ*-valued monodromy factorizations* $\bar{\mathfrak{m}}'$, $\bar{\mathfrak{m}}''$ *of length* r *are weakly equivalent. Then there is a permutation* $\sigma \in \mathbb{S}_r$ *such that, for any coloring* ℓ *of length* r, *one has* $\mathcal{T}(\bar{\mathfrak{m}}',\ell) \cong \mathcal{T}(\bar{\mathfrak{m}}'',\ell\circ\sigma)$. □

7.2.6. Example. In [16], the lattices $\mathcal{T}(\bar{\mathfrak{m}},\ell)$ are computed for all Γ-valued monodromy factorizations given by Theorem 1.2.1, see (5.2.2), and all colorings ℓ taking exactly one value -1. It turns out that the isomorphism class of $\mathcal{T}(\bar{\mathfrak{m}},\ell)$ depends on k only. The corresponding quadratic forms $q\colon \mathcal{H}\otimes\bar{\mathfrak{m}} \to \mathbb{Z}$ are also computed; in general, they do *not* extend to integral symmetric bilinear forms.

7.2.7. Problem. Does Proposition 7.2.5 distinguish the weak equivalence classes given by Theorem 1.2.2?

7.2.8. Example. We conclude with the only known to me example of a direct computation of the transcendental lattice using a representation other than $\tilde{\Gamma} = \operatorname{Sp}\mathcal{H}$. In [1], the authors give an explicit construction of a pair of reducible sextics (each splitting into an irreducible quintic Q and a line L) with the set of singularities $\mathbf{A}_{10} \oplus \mathbf{A}_9$ and compute their $\mathbb{B}_5$-valued braid monodromies with respect to the pencil of lines through a generic point in L. Then, following more or less the lines of Definition 7.1.1 and using the obvious representation $\mathbb{B}_5 \to \operatorname{Sp} H_1(F)$, where F is a punctured surface of genus 2, they compute the transcendental lattices and show that they are distinct (the latter fact being predicted beforehand using theory of $K3$-surfaces). It is worth mentioning that the two sextics are conjugate over $\mathbb{Q}(\sqrt{5})$; thus, $\mathcal{T}$ is a topological, but not algebraic, invariant.

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