SMOOTH MODELS OF SINGULAR $K3$-SURFACES

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Abstract. We show that the classical Fermat quartic has exactly three smooth spatial models. As a generalization, we give a classification of smooth spatial (as well as some other) models of singular $K3$-surfaces of small discriminant. As a by-product, we observe a correlation (up to a certain limit) between the discriminant of a singular $K3$-surface and the number of lines in its models. We also construct a $K3$-quartic surface with 52 lines and singular points, as well as a few other examples with many lines or models.

1. Introduction

All algebraic varieties considered in the paper (except §1.5) are over $\mathbb{C}$.

1.1. **Fermat quartics.** The original motivation for this paper was the classical *Fermat quartic* $\Phi_4 = X_{48} \subset \mathbb{P}^3$ given by the equation

$$z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0.$$

It is immediate from the equation that X_{48} contains 48 straight lines, *viz.*

$$z_a - \epsilon_1 z_b = z_c - \epsilon_2 z_d = 0,$$

where $\epsilon_1^4 = \epsilon_2^4 = -1$ and $\{\{a, b\}, \{c, d\}\}$ is an unordered partition of the index set $\{0, \ldots, 3\}$ into two unordered pairs. The maximal possible number of lines in a smooth quartic surface is 64 (see [24, 20]) and there are but ten (eight up to complex conjugation) quartics with more than 52 lines (see [5]). When [5] appeared, it was immediately observed by T. Shioda that one of these extremal quartic, *viz.* X_{56} in the notation of [5], is isomorphic, as an abstract $K3$-surface, to the classical Fermat quartic X_{48}. This observation resulted in a beautiful paper [25], which provides explicit defining equations for the surface X_{56} and isomorphism $X_{48} \cong X_{56}$ and studies further geometric properties of X_{56}. This explicit construction (first to my knowledge) is particularly interesting due to the fact (see [18, Theorem 1.8] with a further reference to [12]) that $(d, n) = (4, 3)$ is the only pair with $n \geqslant 3$ for which two smooth hypersurfaces of the same degree d in $\mathbb{P}^n$ may be isomorphic as abstract algebraic varieties but not projectively equivalent.

Smooth quartics in $\mathbb{P}^3$ are $K3$-surfaces, and we define a *smooth spatial model* of a $K3$-surface X as an embedding $X \hookrightarrow \mathbb{P}^3$ defined by a very ample line bundle of degree 4. Two models are *projectively equivalent* if so are their images. According to the authors of [25], an extensive search for other smooth spatial models of the Fermat quartic Φ_4 did not produce any results, suggesting that such models do not exist. This assertion is one of the principal results of the present paper.

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Theorem 1.1 (see §5)**.** *Up to projective equivalence, there are three smooth models* $\Phi_4 \hookrightarrow \mathbb{P}^3$ *of the Fermat quartic*: *they are* X_{48}, X_{56}, *and* $\bar{X}_{56}$.

A detailed proof of this theorem is given in §4 and §5.

This statement is rather surprising, because there must be several thousands of *singular* spatial models of the Fermat quartic (*cf.* Remark 4.5; we do not attempt their classification) and because the number of distinct spatial models of a singular $K3$-surface growth rather fast with the discriminant (*cf.* Remark 8.4 in §8.2 or the discussion of the smooth models of $X([2, 1, 82])$ in §8.4).

1.2. **Quartics of small discriminant.** The approach used in the classification of the spatial models of Φ_4 applies to other $K3$-surfaces. We confine ourselves to the most interesting case of the so-called *singular* $K3$-surfaces X, *i.e.*, those of the maximal Picard rank $\operatorname{rk} NS(X) = 20$. Due to the global Torelli theorem, such a $K3$-surface X is determined by its *transcendental lattice* $T := NS(X)^\perp \subset H_2(X)$, considered up to orientation preserving isometry (see, *e.g.*, [26] and §3.1). This is a positive definite even lattice of rank 2; we use the notation $X := X(T)$ and the single line (instead of a matrix) notation $T := [a, b, c]$ for the lattice $T = \mathbb{Z}u + \mathbb{Z}v$, $u^2 = a$, $v^2 = c$, $u \cdot v = b$ (see §2.4 and §3.1). To illustrate the approach, we outline the proof of the following theorem, which formally incorporates Theorem 1.1. For a technical reason (see Remark 3.8), we omit the case $T = [4, 0, 16]$; conjecturally, the corresponding $K3$-surface $X(T)$ has no smooth spatial models.

Theorem 1.2 (see §6.3)**.** *Let* T *be a positive definite even lattice of rank* 2, *and assume that* $\det T \leqslant 80$ *and* $T \neq [4, 0, 16]$. *Then, up to projective equivalence, any smooth spatial model* $X(T) \hookrightarrow \mathbb{P}^3$ *is one of those listed in Table 1.*

In Table 1, the lattices T are grouped according to their genus (or, equivalently, isomorphism class of the discriminant form, see §2.1 for details). For each genus, we list the discriminant $\det := \det T$, the isomorphism classes of lattices T (line by line), the smooth spatial models X (using, whenever possible, the notation introduced in [5] for the configurations of lines, the subscript always indicating the number of lines; all models apply to all lattices in the genus), and the pencil structure $\mathfrak{ps}(X)$, which can be used to identify quartics (see §3.2). The number of models of $X(T)$ is constant within each genus (see Remark 3.6). Lattices admitting no orientation reversing automorphism are marked with a *; the corresponding $K3$-surfaces are not real, and neither are their models. The * next to a model designates the fact that, although the $K3$-surface itself is defined over $\mathbb{R}$, the model is not.

The notation X_*, Y_*, *etc.* refers to the diagrams found on pages 27–28 (unless indicated otherwise in the table); diagrams corresponding to several models are denoted by $(D*)$. In principle, these diagrams suffice to recover the lattices; details are explained in the relevant parts of the proof.

Two of the quartics, *viz.* X_{64} and X_{48}, carry faithful actions of *Mukai groups* (*i.e.*, maximal finite groups of symplectic automorphisms, see [13] and §7.3); these are all quartics with this property that contain lines (see Theorem 7.5).

Note that there are 97 lattices T, constituting 78 genera, satisfying the condition $\det T \leqslant 80$. According to Theorem 1.2, very few of the corresponding $K3$-surfaces admit smooth spatial models.

The configurations of lines not found in [5] are X_{48} (the classical Fermat quartic), Q'''_{52}, the alternative models $\tilde{Y}'_{48}$, $\tilde{Y}''_{48}$ of Y'_{52}, Y''_{52}, respectively, and the alternative models Z'_{48}, Z''_{48} of Z_{52}, Z_{50} and Q_{48} of X'_{52}, Q''_{52}. (Following [5], we denote by Z_*

Table 1. Nonsingular spatial models (see Theorem 1.2)

det	T	X	Pencils, remarks
48	$[8,4,8]$	X_{64}	$(6,0)^{16}(4,6)^{48}$; Mukai group T_{192}
55	$[4,1,14]^*$	X''_{60}	$(4,5)^{60}$
60	$[4,2,16]$	X'_{60}	$(6,2)^{10}(4,4)^{30}(3,7)^{20}$
		Q_{56}	$(4,4)^{24}(3,7)^{32}$
64	$[2,0,32]$	Y_{56}	$(4,4)^{32}(3,7)^{24}$
64	$[4,0,16]$	??	Conjecturally, none
64	$[8,0,8]$	$X_{56}{}^*$	$(4,6)^{8}(4,4)^{32}(2,8)^{16}$ (see page 22)
		X_{48}	$(2,8)^{48}$ (see page 22); Mukai group F_{384}
75	$[10,5,10]$	Q'''_{52}	$(5,0)^{4}(3,6)^{48}$
76	$[2,0,38]$	Y'_{52}	$(4,6)^{2}(4,4)^{16}(3,5)^{20}(2,8)^{14}$
	$[8,2,10]^*$	$\tilde{Y}'_{48}$	$(3,5)^{24}(2,8)^{24}$
76	$[4,2,20]$	Q_{54}	$(4,4)^{24}(4,3)^{24}(0,12)^{6}$; see (D1)
		X''_{52}	$(6,0)^{1}(4,4)^{9}(4,3)^{18}(3,5)^{18}(0,12)^{6}$; see (D1)
79	$[2,1,40]$	Y''_{52}	$(4,5)^{8}(4,3)^{12}(3,6)^{16}(2,7)^{16}$
	$[4,1,20]^*$	$\tilde{Y}''_{48}$	$(4,3)^{6}(3,6)^{12}(2,7)^{30}$
	$[8,1,10]^*$		
80	$[4,0,20]$	Z_{52}	$(6,0)^{4}(4,4)^{12}(4,2)^{24}(2,8)^{12}$; see (D2), (D5)
		$Z_{50}{}^*$	$(4,4)^{10}(3,5)^{40}$
		Z'_{48}	$(4,2)^{16}(2,8)^{32}$; see (D2)
		$Z''_{48}{}^*$	$(3,5)^{48}$; see (D2), (D5)
80	$[8,4,12]$	X'_{52}	$(6,0)^{1}(4,4)^{12}(4,3)^{12}(4,2)^{3}(3,5)^{18}(0,12)^{6}$
		$Q''_{52}{}^*$	$(4,4)^{8}(4,3)^{32}(4,2)^{8}(0,12)^{4}$
		Q_{48}	$(4,2)^{8}(3,5)^{32}(0,12)^{8}$; see (D4)

the configurations of lines generating a sublattice of rank 19 in $NS(X)$. In all other cases in Table 1, the space $NS(X)\otimes\mathbb{Q}$ is generated by the classes of the lines contained in X.) All quartics given by Theorem 1.2 contain many (at least 48) lines and, conversely, all eight quartics with at least 56 lines (see [5]) do appear in the table. This observation may shed new light on the line counting problem. For example, the following statement is an alternative characterization of Schur's quartic X_{64}, given by the equation

$$z_0(z_0^3-z_1^3)=z_2(z_2^3-z_3^3).$$

(Recall that, according to [5], Schur's quartic X_{64} is the only smooth quartic that contains the maximal number 64 of lines).

Corollary 1.3. *If a singular $K3$-surface $X(T)$ admits a smooth spatial model, then either $T=[8,4,8]$ and the model is Schur's quartic X_{64}, or $\det T\geqslant 55$.* ◁

In other words, 48 is the minimal discriminant of a singular $K3$-surface admitting a smooth spatial model, and Schur's quartic X_{64} is the only one minimizing this discriminant, just as it is the only quartic maximizing the number of lines.

This statement, together with the next corollary (which follows from the proof of Theorem 1.2), substantiates Conjecture 1.6 below. (Though, see also Remark 1.7.)

Corollary 1.4 (see §6.4)**.** *Let T be as above, $\det T \leqslant 80$, and $T \neq [4,0,16]$. Then, $X(T)$ admits a degree 2 map $X(T) \to Q := \mathbb{P}^1 \times \mathbb{P}^1$ with smooth ramification locus $C \subset Q$ if and only if either*

- *$T = [4,2,20]$, and the model is V'_{48}, see page 27, or*
- *$T = [8,4,12]$, and the model is V''_{48}, see (D3) on page 27.*

In each case, the model is unique up to projective equivalence and C has the maximal number 12 of bitangents in each of the two rulings of Q (see also Remark 6.7).

Certainly, when speaking about bitangents of the ramification locus C, we admit the degeneration of a bitangent to a generatrix intersecting C at a single point with multiplicity 4 (*cf.* the discussion of tritangents right before Theorem 1.5 below). Note, though, that in the extremal case as in Corollary 1.4, all twelve generatrices are true bitangents, as follows immediately (together with the bound 12 itself) from the Riemann–Hurwitz formula.

With few exceptions, any pair $X', X'' \subset \mathbb{P}^3$ of smooth models (not necessarily distinct) of the same $K3$-surface $X(T)$ appearing in Table 1 constitutes a so-called *Oguiso pair* (see [18] and §6.5); in particular, the models are Cremona equivalent. The exceptions are:

- the quadruple Z_* of models of $X([4,0,20])$, which splits into two pairs, *viz.* (Z_{52}, Z'_{48}) and (Z_{50}, Z''_{48}), each having the above property, and
- the Fermat quartic X_{48}: there is no Oguiso pair (X_{48}, X_{48}) (*cf.* [25]).

If $X' = X''$, then Oguiso's construction [18] gives us a Cremona self-equivalence $X' \to X''$ that is not regular on the ambient space $\mathbb{P}^3$. It follows also that each model is a *Cayley $K3$-surface*, *i.e.*, a smooth determinantal quartic (see [1, 18]). These phenomena are specific to small discriminants, see Theorem 1.8 below.

1.3. **Other polarizations.** The approach applies as well to other polarizations of $K3$-surfaces, *i.e.*, projective models $\varphi\colon X \hookrightarrow \mathbb{P}^n$ defined by linear systems $|h|$, $h \in NS(X)$, $h^2 = 2n-2$. We will consider the following commonly used models.

(1) $h^2 = 2$: a *planar* model $\varphi\colon X \to \mathbb{P}^2$. The model is a degree 2 map ramified at a sextic curve $C \subset \mathbb{P}^2$; it is called *smooth* if so is C (*cf.* Corollary 1.4).
(2) $h^2 = 4$: a *spatial* or *quartic* model $\varphi\colon X \to \mathbb{P}^3$ considered in Theorem 1.2.
(3) $h^2 = 6$: a *sextic* model $\varphi\colon X \to \mathbb{P}^4$. The image of φ is a complete intersection (regular if φ is smooth) of a quadric and a cubic.
(4) $h^2 = 8$: an *octic* model $\varphi\colon X \to \mathbb{P}^5$. Typically, the image of φ is a complete intersection (regular if φ is smooth) of three quadrics (*cf.* Lemma 7.4).

In the first case, a *line* in X is a smooth rational curve that projects isomorphically to a line in $\mathbb{P}^2$. The projection establishes a two-to-one correspondence between lines and tritangents of C. (Here, we admit the possibility that a tritangent degenerates to a line intersecting C at two points with multiplicities 2 and 4 or at a single point with multiplicity 6. More generally, if C is allowed to be singular, the "tritangents" are the so-called *splitting lines*, *i.e.*, lines whose local intersection index with C is even at each intersection point.)

Theorem 1.5 (see §7.2)**.** *Let T be a positive definite even lattice of rank 2.*

(1) *If $\det T \leqslant 116$ and $X(T)$ admits a smooth planar model $X(T) \to \mathbb{P}^2$, then $T = [12,6,12]$ and the only model is 2_{144}, see page 33.*

Table 2. Smooth sextic models (see Theorem 1.5(3))

det	T	X	Ranks	Pencils, remarks
39	$[2,1,20]$	6_{42}	$(19,19)$	$(9)^{42}$
	$[6,3,8]$			
48	$[6,0,8]$	$6'_{42}$	$(19,19)$	$(9)^{42}$; see (D6)
		6_{38}	$(19,19)$	$(11)^2(9)^{16}(7)^{20}$
		$6'_{36}$	$(19,19)$	$(8)^{36}$; see (D6)
48	$[8,4,8]$	$6''_{36}$	$(18,18)$	$(8)^{36}$

Table 3. Smooth octic models (see Theorem 1.5(4))

det	T	X	Ranks	Pencils, remarks
32	$[4,0,8]$	$8_{36}{}^*$	$(20,20)$	$(7)^{16}(6)^{16}(4)^4$
		8_{32}	$(17,17)$	$(6)^{32}$; Mukai group F_{384}
36	$[6,0,6]$	$8'_{36}$	$(19,20)$	$(6)^{36}$; see (D7)
		$8_{33}{}^\diamond$	$(18,18)$	$(8)^9(5)^{24}$; see (D7)
		$8'_{32}$	$(18,18)$	$(6)^{32}$
39	$[2,1,20]$	8_{30}	$(18,19)$	$(6)^{12}(5)^{12}(4)^6$
	$[6,3,8]$			

(3) *If* $\det T \leqslant 48$*, then, up to projective equivalence, any smooth sextic model* $X(T) \hookrightarrow \mathbb{P}^4$ *is one of those listed in Table 2.*

(4) *If* $\det T \leqslant 40$*, then, up to projective equivalence, any smooth octic model* $X(T) \hookrightarrow \mathbb{P}^5$ *is one of those listed in Table 3.*

In the tables, we use conventions similar to Table 1, referring to the diagrams found on pages 33–34. As an additional invariant, we list the ranks of the sublattice $F \subset NS(X)$ generated by the classes of lines and its extension $F + \mathbb{Z}h$. Instead of the pencil structure, we merely list the valencies of the vertices of the dual adjacency graph of the lines. The model 8_{33} marked with a diamond $^\diamond$ is the only one (in Table 3) whose defining ideal is not generated by polynomials of degree 2 (see Lemma 7.4). With the exception of 6_{42} and $6'_{42}$, all configurations found in the tables are pairwise distinct, as the invariants show.

The ramification locus of 2_{144} admits a faithful action of the Mukai group M_9; hence, according to Sh. Mukai [13], its equation is

$$z_0^6 + z_1^6 + z_2^6 = 10(z_0^3 z_1^3 + z_1^3 z_2^3 + z_2^3 z_0^3).$$

In §7.3, we show that no sextic model containing lines carries a faithful action of a Mukai group; the two octics with this property are 8_{32} and a model of $X([4,0,12])$, with the same configuration of lines and an action of H_{192} (see Theorem 7.5).

As a by-product, we obtain new lower bounds on the maximal number of lines in a model. (According to S. Rams, private communication, no interesting examples of sextics are known, whereas the best known example of an octic has 32 lines; octics with 32 lines have been studied in [6].) Comparing Theorem 1.5 and Corollaries 1.3 and 1.4, we conjecture that these new bounds are sharp.

Conjecture 1.6. A smooth sextic curve $C \subset \mathbb{P}^2$ has at most 72 tritangents. A smooth sextic (octic) model of a $K3$-surface has at most 42 (respectively, 36) lines.

Remark 1.7. The cases of sextic and octic models in Conjecture 1.6 are settled, in the affirmative, in [4], where we also give a sharp bound on the maximal number of lines in a smooth $K3$-surface $X \hookrightarrow \mathbb{P}^{D+1}$ for all $2 \leqslant D \leqslant 15$. (For $D \geqslant 16$, the bound is 24 and its sharpness depends on the residue $D \bmod 12$; for all $D \gg 0$, large configurations of lines are fiber components of elliptic pencils.) Thus, the only case that remains open is that of plane sextic curves.

Curiously, the motivating observation, *viz.* the fact that the number of lines is maximized by the discriminant minimizing singular $K3$-surfaces, does not persist for higher polarizations: for degree 10 surfaces in $\mathbb{P}^6$, the discriminant minimizing surface $X([2,0,16])$ has fewer (28) lines than the maximum 30.

1.4. **Further examples.** Each $K3$-surface $X(T)$ found in Table 1 has at most three (two up to projective equivalence and complex conjugation) distinct smooth spatial models. The number of models of any particular $K3$-surface is always finite, but we show that this number is not bounded. (The former statement, which is an immediate consequence from the finiteness of each genus of lattices, was first obtained by Sterk [27].)

Theorem 1.8 (see §8.2)**.** *For each integer d, the number of projective equivalence classes of spatial models (smooth or not) $X(T) \to \mathbb{P}^3$ with $\det T \leqslant d$ is finite.*

However, for each integer $M > 0$, there exist a lattice T and M smooth spatial models $X(T) \hookrightarrow \mathbb{P}^3$ that are pairwise not Cremona equivalent.

An almost literate analogue of this statement for the other three polarizations considered in §1.3 is discussed at the end of §8.2.

Another misleading observation suggested by Table 1 is the fact that the number of lines in smooth spatial models of $X(T)$ tends to decrease when $\det T$ increases. This tendency persists only up to a certain limit, *viz.* 52 lines (which is the maximal number of lines realized by an equilinear 1-parameter family of quartics, see [5]). Furthermore, the number of such models of a given $K3$-surface is not bounded, providing an alternative series of examples, with a large number of lines but much lower growth rate (*cf.* Remark 8.4), for the statement of Theorem 1.8.

Proposition 1.9 (see §8.3)**.** *For each integer $n > 1$, there is a smooth spatial model $X([4n,0,24]) \hookrightarrow \mathbb{P}^3$ containing 52 lines, namely, the configuration Z_{52}. Denoting by $N(n)$ the number of such models, we have $\limsup_{n\to\infty} N(n) = \infty$.*

The proof of Proposition 1.9 is based on the fact that the lines constituting the configuration Z_{52} span a lattice of rank 19. Similar arguments would apply to all sextics in Table 2 and most octics in Table 3, producing infinitely many smooth models $X \hookrightarrow \mathbb{P}^4$ or $\mathbb{P}^5$ with many (42 or 33, respectively) lines. (In fact, as shown in [4], there also is a 1-parameter family of octics with 34 lines.)

We conclude with an example of a *singular* quartic containing many lines. At present, the best known bound on the number of lines in a quartic surface with at worst simple singularities is 64, and the best known example of a singular quartic has 40 lines (and one simple node). Both statements are due to D. Veniani [28].

Theorem 1.10 (see §8.5)**.** *The $K3$-surface $X([4,0,12])$ has a spatial model with two simple nodes that contains 52 lines.*

Recently, D. Veniani (private communication) has found an explicit equation of this quartic:

$$z_0z_1(z_0^2+z_1^2+z_2^2+z_3^2)+z_2z_3(z_0^2+z_1^2-z_2^2-z_3^2)=2z_2^2z_3^2-2z_0^2z_1^2-2z_0z_1z_2z_3.$$

He also observed that, when reduced modulo 5, the quartic has four simple nodes and 56 lines: the best known example in characteristics other than 2 and 3.

Remarkably, the discriminant $\det[4,0,12]=48$ equals that of Schur's quartic. A few other singular quartics with many lines are also discussed in §8.5.

1.5. **Other fields of definition.** Given a field $\Bbbk\subset\mathbb{C}$, one can define the maximal number $M_\Bbbk$ of lines defined over $\Bbbk$ in a smooth quartic $X\subset\mathbb{P}^3$ defined over $\Bbbk$. Thus,

$$M_{\mathbb{C}}=64,\qquad M_{\mathbb{R}}=56,\qquad M_{\mathbb{Q}}\leqslant 52,$$

see [5, 20]. (Similar numbers can be defined for other polarizations as well, but very little is known about them.) The precise value of $M_{\mathbb{Q}}$ was left unsettled in [5], as in all interesting examples for which defining equations are known the lines are only defined over quadratic algebraic number fields (which can usually be shown by computing the cross-ratios of appropriate quadruples of intersection points). We discuss this problem in §8.4 and show that $M_{\mathbb{Q}}\geqslant 46$.

Theorem 1.11 (see §8.4)**.** *Given a smooth quartic $X\subset\mathbb{P}^3$ defined over $\mathbb{Q}$, denote by $\operatorname{Fn}_{\mathbb{Q}}X$ the set of lines in X defined over $\mathbb{Q}$ and let $\mathcal{F}_{\mathbb{Q}}(X)\subset NS(X)$ be the lattice spanned by h and the classes $[\ell]$, $\ell\in\operatorname{Fn}_{\mathbb{Q}}X$. Then one has:*

1. *if $\operatorname{rk}\mathcal{F}_{\mathbb{Q}}(X)=20$, then $|\operatorname{Fn}_{\mathbb{Q}}X|\leqslant 41$, and this bound is sharp;*
2. *there exists a model Q_{46} of $X([2,1,82])$ with $|\operatorname{Fn}_{\mathbb{Q}}Q_{46}|=|\operatorname{Fn}Q_{46}|=46$; one has $\operatorname{rk}\mathcal{F}_{\mathbb{Q}}(Q_{46})=19$ and $\mathfrak{ps}(Q_{46})=(4,3)^2(3,5)^{16}(3,4)^{16}(2,7)^{12}$.*

In the course of the proof of Theorem 1.11 we also observe that the $K3$-surface $X([2,1,82])$ has more than three thousands of distinct smooth spatial models. All models and lines therein are defined over $\mathbb{Q}$ (see §8.4).

The bound $M_\Bbbk$ can be defined for any field $\Bbbk$, including the case $\operatorname{char}\Bbbk>0$. We have

$$M_{\bar{\mathbb{F}}_2}=60,\qquad M_{\bar{\mathbb{F}}_3}=112,\qquad M_{\bar{\mathbb{F}}_p}=64\quad\text{for } p\geqslant 5,$$

see [3], [19], and [20], respectively. (In fact, lines in the maximizing examples are also defined over quadratic extensions of $\mathbb{F}_p$.) Most other questions considered in this paper also make sense over fields of positive characteristic. If $\operatorname{rk}NS(X)\leqslant 20$, the lattice $NS(X)$ lifts to characteristic 0 (see, *e.g.*, [11]) and appropriate versions of Theorems 1.2 and 1.5 still hold as *upper bounds* on the number of models. (We can no longer assert the existence of each model over each field because of the lack of surjectivity of the period map. Note also that, in the arithmetical settings, the transcendental lattice T is not defined; however, we can still speak about its genus and, in particular, the discriminant $\det T$.) Corollary 1.3 also holds if $\operatorname{char}\Bbbk\geqslant 5$.

In the case $\operatorname{char}\Bbbk>0$, a more interesting phenomenon is that of *supersingular* $K3$-surfaces X, *i.e.*, such that $\operatorname{rk}NS(X)=22$. (Note, though, that the quartics over $\bar{\mathbb{F}}_p$, $p\neq 3$, maximizing the number of lines are *not* supersingular; for example, if $p=2$, the bound for supersingular quartics is 40 lines, see [3].) Up to isomorphism, a supersingular $K3$-surface X is determined by its *Artin invariant*

$$\sigma(X):=\frac{1}{2}\dim_{\mathbb{F}_p}\operatorname{discr}NS(X)\in\{1,\dots,10\},$$

cf. §2.1 below. Unfortunately, the approach outlined in §1.6, *viz.* embedding the orthogonal complement $S := h^\perp \subset NS(X)$ to a Niemeier lattice, does not work unless $\sigma(X) = 1$. As an alternative, one can probably start from all, not necessarily smooth, models with $\sigma(X) = 1$ and study root-free finite index sublattices of S. We postpone this question until a future paper. The intuition (*cf.* the discussion of long vectors in §1.6 below) suggests that there should be a large number of smooth models whenever $\operatorname{char} \Bbbk \gg 0$ or $h^2 \gg 0$; probably, the two most interesting cases would be $\operatorname{char} \Bbbk = 2$ or 3 (*cf.* [3, Theorems 1.1, 1.2 and Remark 7.7]).

1.6. **Idea of the proof.** Most proofs in the paper reduce to a detailed study of the lattice $S := h^\perp \subset NS(X)$, where h is the polarization. We can easily control the genus of S; however, since S is negative definite of rank 19, this genus typically consists of a huge number of isomorphism classes. To list the classes, we represent S as the orthogonal complement of a certain fixed lattice $\mathbf{V}$ in a Niemeier lattice N (see §3.3). Certainly, this approach is not new, *cf.*, *e.g.*, Kondō [10], dealing with the Mukai groups, or Nishiyama [17], where Jacobian elliptic $K3$-surfaces are studied by means of the orthogonal complement $\mathbf{U}^\perp \subset NS(X)$ of the sublattice generated by the distinguished section and a generic fiber.

Typically, there are many isometric embeddings $\mathbf{V} \hookrightarrow N$, hence, many models (*cf.* Theorem 1.8 and Proposition 1.9). However, if $\mathbf{V}$ is sufficiently "small", all or most orthogonal complements $S \cong \mathbf{V}^\perp \subset N$ contain roots and the corresponding models are singular; this phenomenon is accountable for the fact that singular $K3$-surfaces of small discriminant admit very few *smooth* models. A good quantitative restatement of this intuitive observation might be an interesting lattice theoretical problem shedding more light to the models of $K3$-surfaces. At present, I can only suggest the computation of the so-called *minimal dense square*, covering partially the special case of a single long vector (see the proof of Theorem 1.11 in §8.4); most other cases are handled by a routine `GAP`-aided enumeration.

1.7. **Contents of the paper.** In §2, we recall briefly a few notions and known results concerning integral lattices and their extensions, which are the principal technical tools of the paper. In §3, Theorems 1.1 and 1.2 are partially reduced to the classification of root-free lattices in certain fixed genera; this, in turn, amounts to the study of appropriate sublattices in the Niemeier lattices.

A detailed proof of Theorem 1.1 is contained in §4 (elimination of most Niemeier lattices) and §5 (a thorough study of the few lattices left). A similar, but less detailed, proof of Theorem 1.2 and Corollary 1.4 is outlined in §6; at the end of this section, we also discuss Oguiso pairs and Cremona equivalence. In §7, we extend the approach to the other polarizations of $K3$-surfaces, prove Theorem 1.5, and discuss smooth polarized $K3$-surfaces that carry a faithful action of one of the Mukai groups by symplectic projective automorphisms (for short, *models of Mukai groups*). Finally, in §8, we consider a few sporadic examples, in particular those constituting Theorems 1.8, 1.10, 1.11 and Proposition 1.9.

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2. Lattices

We recall briefly a few notions and known results concerning integral lattices and their extensions. Principal references are [15] and [2].

2.1. **Integral lattices** (see [15])**.** An (*integral*) *lattice* is a finitely generated free abelian group L equipped with a symmetric bilinear form

$$L \otimes L \to \mathbb{Z}, \quad x \otimes y \mapsto x \cdot y.$$

We abbreviate $x \cdot x = x^2$. In this paper, all lattices are nondegenerate and *even*, *i.e.*, $x^2 = 0 \bmod 2$ for each $x \in L$. The group of autoisometries of a lattice L is denoted by $O(L)$.

We also consider $\mathbb{Q}$-valued symmetric bilinear forms, possibly degenerate, on free abelian groups; to avoid confusion, they are referred to as *forms*. The kernel of a form Q is the subgroup

$$\ker Q = Q^\perp := \bigl\{x \in Q \bigm| x \cdot y = 0 \text{ for each } y \in Q\bigr\}.$$

The quotient $Q/\ker Q$ (often abbreviated to $Q/\ker$) is a nondegenerate form.

An example of a form is the *dual group* $L^\vee$ of a lattice L,

$$L^\vee := \bigl\{x \in L \otimes \mathbb{Q} \bigm| x \cdot y \in \mathbb{Z} \text{ for each } y \in L\bigr\},$$

with the bilinear form inherited from $L \otimes \mathbb{Q}$. We have an obvious inclusion $L \subset L^\vee$; the finite quotient group $\operatorname{discr} L := L^\vee/L$ is called the *discriminant group* of L. The lattice L is said to be *unimodular* if $\operatorname{discr} L = 0$, *i.e.*, if $L = L^\vee$. In general, one has $\ell(\operatorname{discr} L) \leqslant \operatorname{rk} L$, where the *length* $\ell(A)$ of an abelian group A is defined as the minimal number of elements generating A.

The discriminant group inherits from $L \otimes \mathbb{Q}$ a symmetric bilinear form

$$b\colon \operatorname{discr} L \otimes \operatorname{discr} L \to \mathbb{Q}/\mathbb{Z}, \quad (x \bmod L) \otimes (y \bmod L) \mapsto (x \cdot y) \bmod \mathbb{Z},$$

and its quadratic extension

$$q := q_L\colon \operatorname{discr} L \to \mathbb{Q}/2\mathbb{Z}, \quad (x \bmod L) \mapsto x^2 \bmod 2\mathbb{Z},$$

called, respectively, the *discriminant bilinear and quadratic forms.* If there is no confusion, we use the abbreviation $b(\alpha, \beta) = \alpha \cdot \beta$ and $q(\alpha) = \alpha^2$. The discriminant form is *nondegenerate* in the sense that the homomorphism

$$\operatorname{discr} L \to \operatorname{Hom}(\operatorname{discr} L, \mathbb{Q}/\mathbb{Z}), \quad \alpha \mapsto (\beta \mapsto \alpha \cdot \beta),$$

is an isomorphism. When speaking about (auto-)morphisms of discriminant groups $\operatorname{discr} L$, *the discriminant forms are always taken into account.*

Note that *not* any sublattice $S \subset L$ is an orthogonal direct summand. However, if S is nondegenerate, we have well-defined orthogonal projections

$$\operatorname{pr}_S\colon L \to S^\vee, \qquad \operatorname{pr}_S^\perp\colon L \to (S^\perp)^\vee \tag{2.1}$$

to the dual forms; these projections extend to $L^\vee$.

Lattices are naturally grouped into *genera*. Omitting the precise definition, we merely use [15, Corollary 1.9.4] which states that two nondegenerate even lattices L', L'' are in the same genus if and only if one has $\operatorname{rk} L' = \operatorname{rk} L''$, $\sigma(L') = \sigma(L'')$ (where $\sigma(L)$ is the usual signature of $L \otimes \mathbb{R}$), and $\operatorname{discr} L' \cong \operatorname{discr} L''$. Each genus consists of finitely many isomorphism classes.

2.2. **The homomorphism $O(L) \to \operatorname{Aut}\operatorname{discr} L$.** When speaking about isometries of discriminant groups, we always take q into account. The group of autoisometries of $(\operatorname{discr} L, q)$ is denoted by $\operatorname{Aut}\operatorname{discr} L$. The action of $O(L)$ extends to $L \otimes \mathbb{Q}$ by linearity, and the latter extension descends to $\operatorname{discr} L$. Hence, there is a canonical homomorphism $O(L) \to \operatorname{Aut}\operatorname{discr} L$. In general, this map is neither one-to-one nor onto; nevertheless, we do not introduce a dedicated notation and freely apply autoisometries $g \in O(L)$ to objects in $\operatorname{discr} L$. The abbreviation $[\operatorname{Aut}\operatorname{discr} L : O(L)]$ stands for the index of the image of the above canonical homomorphism. Given an element $\gamma \in \operatorname{discr} L$, we denote by $\operatorname{stab}\gamma \subset \operatorname{Aut}\operatorname{discr} L$ and $\operatorname{Stab}\gamma \subset O(L)$ the stabilizer of γ and its pull-back in $O(L)$, respectively.

Let $\mathcal{L} := \operatorname{discr} L$ and $A := \operatorname{Aut}\mathcal{L}$. Denote by $\langle\gamma\rangle \subset \mathcal{L}$ the subgroup generated by an element $\gamma \in \mathcal{L}$. The restriction $q|_{\langle\gamma\rangle}$ is nondegenerate if and only if the order of γ^2 in $\mathbb{Q}/\mathbb{Z}$ equals that of γ in $\mathcal{L}$. If this is the case, we have an orthogonal direct sum decomposition $\mathcal{L} = \langle\gamma\rangle \oplus \gamma^\perp$ and $\gamma^\perp$ is also nondegenerate. Fix $s \in \mathbb{Q}/2\mathbb{Z}$ and a nondegenerate quadratic form $\mathcal{V}$ and consider the set

$$\mathcal{L}_s(\mathcal{V}) := \bigl\{\gamma \in \mathcal{L} \bigm| q|_{\langle\gamma\rangle} \text{ is nondegenerate, } \gamma^2 = s,\ \gamma^\perp \cong \mathcal{V}\bigr\}.$$

It is immediate from the definitions that $\mathcal{L}_s(\mathcal{V})$ consists of a single A-orbit and that, for each $\gamma \in \mathcal{L}_s(\mathcal{V})$, the stabilizer $\operatorname{stab}\gamma$ is canonically identified with the full automorphism group $\operatorname{Aut}\gamma^\perp$. Hence, denoting by $\gamma\, O(L)$ the orbit, we have

$$|\mathcal{L}_s(\mathcal{V})| \cdot [\operatorname{Aut}\gamma^\perp : \operatorname{Stab}\gamma] = |\gamma\, O(L)| \cdot [\operatorname{Aut}\mathcal{L} : O(L)]. \tag{2.2}$$

2.3. **Extensions** (see [15])**.** An *extension* of a lattice S is an even lattice $L \supset S$. Two extensions $L', L'' \supset S$ are *isomorphic* if there is a bijective isometry $L' \to L''$ preserving S *as a set*; they are *strictly isomorphic* if this isometry can be chosen identical on S. In the latter case, if L is fixed, we will also speak about the $O(L)$-orbits of isometries $S \hookrightarrow L$.

A *finite index extension* of S is an even lattice $L \supset S$ such that $\operatorname{rk} L = \operatorname{rk} S$, *i.e.*, L contains S as a subgroup of finite index. An extension gives rise to an isometry $L \hookrightarrow S \otimes \mathbb{Q}$ and, since L is also a lattice, we have $L \subset S^\vee$. Furthermore, the subgroup $\mathcal{K} := L/S \subset \operatorname{discr} S = S^\vee/S$, called the *kernel* of the extension, is *isotropic*, *i.e.*, $q|_\mathcal{K} = 0$. Conversely, if $\mathcal{K} \subset \operatorname{discr} S$ is isotropic, the group

$$L := \bigl\{x \in S \otimes \mathbb{Q} \bigm| x \bmod S \in \mathcal{K}\bigr\}$$

is an integral lattice. This can be summarized in the following statement.

Proposition 2.3 (see [15])**.** *Given a lattice S, the correspondence*

$$L \mapsto \mathcal{K} := L/S, \qquad \mathcal{K} \mapsto L := \bigl\{x \in S \otimes \mathbb{Q} \bigm| x \bmod S \in \mathcal{K}\bigr\}$$

is a bijection between the set of strict isomorphism classes of finite index extensions $L \supset S$ and that of isotropic subgroups $\mathcal{K} \subset \operatorname{discr} S$. Under this correspondence, one has $\operatorname{discr} L = \mathcal{K}^\perp/\mathcal{K}$. Furthermore,

1. *two extensions L', L'' are isomorphic if and only if their kernels $\mathcal{K}'$, $\mathcal{K}''$ are in the same $O(S)$-orbit,* i.e.*, there is $g \in O(S)$ such that $g(\mathcal{K}') = \mathcal{K}''$;*
2. *an autoisometry $g \in O(S)$ extends to L if and only if $g(\mathcal{K}) = \mathcal{K}$.* ◁

A *primitive extension* is an extension $L \supset S$ such that S is primitive in L, *i.e.*, $(S \otimes \mathbb{Q}) \cap L = S$. In [15], such extensions are studied by fixing (the isomorphism class of) the orthogonal complement $T := S^\perp \subset L$. Then, L is a finite index extension of $S \oplus T$ in which both S and T are primitive. According to Proposition 2.3, this extension is described by its kernel

$$\mathcal{K} \subset \operatorname{discr}(S \oplus T) = \operatorname{discr} S \oplus \operatorname{discr} T,$$

and the primitivity of S and T in L implies that

$$\mathcal{K} \cap \operatorname{discr} S = \mathcal{K} \cap \operatorname{discr} T = 0.$$

In other words, $\mathcal{K}$ is the graph of a certain monomorphism $\psi\colon \mathcal{D} \to \operatorname{discr} T$, where $\mathcal{D} \subset \operatorname{discr} S$; since $\mathcal{K}$ is isotropic, ψ is an anti-isometry.

To keep track of the sublattice $S \subset L$, Statements (1) and (2) of Proposition 2.3 should be restricted to the subgroup $O(S) \times O(T) \subset O(S \oplus T)$. Below, we use freely a number of other similar restrictions taking into account additional structures.

An extension $L \supset S \oplus T$ is unimodular if and only if $\mathcal{K}^\perp/\mathcal{K} = 0$, *i.e.*, $\mathcal{K}^\perp = \mathcal{K}$. Then, $|\mathcal{K}|^2 = |\operatorname{discr} S| \cdot |\operatorname{discr} T|$, which implies that $|\mathcal{K}| = |\operatorname{discr} S| = |\operatorname{discr} T|$ and ψ above is a group isomorphism $\operatorname{discr} S \to \operatorname{discr} T$.

Corollary 2.4 (see [15])**.** *Given a pair of lattices S, T, there is a natural one-to-one correspondence between the strict isomorphism classes of unimodular finite index extensions $N \supset S \oplus T$ in which both S and T are primitive and bijective isometries $\psi\colon \operatorname{discr} S \to -\operatorname{discr} T$. If an isometry ψ (hence, an extension N) is fixed, then*

1. *the isomorphism classes of extensions are in a one-to-one correspondence with the double cosets $O(S)\backslash \operatorname{Aut} \operatorname{discr} S/O(T)$;*
2. *a pair of autoisometries $g \in O(S)$, $h \in O(T)$ extends to N if and only if one has $g = \psi^{-1} h \psi$ in $\operatorname{Aut} \operatorname{discr} S$.* ◁

2.4. **Lattices of rank 2.** According to Gauss [8], any positive definite even lattice T of rank 2 has a basis in which the Gram matrix of T is of the form

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix}, \qquad 0 < a \leqslant c, \quad 0 \leqslant 2b \leqslant a, \quad a = c = 0 \bmod 2,$$

and the ordered triple (a, b, c) satisfying the above conditions is uniquely determined by T. We use the notation $[a, b, c]$ for the isomorphism class of T. This description implies that $\frac{3}{4}c \leqslant \det T \leqslant c^2$, which makes the enumeration of lattices within a given genus an easy task.

Let $O^+(T) \subset O(T)$ be the group of orientation preserving autoisometries of T. Using the description above, one can see that, with few exceptions, $O^+(T) = \{\pm \mathrm{id}\}$. The exceptions are the lattices $T = [2n, 0, 2n]$ (with $O^+(T) \cong \mathbb{Z}/4$) and $[2n, n, 2n]$ (with $O^+(T) \cong \mathbb{Z}/6$), where n is any positive integer. Since the greatest common divisor of the entries of a Gram matrix is a genus invariant, each of the exceptional lattices is unique in its genus. It follows that, for a positive definite even lattice T of rank 2, the image of $O^+(T)$ in $\operatorname{Aut} \operatorname{discr} T$ depends on the genus of T only.

2.5. **Root systems** (see [2, Chapter 4])**.** A *root system*, or *root lattice*, is a positive definite lattice R generated by its *roots*, *i.e.*, vectors $a \in R$ of square 2. (Recall that we consider even lattices only.) Any positive definite lattice L contains its maximal root lattice $\operatorname{rt}(L)$, which is generated by all roots $a \in L$.

Each root lattice decomposes uniquely into an orthogonal direct sum of irreducible ones, which are of type $\mathbf{A}_n$, $n \geqslant 1$, $\mathbf{D}_n$, $n \geqslant 4$, or $\mathbf{E}_n$, $n = 6, 7, 8$. One has

$$\operatorname{discr} \mathbf{A}_n \cong \mathbb{Z}/(n+1), \quad |\operatorname{discr} \mathbf{D}_n| = 4, \quad |\operatorname{discr} \mathbf{E}_n| = 9 - n.$$

For these groups, we use the numbering $\operatorname{discr} R = \{0 = \alpha_0, \alpha_1, \ldots\}$ as in [2].

The lattices $\mathbf{A}_{n-1}$ and $\mathbf{D}_n$ are, respectively, the orthogonal complement and (mod 2)-orthogonal complement of the characteristic vector $\bar{\mathbf{e}} := \mathbf{e}_1 + \ldots + \mathbf{e}_n$ in the odd unimodular lattice $\mathbf{H}_n := \bigoplus_{i=1}^{n} \mathbb{Z}\mathbf{e}_n$, $\mathbf{e}_i^2 = 1$. Then, $\mathbf{E}_8 \supset \mathbf{D}_8$ and $\mathbf{E}_7 \supset \mathbf{A}_7$ are the index 2 extensions by the vector $\frac{1}{2}\bar{\mathbf{e}} - 4\mathbf{e}_8$. Alternatively, the lattices $\mathbf{E}_n$, $n = 6, 7$, can be described as $\mathbf{E}_n = \mathbf{A}_{8-n}^{\perp} \subset \mathbf{E}_8$.

If n is large, "short" vectors in $\mathbf{H}_n \otimes \mathbb{Q}$ tend to have many equal coordinates, and we follow [2] and use the "run-length encoding" for the $\mathbb{S}_n$-orbits of such vectors: the notation

$$(s_1)^{u_1} \ldots (s_t)^{u_t}, \quad s_1 < \ldots < s_t, \quad u_i > 0, \quad u_1 + \ldots + u_t = n$$

designates the orbit whose representatives have u_i coordinates s_i, $i = 1, \ldots, t$.

In this notation, the shortest representatives of the nonzero elements of the discriminant groups are as follows (for $\mathbf{E}_6$ and $\mathbf{E}_7$, we only indicate the squares):

$$\mathbf{A}_n\colon \quad \alpha_p = \frac{1}{n+1}\big((-p)^q (q)^p\big), \quad \alpha_p^2 = \frac{pq}{n+1} \quad (q := n+1-p); \tag{2.5}$$

$$\mathbf{D}_n\colon \quad \begin{array}{c} \alpha_{2k+1} = \dfrac{1}{2}\big((-1)^p (1)^{n-p}\big), \quad \alpha_{2k+1}^2 = \dfrac{n}{4} \quad (p = k \bmod 2), \\ \alpha_2 = \big((0)^{n-1}(\pm 1)^1\big), \quad \alpha_2^2 = 1; \end{array} \tag{2.6}$$

$$\mathbf{E}_6\colon \quad \alpha_1^2 = \alpha_2^2 = \frac{4}{3}; \qquad \mathbf{E}_7\colon \quad \alpha_1^2 = \frac{3}{2}. \tag{2.7}$$

The groups $O(\mathbf{A}_{n-1})$ and $O(\mathbf{D}_n)$ are semi-direct products $RG \rtimes \mathbb{S}_n$, where RG is generated by $-\operatorname{id}$ and, for $\mathbf{D}_n$, by the reflections against all basis elements $\mathbf{e}_i$. If n is large, `GAP`'s built-in orbit/stabilizer routines do not work very well, and we use the run-length encoding to classify the pairs and triples of vectors (*cf.* §4.2 below). More precisely, given two $\mathbb{S}_n$-orbits encoded by $(s_1^*)^{u_1^*} \ldots (s_{t^*}^*)^{u_{t^*}^*}$, $* = {}'$ or ${}''$, the $\mathbb{S}_n$-orbits of pairs of vectors can be encoded by the sequences of the form

$$\ldots (s_i', s_j'')^{u_{ij}} \ldots, \quad 1 \leqslant i \leqslant t', \quad 1 \leqslant j \leqslant t''$$

(after disregarding the entries with $u_{ij} = 0$), where $u_{ij} \geqslant 0$ are integers such that $\sum_j u_{ij} = u_i'$ for each i and $\sum_i u_{ij} = u_j''$ for each j. The classification of such balanced matrices $[u_{ij}]$ is straightforward. The further passage from pairs to triples of vectors is done in a similar way.

2.6. **The Niemeier lattices** (see [2, Chapter 16] and [14]). A *Niemeier lattice* is a positive definite unimodular even lattice of rank 24. Up to isomorphism, there are 24 Niemeier lattices. One of them, the so-called *Leech lattice*, has no roots, and each of the other 23 lattices N is a finite index extension of $\operatorname{rt}(N)$. Furthermore, the isomorphism class of N is determined by that of $\operatorname{rt}(N)$, see Table 4 (where we also refer to the relevant parts of the proof of Theorem 1.1); for this reason, the Niemeier lattice with $\operatorname{rt}(N) = R$ is often denoted by $N(R)$.

Consider two positive definite lattices S, V and assume that $\operatorname{rk} S + \operatorname{rk} V = 24$ and $\operatorname{discr} S \cong -\operatorname{discr} V$, so that $V \oplus S$ admits a finite index extension to a Niemeier lattice, see Corollary 2.4.

Lemma 2.8. *Given S, V as above, the index* $[\operatorname{Aut} \operatorname{discr} S : O(S)]$ *equals the number of strict isomorphism classes of extensions $N \supset V$, to* all *Niemeier lattices N, with the property that $V^{\perp} \cong S$.*

TABLE 4. The 24 Niemeier lattices

Roots	Details	Roots	Details
$\mathbf{D}_{24}$	see §4.3	$\mathbf{A}_9^2 \oplus \mathbf{D}_6$	see §4.2.6
$\mathbf{D}_{16} \oplus \mathbf{E}_8$	see §4.2.1	$\mathbf{D}_6^4$	see §4.2.6
$\mathbf{E}_8^3$	see §4.2.1	$\mathbf{A}_8^3$	see §4.2.6
$\mathbf{A}_{24}$	see §4.3	$\mathbf{A}_7^2 \oplus \mathbf{D}_5^2$	see §4.2.6
$\mathbf{D}_{12}^2$	see §4.2.4	$\mathbf{A}_6^4$	see §4.2.6
$\mathbf{A}_{17} \oplus \mathbf{E}_7$	see §4.2.2	$\mathbf{A}_5^4 \oplus \mathbf{D}_4$	see §4.2.6
$\mathbf{D}_{10} \oplus \mathbf{E}_7^2$	see §4.2.2	$\mathbf{D}_4^6$	see §4.2.6
$\mathbf{A}_{15} \oplus \mathbf{D}_9$	see §4.2.5	$\mathbf{A}_4^6$	see §4.2.6
$\mathbf{D}_8^3$	see §4.2.6	$\mathbf{A}_3^8$	see §4.5
$\mathbf{A}_{12}^2$	see §4.2.4	$\mathbf{A}_2^{12}$	see §4.6
$\mathbf{A}_{11} \oplus \mathbf{D}_7 \oplus \mathbf{E}_6$	see §4.2.3	$\mathbf{A}_1^{24}$	see §4.7
$\mathbf{E}_6^4$	see §4.2.3	Leech	no roots

Proof. Let $A := \operatorname{Aut} \operatorname{discr} S$, and let $H, G \subset A$ be the images of $O(S)$ and $O(V)$, respectively. (For the latter, we fix an anti-isometry $\psi\colon \operatorname{discr} S \to \operatorname{discr} V$.) The statement of the lemma follows from the *double coset formula*

$$|A/G \times A/H| = \sum_{g \in G\backslash A/H} |A/(G^g \cap H)|,$$

where we let $G^g := g^{-1}Gg$. Indeed, the formula implies $[A : H] = \sum [G^g : G^g \cap H]$, the summation running over all double cosets $g \in G\backslash A/H$, *i.e.*, over all isomorphism classes of extensions $N \supset V \oplus S$, see Corollary 2.4. By the same corollary, $G^g \cap H$ is the group of autoisometries of V extending to N; hence, the contribution of a class g in the above sum is the number of strict isomorphism classes of extensions $N \supset V$ that are contained in g. □

3. THE REDUCTION

Theorems 1.1 and 1.2 can partially be reduced to the classification and study of root-free lattices in certain fixed genera, see Theorem 3.4. This, in turn, amounts to the study of appropriate sublattices in the Niemeier lattices, see Lemma 3.7.

3.1. **Quartic $K3$-surfaces.** Consider a $K3$-surface X, and let $L := H_2(X)$ be its homology group, equipped with the intersection paring. It is a unimodular even lattice of signature $(3, 19)$; such a lattice is unique up to isomorphism.

For the sake of simplicity, we confine ourselves to the *singular* $K3$-surfaces, *i.e.*, we assume that $\operatorname{rk} NS(X) = 20$. A singular $K3$-surface X is characterized by the *oriented* isomorphism type of its *transcendental lattice*

$$T := NS(X)^{\perp} \subset L = H_2(X),$$

which is a positive definite even lattice of rank 2, see [26]. (The vector space $T \otimes \mathbb{R}$ is spanned by the real and imaginary parts of a holomorphic 2-form on X, and this basis defines a distinguished orientation.) This correspondence is emphasized by the notation $X := X(T)$. The $K3$-surface $X(\bar{T})$ corresponding to the lattice T with the opposite orientation is the complex conjugate surface $\overline{X(T)}$. The surface $X(T)$

is *real* (defined over $\mathbb{R}$) if and only if T has an orientation reversing automorphism (which, in rank 2, is always involutive).

A *spatial model* of a singular $K3$-surface X is a map $\varphi\colon X \to \mathbb{P}^3$ defined by a fixed point free ample linear system of degree 4. Two models φ_1, φ_2 are *projectively equivalent* if there exist automorphisms $a\colon \mathbb{P}^3 \to \mathbb{P}^3$ and $a_X\colon X \to X$ such that $\varphi_2 \circ a_X = a \circ \varphi_1$. Denote by $h := h_\varphi \in NS(X)$, $h^2 = 4$, the class of a hyperplane section and let $S_\varphi := h^\perp \subset NS(X)$; it is an even negative definite lattice of rank 19. We can represent S_φ as the orthogonal complement $(T \oplus \mathbb{Z}h)^\perp \subset L$. The new sublattice $T \oplus \mathbb{Z}h \subset L$ is not necessarily primitive, and its primitive hull is the finite index extension determined by a certain isotropic subgroup

$$\mathcal{C} \subset \operatorname{discr} T \oplus \operatorname{discr} \mathbb{Z}h, \qquad \mathcal{C} \cap \operatorname{discr} T = 0. \tag{3.1}$$

(The last identity is due to the fact that T *is* primitive in L.) This subgroup $\mathcal{C}$ is cyclic of order 1, 2, or 4, and we define the *depth* of φ as $\operatorname{dp}\varphi := |\mathcal{C}|$. Then, by Corollary 2.4,

$$\operatorname{discr} S_\varphi \cong -\mathcal{C}^\perp/\mathcal{C}, \qquad |\operatorname{discr} S_\varphi| = 4|\operatorname{discr} T|/(\operatorname{dp}\varphi)^2. \tag{3.2}$$

Fix a lattice $S := S_\varphi$ and let $d = d(S) := \operatorname{dp}\varphi$; this number is recovered from S and T *via* (3.2). Consider the set

$$\mathcal{S}_{dh} := \bigl\{\gamma \in \operatorname{discr} S \bigm| (4/d)\gamma = 0,\ \gamma^2 = -d^2/4 \bmod 2\mathbb{Z}\bigr\}.$$

Each element $\gamma \in \mathcal{S}_{dh}$ gives rise to the isotropic subgroup $\mathcal{K}_\gamma \subset \operatorname{discr} \mathbb{Z}h \oplus \operatorname{discr} S$ generated by $(d/4)h \oplus \gamma$, and we define

$$\mathcal{S}^+_{dh} := \bigl\{\gamma \in \mathcal{S}_{dh} \bigm| \mathcal{K}^\perp_\gamma/\mathcal{K}_\gamma \cong -\operatorname{discr} T\bigr\}.$$

Clearly, $\mathcal{K}_\gamma$ is the kernel of the extension $NS(X) \supset \mathbb{Z}h \oplus S$; one has $|\mathcal{K}_\gamma| = 4/d$. Recall that $\operatorname{stab}\gamma \subset \operatorname{Aut}\operatorname{discr} S$ and $\operatorname{Stab}\gamma \subset O(S)$ are the stabilizers of an element $\gamma \in \mathcal{S}^+_{dh}$. Fixing an isometry $\mathcal{K}^\perp_\gamma/\mathcal{K}_\gamma \cong -\operatorname{discr} T$, we can regard both groups acting on the discriminant $\operatorname{discr} T$.

Remark 3.3. We always have $\mathcal{S}^+_{4h} = \mathcal{S}_{4h} = \{0\}$ and $\mathcal{K}^\perp_0/\mathcal{K}_0 = \operatorname{discr} S$. If $d = 1$ or 2, the inclusion $\mathcal{S}^+_{dh} \subset \mathcal{S}_{dh}$ may be proper. If $d = 1$, then $\mathcal{K}^\perp_\gamma/\mathcal{K}_\gamma = \gamma^\perp \subset \mathcal{S}$, where $\mathcal{S} := \operatorname{discr} S$; hence, in this case, the set $\mathcal{S}_h = \mathcal{S}_{-1/4}(-\operatorname{discr} T)$ is a single orbit of $\operatorname{Aut}\operatorname{discr} S$ and we have $\operatorname{stab}\gamma = \operatorname{Aut}\gamma^\perp \cong \operatorname{Aut}\operatorname{discr} T$, see §2.2.

Theorem 3.4. *The projective equivalence classes of spatial models $\varphi\colon X(T) \to \mathbb{P}^3$ are in a one-to-one correspondence with the triples consisting of*

- *a negative definite lattice S of rank 19 and $\operatorname{discr} S \cong -\mathcal{C}^\perp/\mathcal{C}$ as in* (3.1),
- *an $O(S)$-orbit $[\gamma] \subset \mathcal{S}^+_{dh}$ (where $d = d(S)$ is the depth, see* (3.2)*), and*
- *a double coset $c \in O^+(T)\backslash \operatorname{Aut}\operatorname{discr} T/\operatorname{Stab}\gamma$*

and such that

(1) *$d \geqslant 2$ or $d = 1$ and γ is not represented by a vector $a \in S^\vee$, $a^2 = -\frac{1}{4}$.*

Under this correspondence, the following statements hold:

(2) *a model φ is birational onto its image if and only if either $d = 4$ or $d \leqslant 2$ and the class $(2/d)\gamma$ is not represented by a (-1)-vector in $S^\vee$*;
(3) *a birational model φ is smooth if and only if S is root free*;
(4) *the straight lines contained in the image of a smooth model φ are in a one-to-one correspondence with the vectors $a \in S^\vee$, $a^2 = -\frac{9}{4}$, representing γ.*

Remark 3.5. Note that the hypothesis of Theorem 3.4(2), which is equivalent to the absence of a vector $e \in NS(X)$ such that $e^2 = 0$ and $e \cdot h = 2$, implies Condition (1) in the theorem. Since we are mainly interested in birational spatial models, we will only check the hypothesis of (2).

Remark 3.6. The number of projective equivalence classes of models and their properties (birational/smooth/number of lines and their adjacency graph) depend on the genus of the transcendental lattice T only. The last statement follows directly from Theorem 3.4(1)–(4), and for the number of models one uses, in addition, the description of the group $O^+(T)$ in §2.4 to conclude that, with (S, γ) fixed, the quotient set $O^+(T)\backslash \operatorname{Aut} \operatorname{discr} T / \operatorname{Stab} \gamma$ is independent of T.

This phenomenon has a simple geometric explanation: if T' and T'' are in the same genus, the corresponding $K3$-surfaces $X(T')$ and $X(T'')$ are Galois conjugate over some algebraic number field (see, *e.g.*, [22]), and so are their spatial models.

Proof of Theorem 3.4. Let $X := X(T)$. As explained above in this section, a spatial model gives rise to a lattice $S := h^\perp \subset NS(X)$, a class γ which defines the extension $NS(X) \supset \mathbb{Z}h \oplus S$, and a double coset c defining the extension $L \supset T \oplus NS(X)$, see §2.3. A projective equivalence induces an autoisometry of $H_2(X)$ preserving the pair $h \in NS(X)$ and the orientation of T; hence, these data are defined up to the actions listed in the statement. Conversely, a set of data as in the statement determines the extensions $H_2(X) \cong L \supset NS \ni h$ uniquely up to automorphism of L preserving the orientation of $NS^\perp$. Multiplying, if necessary, by (-1) and applying reflections, we can assume that the *marking* $L \cong H_2(X)$ is chosen so that NS is taken to $NS(X)$ and h is taken to the closure of the Kähler cone, so that h is nef. Condition (1), stating that there is no vector $e \in NS(X)$ such that $e^2 = 0$ and $e \cdot h = 1$, is equivalent to the requirement that the linear system $|h|$ has no fixed components, see [16]. Then, by [21, Corollary 3.2], the system $|h|$ is fixed point free and $\dim|h| = 3$; hence, h does define a spatial model $\varphi\colon X \to \mathbb{P}^3$.

Statement (2) follows from [21, Theorem 5.2].

Statement (3) is well known. By the Riemann–Roch theorem, if $a \in NS(X)$ and $a^2 = -2$, then either a or $-a$ is effective. If also $a \cdot h = 0$, then a represents a curve of projective degree 0; this curve is contracted by φ. Conversely, by the adjunction formula, the genus of a curve C in a $K3$-surface is given by $\frac{1}{2}C^2 + 1$. It follows that all exceptional divisors are rational (-2)-curves (in particular, all singularities are simple); clearly, the classes of these curves are orthogonal to h.

Statement (4) is also known, see, *e.g.*, [5]. If $\varphi(X)$ is smooth, then the classes of lines are the vectors $l \in NS(X)$ such that $l^2 = -2$ and $l \cdot h = 1$. Any such vector is of the form $\frac{1}{4}h \oplus a$, where $a \in S^\vee$ is as in the statement. □

3.2. **The pencil structure.** Most quartics considered in this paper contain many lines. A convenient way to identify/distinguish such quartics is the so-called *pencil structure*, which is an easily computable projective invariant.

Fix a smooth quartic $X \subset \mathbb{P}^3$. Given a line $\ell \subset X$, we can consider the pencil $\{\pi_t, t \in \mathbb{P}^1\}$ of planes containing ℓ. Each intersection $\pi_t \cap X$ is a planar quartic curve which splits into ℓ itself and the *residual cubic* $C_t \subset \pi_t$. All but finitely many residual cubics are irreducible; a certain number p of them split into three lines, and a certain number q split into a line and an irreducible conic. The pair (p, q) is called the *type* of the original line ℓ, and the *pencil structure* of X is the multiset of the types of all lines contained in X. Following [5], we use the partition notation

$$\mathfrak{ps}(X) = (p_1, q_1)^{u_1} \dots (p_r, q_r)^{u_r},$$

meaning that X contains u_i lines of type (p_i, q_i), $i = 1, \dots r$. (The total number of lines in X equals $u_1 + \dots + u_r$; this number is used as the subscript in the notation for quartics and configurations of lines.)

The pencil structure is easily computed in terms of the adjacency matrix of lines. If lines are represented as in Theorem 3.4(4), by vectors $a_i \in S^\vee$, $a_i^2 = -\frac{9}{4}$, then the adjacency matrix is the Gram matrix of $\{a_i\}$ with each entry increased by $\frac{1}{4}$.

If X is singular, the types of the fibers π_t are much more diverse (see [28]) and we do not use the notion of pencil structure.

3.3. **Reduction to Niemeier lattices.** Theorem 3.4 reduces the classification of (smooth) spatial models of a fixed $K3$-surface $X(T)$ to that of (root-free) definite even lattices S within a certain collection of genera (which is determined, *via* (3.1) and (3.2), by the genus of T). For the convenience of the further exposition, we will switch to the positive definite lattice $-S$, so that $\operatorname{discr}(-S) = \mathcal{C}^\perp/\mathcal{C}$.

By the construction, $\ell(\mathcal{C}^\perp/\mathcal{C}) \leqslant \operatorname{rk} T + 1 = 3$. Hence, using [15, Theorem 1.10.1], one can find a positive definite lattice V of rank 5 such that $\operatorname{discr} V \cong -\mathcal{C}^\perp/\mathcal{C}$. (The other conditions of the theorem hold trivially due to the additivity of the signature.) Fix one such lattice V and call it the *test lattice*. Then, according to Corollary 2.4, the direct sum $V \oplus -S$ has a unimodular finite index extension in which both V and $-S$ are primitive. Since $\operatorname{rk} V + \operatorname{rk} S = 24$, this extension is a Niemeier lattice. Thus, we have the following statement.

Lemma 3.7. *Any lattice S in Theorem 3.4 is of the form $-V^\perp$, where V is any fixed test lattice and $V \hookrightarrow N$ is a primitive embedding to a Niemeier lattice.* ◁

The finite index extension $N \supset (V \oplus -S)$ depends on certain extra data (see Corollary 2.4). Hence, *a priori*, distinct embeddings $V \hookrightarrow N'$, $V \hookrightarrow N''$ may result in isomorphic orthogonal complements $S = -V^\perp$, *cf.* Lemma 2.8.

Remark 3.8. In Lemma 3.7, one can choose a "universal" test lattice V satisfying

$$\operatorname{discr} V \cong -\operatorname{discr} T \oplus -\operatorname{discr} \mathbb{Z}h$$

and depending on the genus of T only. Then, lifting the primitivity requirement, one should check an analogue of (3.1) for each embedding $V \hookrightarrow N$. Note also that one can always assume that V has at least one root and, thus, exclude the Leech lattice in Lemma 3.7 (*cf.* Kondō [10]); in general, one cannot assert that $\operatorname{rk} \operatorname{rt}(V) \geqslant 2$ (*cf.* Convention 6.1 below; this is the reason for excluding the lattice $T = [4, 0, 16]$ from the statement of Theorem 1.2).

3.4. **The Fermat quartic: the genus $-\mathfrak{S}$.** The abstract Fermat quartic Φ_4 is characterized by the transcendental lattice $\mathbf{T} := [8, 0, 8]$, see, *e.g.*, [26]. Hence, it is immediate that any spatial model of Φ_4 has depth 1 and one has

$$\operatorname{discr} S \cong -\operatorname{discr} \mathbf{T} \oplus -\operatorname{discr} \mathbb{Z}h$$

in Theorem 3.4. Passing, as above, to $-S$, we define the *genus* $-\mathfrak{S}$ as the set of isomorphism classes of positive definite even lattices of rank 19 and discriminant $\operatorname{discr} \mathbf{T} \oplus \operatorname{discr} \mathbb{Z}h$, $h^2 = 4$.

To use Lemma 3.7, consider the test lattice $\mathbf{V}$ given, in a certain distinguished basis $\mathbf{a}_2^1$, $\mathbf{a}_2^2$, $\mathbf{a}_4$, $\mathbf{c}_4$, $\mathbf{a}_8$, by the Gram matrix

$$\mathbf{V} := \begin{bmatrix} 2 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 4 & 2 & 0 \\ 1 & 1 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 & 8 \end{bmatrix}.$$

One has $\operatorname{rt}(\mathbf{V}) = \mathbb{Z}\mathbf{a}_2^1 \oplus \mathbb{Z}\mathbf{a}_2^2 \cong \mathbf{A}_1^2$, and $\operatorname{rt}(\mathbf{V})^\perp$ is the diagonal lattice

$$\bar{\mathbf{V}} := \mathbb{Z}\bar{\mathbf{a}}_8 \oplus \mathbb{Z}\bar{\mathbf{c}}_8 \oplus \mathbb{Z}\bar{\mathbf{a}}_4, \quad (\bar{\mathbf{a}}_8)^2 = (\bar{\mathbf{c}}_8)^2 = 8, \quad (\bar{\mathbf{a}}_4)^2 = 4,$$

where

$$\bar{\mathbf{a}}_4 = \mathbf{a}_4, \quad \bar{\mathbf{a}}_8 = \mathbf{a}_8, \quad \bar{\mathbf{c}}_8 = 2\mathbf{c}_4 - \mathbf{a}_4 - \mathbf{a}_2^1 - \mathbf{a}_2^2. \tag{3.9}$$

The sublattice $\bar{\mathbf{V}} \oplus \mathbb{Z}\mathbf{a}_2^1 \oplus \mathbb{Z}\mathbf{a}_2^2 \subset \mathbf{V}$ has index 2, since $\frac{1}{2}(\bar{\mathbf{c}}_8 + \bar{\mathbf{a}}_4 + \mathbf{a}_2^1 + \mathbf{a}_2^2) \in \mathbf{V}$. Therefore, Lemma 3.7 can be restated in the following form.

Lemma 3.10. *One has* $-\mathfrak{S} = \{(\bar{\mathbf{V}} \oplus \mathbb{Z}r_1 \oplus \mathbb{Z}r_2)^\perp\}$*, where* $\bar{\mathbf{V}} \hookrightarrow N$ *runs through all primitive embedding to Niemeier lattices* N *and* $r_1, r_2 \in \operatorname{rt}(\bar{\mathbf{V}}^\perp)$ *run through pairs of orthogonal roots such that* $\frac{1}{2}(\bar{\mathbf{c}}_8 + \bar{\mathbf{a}}_4 + r_1 + r_2) \in N$. ◁

In practice, we usually simplify Lemma 3.10 even further and merely analyze a triple $\bar{\mathbf{a}}_8, \bar{\mathbf{c}}_8, \bar{\mathbf{a}}_4 \in N$ of pairwise orthogonal primitive vectors of squares 8, 8, and 4, respectively. Note that $\bar{\mathbf{a}}_4$ is always primitive, whereas $\bar{\mathbf{a}}_8$ or $\bar{\mathbf{c}}_8$ is imprimitive if and only the vector equals $2r$ for a root $r \in N$.

4. Eliminating Niemeier lattices

The principal result of this section is Theorem 4.14 in §4.8. It is proved by eliminating the Niemeier lattices one-by-one, mainly using Lemma 3.10.

4.1. **Projections to the components.**

Fix a Niemeier lattice N and let

$$\operatorname{rt}(N) = \bigoplus R_k, \quad k \in \mathcal{I} := \{1, \ldots, m\},$$

be the decomposition of $\operatorname{rt}(N)$ into irreducible components. For a subset $I \subset \mathcal{I}$ of the index set, we define $R_I := \bigoplus_{k \in I} R_k$. We use repeatedly the following statement, which is, essentially, the definition of $\operatorname{rt}(N)$.

Lemma 4.1. *If* $r \in N$ *is a root, then* $r \in R_k$ *for some index* $k \in \mathcal{I}$. ◁

Consider the orthogonal projections

$$\operatorname{pr}_k \colon N \to R_k^\vee, \quad k \in \mathcal{I}, \quad \text{and} \quad \operatorname{pr}_I \colon N \to R_I^\vee, \quad I \subset \mathcal{I},$$

see (2.1). In most cases, we will analyze a sublattice $\bar{\mathbf{V}} \subset N \subset \bigoplus_k R_k^\vee$ by means of its images $\operatorname{pr}_k(\bar{\mathbf{V}})$. More precisely, we will speak about isometries $V_k \to R_k^\vee$, not necessarily injective, of $\mathbb{Q}$-valued 3-forms *with distinguished bases* $\bar{\mathbf{a}}_8, \bar{\mathbf{c}}_8, \bar{\mathbf{a}}_4$. By means of these bases, we can identify a form and its Gram matrix and consider sums and differences of forms; thus, $\operatorname{pr}_I(\bar{\mathbf{V}}) = \sum_{k \in I} V_k$ for a subset $I \subset \mathcal{I}$, and we must have $\bar{\mathbf{V}} = \sum_{k \in \mathcal{I}} V_k$. The *orthogonal complement* $V^\perp \subset R^\vee$ of an isometry $V \to R^\vee$ is the orthogonal complement of its image.

Definition 4.2. Let R be a direct summand of $\operatorname{rt}(N)$, *i.e.*, $R = R_I$ for some $I \subset \mathcal{I}$. A 3-form $V \to R^\vee$ is said to be *bounded*, $V \leqslant \bar{\mathbf{V}}$, if the difference $\bar{\mathbf{V}} - V$ is positive semi-definite. The form $V \to R^\vee$ is *n-dense*, $n = 0, 1, 2$, if there are n pairwise orthogonal roots $r_i \in V^\perp \cap R$ such that

(1) the orthogonal complement $\bigl(V \oplus \bigoplus_i \mathbb{Z}r_i\bigr)^\perp$ has no roots in R, and
(2) the vector $\bar{\mathbf{c}}_8 + \bar{\mathbf{a}}_4 + \sum_i r_i$ is divisible by 2 in $R^\vee$.

We will denote by $\operatorname{dense}_n(R)$, $n = 0, 1, 2$, the set of the Gram matrices of bounded n-dense 3-forms $V \to R^\vee$ satisfying the additional primitivity condition

(3) if $\bar{\mathbf{a}}_8$ or $\bar{\mathbf{c}}_8$ projects to a square 8 vector in a single summand $R_k^\vee$, then the image is *not* of the form $2r$, $r \in R_k$.

Given $V \in \operatorname{dense}_n R$, the *reduced complement* $\operatorname{red}_n V$ is the *abstract* isomorphism class of the $\mathbb{Q}$-valued quadratic form obtained as follows: start with $(\bar{\mathbf{V}} - V)/\ker$ and, if $n = 2$, pass to its extension *via* $\frac{1}{2}(\bar{\mathbf{c}}_8 + \bar{\mathbf{a}}_4)$.

As usual, we let $\operatorname{dense}_*(R) := \bigcup_n \operatorname{dense}_n(R)$, $n = 0, 1, 2$.

Clearly, n roots r_i as in Definition 4.2(1) exist if and only if the root lattice $R' := \operatorname{rt}(V^\perp \cap R)$ is isomorphic to

- 0, if $n = 0$,
- $\mathbf{A}_1$ or $\mathbf{A}_2$, if $n = 1$, or
- $\mathbf{A}_1^2$, $\mathbf{A}_2 \oplus \mathbf{A}_1$, $\mathbf{A}_2^2$, $\mathbf{A}_3$, or $\mathbf{A}_4$, if $n = 2$;

in particular, $\operatorname{rk} R' \leqslant 2n$. Then, such a collection $\{r_i\}$ is unique up to the action of the Weyl group of R'; hence, Condition (2) in the definition does not depend on the choice of $\{r_i\}$. Furthermore, in view of Lemma 4.1, both conditions are "local" and can be checked independently on irreducible components: if $R = R' \oplus R''$, then the set $\operatorname{dense}_n(R)$ is the union over $n' + n'' = n$ of the subsets

$$\bigl\{V' + V'' \bigm| (V', V'') \in \operatorname{dense}_{n'}(R') \times \operatorname{dense}_{n''}(R''),\ V' + V'' \leqslant \bar{\mathbf{V}}\bigr\}. \tag{4.3}$$

It follows from (3.9) that, if $\mathbf{V} \subset N$ is embedded so that $\mathbf{V}^\perp$ is root free, then the sublattice $\bar{\mathbf{V}} \subset \mathbf{V}$, regarded as a 3-form $\bar{\mathbf{V}} \to \operatorname{rt}(N)^\vee$, is 2-dense. An immediate consequence of this observation and (4.3) is the following simple lemma.

Lemma 4.4. *Assume that any one of the following conditions holds*:

(1) $\operatorname{dense}_n R_I = \varnothing$ *whenever* $n = 0, 1$ *and* $|I| = m - 1$;
(2) $\operatorname{dense}_0 R_I = \varnothing$ *whenever* $|I| = m - 2$;
(3) $\operatorname{dense}_* R_I = \varnothing$ *for a subset* $I \subset \mathcal{I}$;
(4) *there is an index* $k \in \mathcal{I}$ *such that, for each* $V \in \operatorname{dense}_n(R_{\mathcal{I} \smallsetminus k})$, $n = 0, 1, 2$, *and each isometry* $C := \operatorname{red}_n V \to R_k^\vee$, *one has* $\operatorname{rk} \operatorname{rt}(C^\perp \cap R_k) > 4 - 2n$.

Then, a root-free lattice in the genus $-\mathfrak{S}$ *does not admit an embedding to* N *with the orthogonal complement isomorphic to* $\mathbf{V}$. ◁

4.2. **The computation.** The sets $\operatorname{dense}_* R_k$ can be computed by `GAP` [7], using the following straightforward algorithm.

(1) Consider the sets $\mathfrak{Q}_s := \{a \in R_k^\vee \mid a^2 \leqslant s\}$, $s = 4, 8$.
(2) For $\bar{\mathbf{a}}_8$, pick a representative of each $O(R_k)$-orbit of $\mathfrak{Q}_8$.
(3) For $\bar{\mathbf{c}}_8$, pick a representative of each $\operatorname{stab}(\bar{\mathbf{a}}_8)$-orbit of $\mathfrak{Q}_8$.
(4) For $\bar{\mathbf{a}}_4$, pick a representative of each $\operatorname{stab}(\bar{\mathbf{a}}_8, \bar{\mathbf{c}}_8)$-orbit of $\mathfrak{Q}_4$.
(5) Compute $\operatorname{rt}(V^\perp \cap R_k)$ and check if $V \to R_k^\vee$ is dense.

At step (1), we disregard square 8 vectors violating Condition (3) in Definition 4.2, and at steps (3) and (4), we check that the form obtained is bounded. A similar algorithm (with appropriate verification at each step) can be used to enumerate all isometries $V \to R_k^\vee$ of a particular $\mathbb{Q}$-valued form V.

If $\operatorname{rk} R_k > 9$, the built-in group action algorithms are slow and we replace them with the run-length encoding as explained in §2.5.

In this section, we do not make use of any information about the group $N/\operatorname{rt}(N)$ defining the extension $N \supset \operatorname{rt}(N)$ and merely compute the sets $\operatorname{dense}_* R_I$, $I \subset \mathcal{I}$, inductively, using (4.3). Below, we outline a few details.

4.2.1. *The lattices* $\mathbf{D}_{16}^+ \oplus \mathbf{E}_8$ *and* $\mathbf{E}_8^3$. (Here, $\mathbf{D}_{16}^+$, also denoted $\mathbf{E}_{16}$, is the so-called *Barnes–Wall lattice*; it can be defined as the only, up to isomorphism, even index 2 extension of $\mathbf{D}_{16}$.) We have $\operatorname{dense}_n(\mathbf{E}_8) = \varnothing$ unless $n = 2$, and Lemma 4.4(2) eliminates the lattice $\mathbf{E}_8^3$. Furthermore, if $V \in \operatorname{dense}_2(\mathbf{E}_8)$, then $\operatorname{red}_2 V = 0$ or $\mathbb{Z}a$, $a^2 = 4$. On the other hand, for any vector $a \in \mathbf{D}_{16}^+$, $a^2 = 4$, one has $\operatorname{rt}(a^\perp) \neq 0$ (*cf.* §2.5), and we apply Lemma 4.4(4).

4.2.2. *The root systems* $\mathbf{A}_{17} \oplus \mathbf{E}_7$ *and* $\mathbf{D}_{10} \oplus \mathbf{E}_7^2$. We have $\operatorname{dense}_0(\mathbf{E}_7) = \varnothing$. Then, $\operatorname{dense}_*(\mathbf{E}_7^2) = \varnothing$ and Lemma 4.4(3) eliminates $\mathbf{D}_{10} \oplus \mathbf{E}_7^2$. There are several dozens of forms $C := \operatorname{red}_n V$, $V \in \operatorname{dense}_n(\mathbf{E}_7)$, $n = 1, 2$. Listing the isometries $C \to \mathbf{A}_{17}^\vee$, we conclude that $\operatorname{rk}\operatorname{rt}(C^\perp \cap \mathbf{A}_{17}) \geqslant 10$ and apply Lemma 4.4(4).

4.2.3. *The root systems* $\mathbf{A}_{11} \oplus \mathbf{D}_7 \oplus \mathbf{E}_6$ *and* $\mathbf{E}_6^4$. All sets $\operatorname{dense}_*(\mathbf{E}_6)$, $\operatorname{dense}_*(\mathbf{D}_7)$ are nonempty; however, $\operatorname{dense}_0(\mathbf{E}_6^2) = \varnothing$ and Lemma 4.4(2) eliminates the root system $\mathbf{E}_6^4$. Besides, $\operatorname{dense}_n(\mathbf{D}_7 \oplus \mathbf{E}_6) = \varnothing$ unless $n = 2$ and the forms $C := \operatorname{red}_2 V$, $V \in \operatorname{dense}_2(\mathbf{D}_7 \oplus \mathbf{E}_6)$, either contain a vector of square $\frac{1}{4}$ or are $\mathbb{Z}a$, $a^2 = \frac{5}{3}$. In the former case, C admits no isometry to $\mathbf{A}_{11}^\vee$, whereas in the latter case, one has $\operatorname{rt}(a^\perp \cap \mathbf{A}_{11}) = \mathbf{A}_9 \oplus \mathbf{A}_1$ and Lemma 4.4(3) applies. Both assertions on vectors in $\mathbf{A}_{11}^\vee$ follow from (2.5).

4.2.4. *The root systems* $\mathbf{D}_{12}^2$ *and* $\mathbf{A}_{12}^2$. The sets $\operatorname{dense}_*(\mathbf{D}_{12})$ and $\operatorname{dense}_n(\mathbf{A}_{12})$ for $n = 0, 1$ are empty, and the lattices are eliminated by Lemma 4.4(1).

4.2.5. *The root system* $\mathbf{A}_{15} \oplus \mathbf{D}_9$. We have $\operatorname{dense}_n(\mathbf{D}_9) = \varnothing$ unless $n = 2$. Then, for each class $C := \operatorname{red}_2 V$, $V \in \operatorname{dense}_2(\mathbf{D}_9)$ (there are but two dozens of forms not representing 1), we enumerate the isometries $C \to \mathbf{A}_{15}^\vee$, obtaining $\operatorname{rk}\operatorname{rt}(C^\perp) \geqslant 10$. Hence, Lemma 4.4(4) eliminates this lattice.

4.2.6. *Other root systems with* $m \leqslant 6$ *components.* For the lattices listed below, all sets $\operatorname{dense}_*(R_k)$ are nonempty, and we use (4.3) to compute $\operatorname{dense}_*(R_I)$, $I \subset \mathcal{I}$. Then, we eliminate the lattice by applying

- Lemma 4.4(1), if $\operatorname{rt}(N) = \mathbf{D}_8^3$, $\mathbf{A}_9^2 \oplus \mathbf{D}_6$, $\mathbf{A}_8^3$, $\mathbf{A}_7^2 \oplus \mathbf{D}_5^2$, or $\mathbf{A}_6^4$, or
- Lemma 4.4(2), if $\operatorname{rt}(N) = \mathbf{D}_6^4$, $\mathbf{A}_5^4 \oplus \mathbf{D}_4$, $\mathbf{D}_4^6$, or $\mathbf{A}_4^6$.

4.3. **The lattices** $\mathbf{D}_{24}^+$ **and** $\mathbf{A}_{24}^+$**.** By [2, Chaprer 16], the Niemeier lattice N is the extension of $\operatorname{rt}(N) = \mathbf{D}_{24}$ or $\mathbf{A}_{24}$ by the vector $\alpha_1 \in \mathbf{D}_{24}^\vee$ or $\alpha_5 \in \mathbf{A}_{24}^\vee$, respectively, see (2.6) and (2.5). Thus, we apply the algorithm at the beginning of §4.2, restricted to the subsets $\mathfrak{Q}_s \cap N$, and list all embeddings $\bar{\mathbf{V}} \hookrightarrow N$, arriving at $\operatorname{rk}\operatorname{rt}(\bar{\mathbf{V}}^\perp) \geqslant 15$ or 11 for $N = \mathbf{D}_{24}^+$ or $\mathbf{A}_{24}^+$, respectively. It follows that $\bar{\mathbf{V}}$ cannot be 2-dense.

Remark 4.5. In fact, it is not difficult to list all primitive embeddings $\mathbf{V} \hookrightarrow N$, obtaining more than a thousand of lattices $\mathbf{V}^\perp$ in the genus $-\mathfrak{S}$ that contain roots. Thus, already these two Niemeier lattices give rise to a large number of singular spatial models of the Fermat quartic.

4.4. **Root systems with many components.** Till the end of this section, we consider a root system $R := \operatorname{rt}(N) = \bigoplus_{k=1}^m R_k$, where $m = 8, 12, 24$ and each R_k is a copy of the same irreducible root system $\mathbf{A}_n$, $n := 24/m$. Let $\alpha_{i,k} \in R_k^\vee$, $k \in \mathcal{I}$, $0 \leqslant i \leqslant n$, be a distinguished shortest representative, given by (2.5), of the i-th element in the cyclic group $\operatorname{discr} R_k \cong \mathbb{Z}/(n+1)$; sometimes, we will use the shortcut $\alpha_{i,I} := \sum_{k \in I} \alpha_{i,k}$ for a subset $I \subset \mathcal{I}$.

In all three cases, the kernel $\mathcal{K} := N/R \subset \operatorname{discr} R$ of the extension is generated by the $(m-1)$ elements of the form

$$\textstyle\sum_{k=1}^m \alpha_{p_k,k} \bmod R \in \operatorname{discr} R,$$

where the sequences (p_k) are obtained from the one given below for each lattice by all cyclic permutations of the subset $\{2, \ldots, m\} \subset \mathcal{I}$ (see [2, Chaprer 16]).

An element $\beta \in \operatorname{discr} R$ has the form $\beta_1 + \ldots + \beta_m$, where $\beta_k \in \operatorname{discr} R_k$ is the projection, $k \in \mathcal{I}$. Similarly, an element $b \in R^\vee$ has the form $b_1 + \ldots + b_m$, where $b_k := \operatorname{pr}_k b \in R_k^\vee$. For such an element, we define the *support*

$$\operatorname{supp} \beta := \bigl\{k \in \mathcal{I} \bigm| \beta_k \neq 0 \bmod R_k\bigr\}, \quad \operatorname{supp} b := \bigl\{k \in \mathcal{I} \bigm| b_k \neq 0\bigr\}$$

and *Hamming norm* $\|\cdot\| := |\operatorname{supp}(\,\cdot\,)|$. Clearly, one has $\operatorname{supp} b \supset \operatorname{supp}(b \bmod R)$. The following statement is a consequence of Lemma 4.1.

Lemma 4.6. *If $b \in R^\vee$ and $r \in R$ is a root, then $b \cdot r = 0$ unless $r \in R_k$ for some index $k \in \operatorname{supp} b$. In particular, for any isometry $\mathbf{V} \hookrightarrow N$, the two sets $\operatorname{supp} \mathbf{a}_2^i$, $i = 1, 2$, are singletons contained in $\operatorname{supp} \mathbf{c}_4$.* ◁

We use a version of run-length encoding: an element $b = b_1 + \ldots + b_m \in R^\vee$ is said to be *of the form*

$$\operatorname{rle}(b) := (s_1)^{u_1} \ldots (s_t)^{u_t}, \quad 0 < s_1 < \ldots < s_t, \quad u_i > 0,$$

if, among the projections $0 \neq b_k \in R_k^\vee$, there are exactly u_i vectors of square s_i for each $i = 1, \ldots, t$, and there are no other nonzero projections.

Unlike the previous two sections, below we consider an embedding $\mathbf{V} \hookrightarrow N$ of the original lattice $\mathbf{V}$ of rank 5. Most computations are done up to the group $O(R)$; in fact, we study isometries $\mathbf{V} \hookrightarrow R^\vee$ using some limited information about the sublattice $N \subset R^\vee$ which must contain the image. At the end, when classifying the root-free lattices found, we switch to the finer group

$$O(N) = \bigl\{g \in O(R) \bigm| g(\mathcal{K}) = \mathcal{K}\bigr\} \tag{4.7}$$

of autoisometries of N. The *combinatorial type* of an isometry $\mathbf{V} \hookrightarrow R^\vee$ is defined as its $O(R)$-orbit. (Recall that a basis for $\mathbf{V}$ is assumed fixed; hence, we can merely speak about $O(R)$-orbits of ordered quintuples of vectors in $R^\vee$.)

4.5. **The root system $\mathbf{A}_3^8$.** The kernel $\mathcal{K}$ is described as in §4.4 by

$$(p_k) = (3, 2, 0, 0, 1, 0, 1, 1).$$

Since $\operatorname{discr} \mathbf{A}_3 \cong \mathbb{Z}/4$, we can refine the Hamming norm of $\beta \in \operatorname{discr} R$ to the *type* $\operatorname{tp} \beta := (\|\beta\|, \|2\beta\|)$. We have

$$\operatorname{tp} \beta \in \{(0,0), (4,0), (5,4), (7,4), (8,0), (8,8)\} \quad \text{for each } \beta \in \mathcal{K}. \tag{4.8}$$

From this and (2.5), one can see that the square 4 vectors $b \in N$ are of the form

$$\left(\tfrac{3}{4}\right)^4(1)^1,\ (1)^4,\ (2)^2,\ (4)^1. \tag{4.9}$$

Another simple observation is the fact that, for a vector $b \in \mathbf{A}_3^\vee$, $b^2 \leqslant 8$, one has $\operatorname{rt}(b^\perp) \neq 0$ unless $b^2 = 5$. Using (4.8) again, we conclude that,

$$\text{if } b \in N,\ b^2 = 8,\ \operatorname{rle}(b) \not\ni (5), \text{ then } \operatorname{rt}(b^\perp \cap R_k) \neq \varnothing \text{ for all } k \in \mathcal{I}. \tag{4.10}$$

In the exceptional cases $\operatorname{rle}(b) = \left(\frac{3}{4}\right)^4(5)^1$ or $(1)^3(5)^1$, there is exactly one trivial intersection $\operatorname{rt}(b^\perp \cap R_k)$.

Consider a sublattice $\mathbf{V} \subset N$. By (4.10), a necessary condition for $\operatorname{rt}(\mathbf{V}^\perp) = 0$ is the bound $|\operatorname{supp}\mathbf{a}_4 \cup \operatorname{supp}\mathbf{c}_4| \geqslant 7$. Since $\mathbf{a}_4 \cdot \mathbf{c}_4 = 2$, using (4.9), we find three combinatorial types of pairs, with $\mathbf{a}_4 = \alpha_{2,1} + \alpha_{1,\{2,\ldots,5\}}$ and $\mathbf{c}_4$ one of

$$\alpha_{2,1} + \alpha_{3,2} + \alpha_{1,\{3,6,7\}}, \quad \alpha_{2,7} + \alpha_{1,\{1,2,3,6\}}, \quad \alpha_{2,2} + \alpha_{1,\{3,4,6,7\}}.$$

In each case, the total support has length 7 and $\operatorname{rt}\big((\mathbb{Z}\mathbf{a}_4 + \mathbb{Z}\mathbf{c}_4)^\perp \cap R_k\big) \neq 0$ for each index $k \in \mathcal{I}$. Then, by (4.10) again, $\mathbf{a}_8$ must have a component of length 5 in $R_8^\vee$, and then it has at most four other nonzero components. At most two components contain $\mathbf{a}_2^1$ and $\mathbf{a}_2^2$; hence, there is at least one index $k \in \mathcal{I}$ left for which

$$\operatorname{rt}(\mathbf{V}^\perp \cap R_k) = \operatorname{rt}\big((\mathbb{Z}\mathbf{a}_4 + \mathbb{Z}\mathbf{c}_4)^\perp \cap R_k\big) \neq 0.$$

4.6. **The root system $\mathbf{A}_2^{12}$.** The kernel $\mathcal{K}$ is described as in §4.4 by the sequence

$$(p_k) = (2,1,1,2,1,1,1,2,2,2,1,2);$$

it is the ternary Golay code $\mathcal{C}_{12}$. We have

$$\|\beta\| \in \{0,6,9,12\} \quad \text{for each } \beta \in \mathcal{K}. \tag{4.11}$$

In view of (2.5), it follows that $\operatorname{rle}(b) = \left(\frac{2}{3}\right)^6$ or $(2)^2$ if $b \in N$ and $b^2 = 4$, whereas the square 8 vectors $b \in N$ are of the form

$$\left(\tfrac{2}{3}\right)^{12},\ \left(\tfrac{2}{3}\right)^9(2),\ \left(\tfrac{2}{3}\right)^8\left(\tfrac{8}{3}\right),\ \left(\tfrac{2}{3}\right)^6(2)^2,\ \left(\tfrac{2}{3}\right)^5(2)\left(\tfrac{8}{3}\right),\ \left(\tfrac{2}{3}\right)^5\left(\tfrac{14}{3}\right),\ \left(\tfrac{2}{3}\right)^4\left(\tfrac{8}{3}\right)^2,\ (2)^4,\ (2)(6).$$

If $b \in \mathbf{A}_2^\vee$, $b^2 < 8$, then $\operatorname{rt}(b^\perp \cap \mathbf{A}_2) \neq 0$ unless $b^2 = 2$ or $\frac{14}{3}$. Hence, a necessary condition for $\operatorname{rt}(\mathbf{V}^\perp) = 0$ is that $|\operatorname{supp}\mathbf{a}_4 \cup \operatorname{supp}\mathbf{c}_4| \geqslant 8$. Since $\mathbf{a}_4 \cdot \mathbf{c}_4 = 2$, up to the action of $O(\mathbf{A}_2^{12})$ we have

$$\mathbf{a}_4 = \alpha_{1,\{1,\ldots,6\}} \quad \text{and} \quad \mathbf{c}_4 = \alpha_{1,\{4,\ldots,9\}} \quad \text{or} \quad \alpha_{2,\{1,2\}} + \alpha_{1,\{5,\ldots,8\}}.$$

In order to eliminate the roots in R_{10} through R_{12}, the remaining vector $\mathbf{a}_8$ must be of the form $(2)^4$; then, at least four pairs of roots survive to $\mathbf{V}^\perp$.

4.7. **The root system $\mathbf{A}_1^{24}$.** This is the only Niemeier lattice containing root-free sublattices in the genus $-\mathfrak{S}$. The kernel $\mathcal{K}$ is described as in §4.4 by

$$(p_k) = (1,0,0,0,0,0,1,0,1,0,0,1,1,0,0,1,1,0,1,0,1,1,1,1);$$

it is the binary Golay code $\mathcal{C}_{24}$. We have

$$\|\beta\| \in \{0,8,12,16,24\} \quad \text{for each } \beta \in \mathcal{K}. \tag{4.12}$$

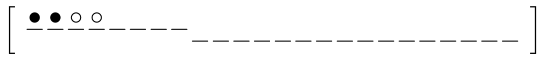
The vectors of Hamming norm 8 are called *octads*; their supports are complementary to those of norm 16. The vectors $b \in N$ of interest are of the form

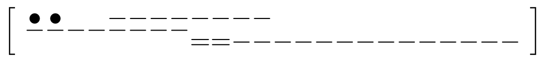
$$\left(\tfrac{1}{2}\right)^8,\ (2)^2 \text{ if } b^2 = 4; \quad \left(\tfrac{1}{2}\right)^{16},\ \left(\tfrac{1}{2}\right)^{12}(2),\ \left(\tfrac{1}{2}\right)^8(2)^2,\ (2)^4 \text{ if } b^2 = 8.$$

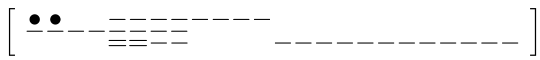
If $\mathbf{V} \subset N$, then

- $\mathbf{c}_4$ is of the form $\left(\frac{1}{2}\right)^8$ and $\operatorname{supp}\mathbf{c}_4 \supset \operatorname{supp}\mathbf{a}_2^i$, $i = 1, 2$, *cf.* Lemma 4.6, and
- $\operatorname{rt}(\mathbf{V}^\perp) = 0$ if and only if $\operatorname{supp}\mathbf{a}_4 \cup \operatorname{supp}\mathbf{c}_4 \cup \operatorname{supp}\mathbf{a}_8 = \mathcal{I}$.

Taking into account the fact that $\mathbf{a}_4 \cdot \mathbf{c}_4 = 2$, we arrive at three combinatorial types, which can be encoded by the following diagrams:

(X_{48})  ,

(X_{56})  ,

($X_{56}{}^\dagger$)  .

In the diagrams, we list, line by line, the images of the basis vectors of $\mathbf{V}$ other than roots, indicating their nonzero coordinates in a standard basis for $\mathbf{A}_1^{24}$ with the following symbols (some of which are used later in the paper):

$$(-) \mapsto \tfrac{1}{2}, \quad (=) \mapsto -\tfrac{1}{2}, \quad (+) \mapsto \tfrac{3}{2}, \quad (\circ) \mapsto 1. \tag{4.13}$$

The images of the basis elements of square 2 are shown by $\bullet$, which can appear in any line; we assume that the only nonvanishing coordinate of each root equals 1. Similar diagrams will be used throughout the paper.

4.8. **The classification.** The principal result of this section can be summarized in the following statement.

Theorem 4.14. *There are two isomorphism classes of extensions $N \supset S$, where N is a Niemeier lattice, S is a root-free lattice in the genus $-\mathfrak{G}$, and $S^\perp \cong \mathbf{V}$. In both cases, $N = N(\mathbf{A}_1^{24})$; the isometry $\mathbf{V} = S^\perp \hookrightarrow N(\mathbf{A}_1^{24})$ has combinatorial type as in diagram (X_{48}) or (X_{56}) in §4.7.*

Proof. The Leech lattice has no roots and, hence, cannot contain $\mathbf{V}$ as a sublattice. All other Niemeier lattices except $N(\mathbf{A}_1^{24})$ have been eliminated in §4.2–§4.6, and the embeddings $\mathbf{V} \hookrightarrow N := N(\mathbf{A}_1^{24})$ have been reduced to three combinatorial types in §4.7. The type ($X_{56}{}^\dagger$) differs from (X_{56}) by the basis change $\mathbf{a}_4 \mapsto -\mathbf{a}_4$, $\mathbf{c}_4 \mapsto \mathbf{c}_4 - \mathbf{a}_4$; hence, it results in an isomorphic pair (N, S).

It is straightforward that any isometry $\mathbf{V} \hookrightarrow R^\vee$ as in the statement is $O(R)$-equivalent to an isometry $\mathbf{V} \hookrightarrow N$, and there only remains to show that the latter is unique up to the action of $O(N)$. Recall that the group of automorphisms of the Golay code $\mathcal{C}_{24}$ is the Mathieu group $M_{24} \subset \mathbb{S}_{24}$, and the action of M_{24} on $\mathcal{I}$ has the following properties (see [2, Chapter 10]):

(1) the action is transitive on the 759 octads in $\mathcal{C}_{24}$;
(2) the stabilizer of an octad $b \in \mathcal{C}_{24}$ factors to $\mathbb{A}_8 \subset \mathbb{S}(\operatorname{supp} b) \cong \mathbb{S}_8$.

By (1), we can fix the octad $\mathbf{c}_4$. Then, the uniqueness of all further choices in the case (X_{48}) follows immediately from Statement (2). For the other case (X_{56}), we need an additional observation, which is easily confirmed by `GAP` [7]:

(3) the action of M_{24} is transitive on the set of ordered pairs $(b_1, b_2) \in \mathcal{C}_{24} \times \mathcal{C}_{24}$ of octads such that the set $C := \operatorname{supp} b_1 \cap \operatorname{supp} b_2$ is of size 4;
(4) the stabilizer of an ordered pair $(b_1, b_2) \in \mathcal{C}_{24} \times \mathcal{C}_{24}$ as above factors to an index 2 subgroup of $\mathbb{S}(C_1) \times \mathbb{S}(C_2)$, where $C_i := \operatorname{supp} b_i \smallsetminus C$.

Thus, by (3), we can fix the pair $(\mathbf{a}_4, \mathbf{c}_4)$; then, by (4), there is a unique choice for the two singletons $\operatorname{supp}\mathbf{a}_2^{1,2} \subset \operatorname{supp}\mathbf{c}_4 \smallsetminus C$ and the subset $V_- \subset \operatorname{supp}\mathbf{a}_4 \smallsetminus C$ where the coordinates of $\mathbf{a}_8$ are equal to $-\frac{1}{2}$ (the two "=" in the diagram). ∎

Corollary 4.15 (of the proof)**.** *Let $\mathbf{V} \hookrightarrow N$ be an isometry as in Theorem 4.14. Then, an autoisometry $g \in O(\mathbf{V})$ extends to to an autoisometry of N if and only if g preserves the combinatorial type of $\mathbf{V} \hookrightarrow N$.* ◁

5. Proof of Theorem 1.1

In this section, we complete the proof of Theorem 1.1 by analyzing extensions $H_2(\Phi_4) \supset NS(\Phi_4) \supset \mathbf{S}_n$ of the two lattices $\mathbf{S}_n$, $n = 48, 56$, constructed in §4.7.

5.1. **Automorphisms of V.** The discriminant $\mathcal{S} := \operatorname{discr} \mathbf{V} \cong (\mathbb{Z}/4) \oplus (\mathbb{Z}/8)^2$ is generated by three pairwise orthogonal elements α, β_1, β_2 of orders $4, 8, 8$ and squares $-\frac{1}{4}, -\frac{1}{8}, -\frac{1}{8} \bmod 2\mathbb{Z}$, respectively. In the notation of §3.1, we have

$$\mathcal{S}_h = \mathcal{S}_h^+ = \bigl\{\pm\alpha + c_1\beta_1 + c_2\beta_2 \bigm| c_1, c_2 = 0, 4\bigr\}.$$

The group $A := \operatorname{Aut} \mathcal{S}$ has order 128, and its image A_h in $\mathbb{S}(\mathcal{S}_h)$ is isomorphic to $(\mathbb{Z}/2) \times \mathbb{D}_8$. In an appropriate ordering of $\mathcal{S}_h$, this image is generated by

$$a := (1,2)(3,4)(5,6)(7,8), \quad u := (2,3)(6,7), \quad -\mathrm{id} = (1,5)(2,6)(3,7)(4,8).$$

In particular, A_h is transitive on $\mathcal{S}_h$, *cf.* Remark 3.3.

The following few statements are straightforward.

(1) The group $O(\mathbf{V})$ has order 64.
(2) The image $G \subset A$ of $O(\mathbf{V})$ is isomorphic to $(\mathbb{Z}/2)^4$.
(3) The image $G_h \subset A_h$ of G is the order 8 subgroup generated by a, $u^{-1}au$, and $-\mathrm{id}$; it acts on $\mathcal{S}_h$ simply transitively.

Below, we fix an isometry $\iota\colon \mathbf{V} \hookrightarrow N := N(\mathbf{A}_1^{24})$ as in Theorem 4.14, consider the lattice $S := -\mathbf{V}^\perp$, and identify $\operatorname{discr} S$ and $\mathcal{S}$ *via* the isometry ψ corresponding to the extension $N \supset (-S) \oplus \mathbf{V}$, see Corollary 2.4. By means of this identification, we can speak about the images $H \subset A$ and $H_h \subset A_h$ of the group $O(S)$. Consider also the subgroup $O^\iota(\mathbf{V}) \subset O(\mathbf{V})$ consisting of the automorphisms preserving the combinatorial type of ι, see §4.4, and its images $G^\iota \subset G$ and $G_h^\iota \subset G_h$. We have

(4) $G^\iota = G \cap H$ and, hence, $G_h^\iota \subset G_h \cap H_h$, see Corollaries 2.4(2) and 4.15.

On the other hand, the two lattices S corresponding to the two isometries given by Theorem 4.14 are not isomorphic, see Lemma 5.6 below; hence, each extension $N \supset (-S) \oplus \mathbf{V}$ is unique up to isomorphism and, by Corollary 2.4(1), the group A is a single double coset, *i.e.*, $A = \{gh \mid g \in G,\ h \in H\}$. Then, by (4) and (3) above,

$$[A : H] \leqslant [G : G^\iota], \tag{5.1}$$

$$H_h \supset G_h^\iota \cdot \langle \bar{g} \rangle \quad \text{for some } \bar{g} \in A_h \smallsetminus G_h. \tag{5.2}$$

For each $\gamma \in \mathcal{S}_h$, we have $\mathcal{K}_\gamma^\perp/\mathcal{K}_\gamma = \gamma^\perp \cong -\operatorname{discr} \mathbf{T}$, see Remark 3.3, and the restriction establishes an isomorphism

$$\operatorname{stab} \gamma = \operatorname{Aut} \gamma^\perp \cong (\mathbb{Z}/2) \times \mathbb{D}_8. \tag{5.3}$$

The intersection $G \cap \operatorname{stab} \gamma$ is a subgroup of order 2.

5.2. **The lattice $\mathbf{S}_{48}$.** Let $\iota\colon \mathbf{V} \hookrightarrow N$ be the isometry with the combinatorial type as in (X_{48}) in §4.7, and denote $\mathbf{S}_{48} := -\mathbf{V}^\perp$. The following lemma is proved by a straightforward computation.

Lemma 5.4. *The form $\mathbf{S}_{48}^\vee$ contains a unique pair $\pm a$ of vectors of square (-1); they represent the element $2\alpha + 4(\beta_1 + \beta_2) \in \operatorname{discr} \mathbf{S}_{48}$. Furthermore, each element $\gamma \in \mathcal{S}_h$ is represented by exactly 48 vectors $a \in \mathbf{S}_{48}^\vee$, $a^2 = -\frac{9}{4}$.* ◁

Lemma 5.5. *The canonical homomorphism $O(\mathbf{S}_{48}) \to \operatorname{Aut}\mathcal{S}$ is surjective. Hence, the action of $O(\mathbf{S}_{48})$ is transitive on $\mathcal{S}_h$ and, for each $\gamma \in \mathcal{S}_h$, the stabilizer $\operatorname{Stab}\gamma$ projects to the full automorphism group $\operatorname{Aut}\gamma^\perp$.*

Proof. Any autoisometry of $\mathbf{V}$ preserves the combinatorial type of ι. Hence, we have $G^\iota = G$ and, by (5.1), $H = A$ (*cf.* also Lemma 2.8). The other statements follow from the properties of the group $A = \operatorname{Aut}\mathcal{S}$ discussed in §5.1. □

5.3. **The lattice $\mathbf{S}_{56}$.** Let $\iota\colon \mathbf{V} \hookrightarrow N$ be the isometry with the combinatorial type as in (X_{56}) in §4.7, and denote $\mathbf{S}_{56} := -\mathbf{V}^\perp$. The following lemma is proved by a straightforward computation.

Lemma 5.6. *The form $\mathbf{S}_{56}^\vee$ has no vectors of square (-1). Each element $\gamma \in \mathcal{S}_h$ is represented by exactly 56 vectors $a \in \mathbf{S}_{56}^\vee$, $a^2 = -\frac{9}{4}$. As a consequence, we have $\mathbf{S}_{56} \not\cong \mathbf{S}_{48}$,* cf. *Lemma 5.4.* ◁

Lemma 5.7. *The image H_{56} of the canonical homomorphism $O(\mathbf{S}_{56}) \to \operatorname{Aut}\mathcal{S}$ is a subgroup of index 2. This subgroup has the following properties:*

(1) *the subgroup H_{56} is transitive on $\mathcal{S}_h$;*
(2) *the subgroup H_{56} is transitive on the set $\mathcal{S}_{56} := \{\gamma \in \mathcal{S} \mid \gamma^2 = -\frac{1}{8} \bmod 2\mathbb{Z}\}$;*
(3) *the subgroup H_{56} contains the index 4 subgroup H' generated by reflections defined by the elements of $\mathcal{S}_r := \{\alpha \in \mathcal{S} \mid \alpha^2 = \frac{3}{8}$ or $\frac{3}{4} \bmod 2\mathbb{Z},\ 4\alpha \neq 0\}$.*

Proof. We have $[G : G^\iota] = 2$, as the combinatorial type depends on the choice of a basis, see ($X_{56}{}^\dagger$) *vs.* (X_{56}) in §4.7. Hence, $[A : H] = 2$ by (5.1). (Alternatively, the same conclusion follows from Lemma 2.8.) Precise computation shows that $G_h^\iota \subset G_h$ is the index 2 subgroup generated by a and $-\operatorname{id}$ (see §5.1 for the notation), and one can check that $G_h^\iota \cdot \langle g \rangle = A_h$ for *each* element $\bar{g} \in A_h \smallsetminus G_h$. Hence, (5.2) gives us $H_h = A_h$, and Statement (1) follows from Statement (3) in §5.1.

Fix $\gamma \in \mathcal{S}_h$. The group $\operatorname{Stab}\gamma$ has been studied in [5, Lemma 6.19] as $O_h(\mathbf{X}_{56})$; its image in $\operatorname{Aut}\gamma^\perp$ is the index 2 subgroup generated by reflections defined by the elements $\alpha \in \gamma^\perp \cap \mathcal{S}_r$. Any element of $\mathcal{S}_r$ is orthogonal to some $\gamma \in \mathcal{S}_h$; hence, $H' \subset H_{56}$. There remains to observe that H' acts transitively on $\mathcal{S}_{56}$. □

5.4. **End of the proof.** Since we are interested in a smooth model, the lattice $S \in -\mathfrak{S}$ in Theorem 3.4 must be root free, *i.e.*, one of the two lattices $\mathbf{S}_n$, $n = 48$ or 56, introduced above. By Lemmas 5.5 and 5.7, the action of $O(\mathbf{S}_n)$ on $\mathcal{S}_h = \mathcal{S}_h^+$ is transitive and, for each lattice, it suffices to consider one representative $\gamma \in \mathcal{S}_h$. The models obtained are birational (and then smooth) due to Theorem 3.4(2), (3) and Lemmas 5.4 and 5.6; by the same lemmas and Theorem 3.4(4), the quartic obtained contains n lines.

To complete the proof, we need to analyze the set

$$O^+(\mathbf{T})\backslash \operatorname{Aut}\operatorname{discr}\mathbf{T}/\operatorname{Stab}\gamma = O^+(\mathbf{T})\backslash \operatorname{Aut}\gamma^\perp/\operatorname{Stab}\gamma.$$

If $n = 48$, Lemma 5.5 gives us a unique double coset, hence one model. If $n = 56$, Lemma 5.7 implies that the image of $\operatorname{Stab}\gamma$ in $\operatorname{Aut}\gamma^\perp$ is the index 2 subgroup $H_{56} \cap \operatorname{stab}\gamma = H' \cap \operatorname{stab}\gamma$ which contains the image of $O^+(\mathbf{T})$. Hence, there are two double cosets resulting in two quartics. One can easily check (or merely refer to [5]) that the two double cosets are interchanged by the full group $O(\mathbf{T})$; hence, the two quartics are complex conjugate (see the beginning of §3.1).

Finally, the three projective quartics obtained are identified with the classical Fermat quartic X_{48} or the pair X_{56}, $\bar{X}_{56}$ constructed in [5] and studied further in [25] according to the number of lines contained in the surface. □

6. Proof of Theorem 1.2

The approach used in this section is similar, but not identical to that of §4. For some mysterious reason, it does not work very well for the Fermat quartic. Hence, below we refer to Theorem 1.1 and assume that $T \neq [8, 0, 8]$.

6.1. **Niemeier lattices with many roots.** Throughout the proof, $\mathbf{T}$ is a fixed positive definite even lattice of rank 2, $\det \mathbf{T} \leqslant 80$, and $\mathbf{T} \neq [8, 0, 0]$ or $[4, 0, 16]$.

Any test lattice $\mathbf{V}$ (see §3.3) can be decomposed into a sum (direct, but not orthogonal) $\mathbf{V} = \operatorname{rt}(\mathbf{V}) + \bar{\mathbf{V}}$. We assume that the basis for $\mathbf{V}$ is obtained from one for $\operatorname{rt}(\mathbf{V})$ inductively, by adding at each step a shortest vector possible, and then $\bar{\mathbf{V}}$ is spanned by the basis vectors $\mathbf{c}_1, \ldots, \mathbf{c}_p$, $p = 5 - \operatorname{rk}\operatorname{rt}(\mathbf{V})$, that are not in $\operatorname{rt}(\mathbf{V})$.

Convention 6.1. We also assume that

- $\mathbf{c}_i^2 \leqslant 16$ for each $i = 1, \ldots, p$, and
- $\operatorname{rk}\operatorname{rt}(\mathbf{V}) \geqslant 2$ (this is the reason for excluding $\mathbf{T} = [4, 0, 16]$);

among the lattices satisfying these conditions, we choose $\mathbf{V}$ by minimizing the rank $p = 5 - \operatorname{rk}\operatorname{rt}(\mathbf{V})$, then the maximal square $\mathbf{c}_p^2$, and then the trace $\mathbf{c}_1^2 + \ldots + \mathbf{c}_p^2$ (see `http://www.fen.bilkent.edu.tr/~degt/papers/test_matrix.zip`).

With a pair $\bar{\mathbf{V}} \subset \mathbf{V}$ fixed, we call a $\mathbb{Q}$-valued p-form V with a distinguished basis $\mathbf{c}_1, \ldots, \mathbf{c}_p$ *bounded*, $V \leqslant \bar{\mathbf{V}}$, if $\bar{\mathbf{V}} - V$ is positive semi-definite.

Fix $\bar{\mathbf{V}} \subset \mathbf{V}$ as in Convention 6.1. Any isometry $\mathbf{V} \hookrightarrow N$ to a Niemeier lattice N restricts to an isometry $\operatorname{rt}(\mathbf{V}) \hookrightarrow \operatorname{rt}(N)$, and we can consider the root lattice

$$R := \operatorname{rt}\bigl(\operatorname{rt}(\mathbf{V})^\perp\bigr) = \bigoplus R_k, \quad k \in \mathcal{I} := \{1, \ldots, m\},$$

where R_k are the irreducible components. As in §4.1, we let $R_I := \bigoplus_{k \in I} R_k$ for a subset $I \subset \mathcal{I}$. Note that this lattice R differs from the maximal root lattice $\operatorname{rt}(N)$ considered in §4.1, and Lemma 4.1 takes the following form.

Lemma 6.2. *If $r \in \operatorname{rt}(\mathbf{V})^\perp$ is a root, then $r \in R_k$ for some index $k \in \mathcal{I}$.* ◁

Warning 6.3. The lattices $\bar{\mathbf{V}} \subset \mathbf{V}$ and R introduced here differ from $\bar{\mathbf{V}} = \operatorname{rt}(\mathbf{V})^\perp$ in §3.4 and $R = \operatorname{rt}(N)$ in §4.1 (and so are the irreducible components R_k and root lattices R_I). This difference is due to a slight change in the approach: instead of considering all isometries $\operatorname{rt}(\mathbf{V})^\perp \hookrightarrow N$ and selecting those with at most two roots in the orthogonal component, we *start* with an embedding $\operatorname{rt}(\mathbf{V}) \hookrightarrow \operatorname{rt}(N)$ and try to find dense (see below) bounded sublattices in $\operatorname{rt}(\mathbf{V})^\perp \subset N$.

At this stage, we do not insist that the isometry $\mathbf{V} \hookrightarrow N$ or its restriction to $\operatorname{rt}(\mathbf{V})$ should be primitive, *cf.* Remark 3.8. The possible isometries $\operatorname{rt}(\mathbf{V}) \hookrightarrow \operatorname{rt}(N)$ can easily be classified step by step, by embedding an irreducible component $Q \subset \operatorname{rt}(\mathbf{V})$ to an irreducible component $P \subset \operatorname{rt}(N)$ and replacing P with $\operatorname{rt}(Q^\perp)$. Isometries of irreducible root systems are well known; for the reader's convenience, we list them in Table 5, where we stretch the notation and let $\mathbf{A}_{n-1} = \mathbf{D}_n := 0$ for $n \leqslant 1$ and

$$\mathbf{D}_2 := \mathbf{A}_1^2, \quad \mathbf{D}_3 := \mathbf{A}_3, \quad \mathbf{E}_3 := \mathbf{A}_1 \oplus \mathbf{A}_2, \quad \mathbf{E}_4 := \mathbf{A}_4, \quad \mathbf{E}_5 := \mathbf{D}_5.$$

Table 5. Embeddings of irreducible root lattices

P	$Q \hookrightarrow P$	$\operatorname{rt}(Q^\perp)$	P	$Q \hookrightarrow P$	$\operatorname{rt}(Q^\perp)$
$\mathbf{A}_n$	$\mathbf{A}_m,\ 1 \leqslant m \leqslant n$	$\mathbf{A}_{n-m-1}$	$\mathbf{E}_7$	$\mathbf{A}_1$	$\mathbf{D}_6$
$\mathbf{D}_n$	$\mathbf{A}_1$	$\mathbf{A}_1 \oplus \mathbf{D}_{n-2}$		$\mathbf{A}_2$	$\mathbf{A}_5$
	$\mathbf{A}_m,\ 2 \leqslant m < n$	$\mathbf{D}_{n-m-1}$		$\mathbf{A}_3$	$\mathbf{A}_3 \oplus \mathbf{A}_1$
	$\mathbf{D}_m,\ 3 \leqslant m \leqslant n$	$\mathbf{D}_{n-m}$		$\mathbf{A}_4, \mathbf{A}_5$	$\mathbf{A}_2$
$\mathbf{E}_6$	$\mathbf{A}_1$	$\mathbf{A}_5$		$\mathbf{A}_5, \mathbf{D}_5, \mathbf{D}_6$	$\mathbf{A}_1$
	$\mathbf{A}_2$	$\mathbf{A}_2^2$		$\mathbf{A}_6, \mathbf{A}_7, \mathbf{E}_6, \mathbf{E}_7$	0
	$\mathbf{A}_3$	$\mathbf{A}_1^2$		$\mathbf{D}_4$	$\mathbf{A}_1^3$
	$\mathbf{A}_4, \mathbf{A}_5$	$\mathbf{A}_1$	$\mathbf{E}_8$	$\mathbf{A}_m,\ 1 \leqslant m \leqslant 5$	$\mathbf{E}_{8-m}$
	$\mathbf{D}_4, \mathbf{D}_5, \mathbf{E}_6$	0		$\mathbf{A}_6, \mathbf{A}_7, \mathbf{E}_7$	$\mathbf{A}_1$
				$\mathbf{A}_7, \mathbf{A}_8, \mathbf{E}_8$	0
				$\mathbf{D}_m,\ 4 \leqslant m \leqslant 8$	$\mathbf{D}_{8-m}$
				$\mathbf{E}_6$	$\mathbf{A}_2$

(These conventions are based on the structure of the discriminant group.) In a few cases, an isometry $Q \hookrightarrow P$ is not unique; most notably, $\mathbf{A}_3$ can be embedded to $\mathbf{D}_n$ as $\mathbf{A}_3$ or $\mathbf{D}_3$.

Fix an isometry $\operatorname{rt}(\mathbf{V}) \hookrightarrow \operatorname{rt}(N)$ and let R be as above. Assume that there is an extension $\mathbf{V} \hookrightarrow N$ and consider the projections of $\bar{\mathbf{V}}$ to the groups $R_I^\vee$; as in §4.1, these projections are regarded as isometries

$$V_k \to R_k^\vee, \quad k \in \mathcal{I}, \quad \text{and} \quad V_I \to R_I^\vee, \quad I \subset \mathcal{I},$$

and referred to as p-forms. Clearly, all forms V_I are bounded. A p-form $V \to R_I^\vee$ is called *dense* if $\operatorname{rt}(V^\perp \cap R_I) = 0$; we denote by $\operatorname{dense}(R_I)$ the set of the Gram matrices of bounded dense p-forms $V \to R_I$. If $I = I' \cup I''$ and $I' \cap I'' = \varnothing$, then, by Lemma 6.2, we have

$$\operatorname{dense}(R_I) = \bigl\{V' + V'' \bigm| (V', V'') \in \operatorname{dense}(R_{I'}) \times \operatorname{dense}(R_{I''}),\ V' + V'' \leqslant \bar{\mathbf{V}}\bigr\},$$

cf. (4.3), *i.e.*, the sets $\operatorname{dense}(R_I)$ for all subsets $I \subset \mathcal{I}$ can be computed inductively starting from $\operatorname{dense}(R_k)$, $k \in \mathcal{I}$. Note that this computation is reusable, as the set $\operatorname{dense}(R_I)$ depends on R_I and $\bar{\mathbf{V}}$ only, and the sublattice $\bar{\mathbf{V}}$ chosen according to Convention 6.1 is often shared by many transcendental lattices $\mathbf{T}$.

The following statement has been obtained using `GAP` [7] and the algorithm outlined in §4.2 to enumerate the bounded isometries $V \to R_k$.

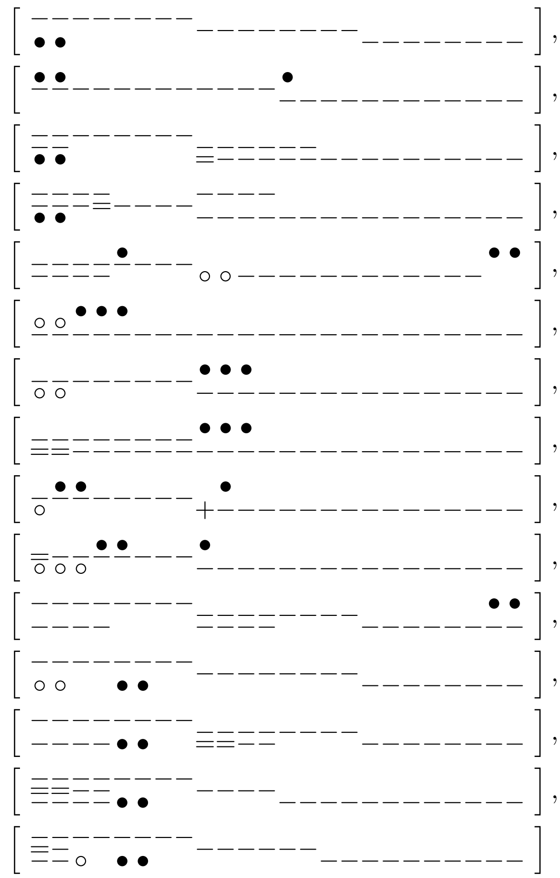
Lemma 6.4. *Assume that the test lattices $\bar{\mathbf{V}} \subset \mathbf{V}$ are chosen as in Convention 6.1 and that the root system $\operatorname{rt}(N)$ has at most six irreducible components. Then, for any isometry $\operatorname{rt}(\mathbf{V}) \hookrightarrow \operatorname{rt}(N)$, one has $\operatorname{dense}(R) = \operatorname{dense}(R_{\mathcal{I}}) = \varnothing$.*

As an immediate consequence, we conclude that, for N as in Lemma 6.4 and any isometry $\mathbf{V} \hookrightarrow N$, the orthogonal complement $\mathbf{V}^\perp$ is not root free.

6.2. **Root systems with many components.** Fix $\bar{\mathbf{V}} \subset \mathbf{V}$ as above and consider one of the remaining three Niemeier lattices (see §4.4), assuming that $\operatorname{rt}(\mathbf{V})$ admits an isometry to the root lattice $R := \operatorname{rt}(N)$ and using the definitions and notation introduced in §4.4–§4.7. This time, we start with building an isometry $\bar{\mathbf{V}} \hookrightarrow R^\vee$, considering the latter up to the action of $O(R)$ and using the list of types/Hamming norms of the elements of $\mathcal{K}$ only, see (4.8), (4.11), (4.12); this restriction is applied to each element of the group $\bar{\mathbf{V}} \bmod R$. It is not difficult to enumerate pairs of vectors $\mathbf{c}_1, \mathbf{c}_2 \in R^\vee$; if a third vector $\mathbf{c}_3$ is to be added, one can usually limit the choices similar to §4.5–§4.7, by analyzing the position of the roots in $\mathbf{V}$ with respect to the supports $\operatorname{supp}(\mathbf{c}_i)$.

There remains to extend the isometries found to $\operatorname{rt}(\mathbf{V}) \hookrightarrow R$ and select those for which the lattice $\mathbf{V}^\perp$ is root free. We arrive at the following fifteen combinatorial types of isometries $\mathbf{V} \hookrightarrow R^\vee$ for $R = \mathbf{A}_1^{24}$:

(X_{64}), (X''_{60}), (X'_{60}), (Q_{56}), ($\tilde{Y}'_{48}$), (V'_{48}), (D1), (D1†), (Y''_{52}), ($\tilde{Y}''_{48}$), (D2), (D3), (D4), (Q''_{52}), (X'_{52})



(see (4.13) for the notation), eight combinatorial types for $R = \mathbf{A}_2^{12}$:

(Y_{56}), (Q'''_{52}), (Y'_{52}), (Z_{50}) (×2), (D5) (×2), ($X'_{52}{}^\dagger$)

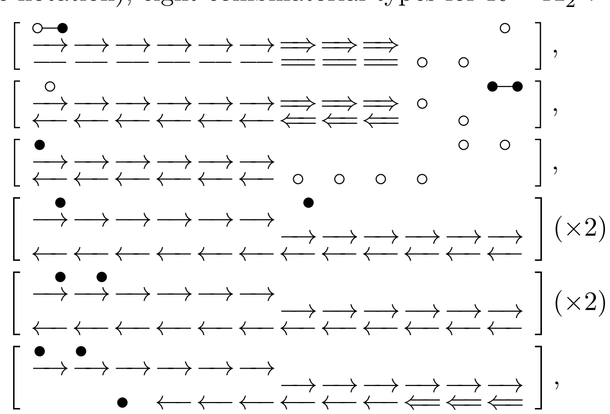
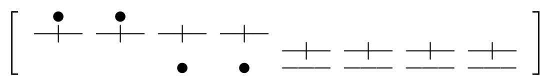


Table 6. Ambiguous combinatorial types

(D1)	(D1†)	(D2)	(D3)	(D4)	(Q''_{52})	(X'_{52})	$(Z_{50})^2$	$(\mathrm{D5})^2$	(${X'_{52}}^\dagger$)	(${Q''_{52}}^\dagger$)
Q_{54}	Q_{54}	$*$	Q''_{52}	Q''_{52}	Q''_{52}	X'_{52}	Z_{50}	$*$	X'_{52}	Q''_{52}
X''_{52}	X''_{52}	Z'_{48}	V''_{48}	Q_{48}						

$*$ stands for the pair of configurations Z_{52} and Z''_{48}
$(Z_{50})^2$ and $(\mathrm{D5})^2$ represent two combinatorial types each

(each of (Z_{50}) and (D5) represents two combinatorial types, which differ by the transposition $\mathbf{c}_1 \leftrightarrow \mathbf{c}_2$), and one combinatorial type for $R = \mathbf{A}_3^8$:

(${Q''_{52}}^\dagger$) .

In the diagrams for $\mathbf{A}_2^{12}$, unlike $\mathbf{A}_1^{24}$, we list the images of the basis vectors of $\mathrm{rt}(\mathbf{V})$ *in the first line*, and those of $\mathbf{c}_1, \mathbf{c}_2, \ldots$ in the other lines. Representing each copy of $\mathbf{A}_2$ as $\mathbb{Z}a_1 + \mathbb{Z}a_2$, $a_1^2 = a_2^2 = 2$, $a_1 \cdot a_2 = -1$, we use the following notation:

(6.5)
$$(\bullet\quad) \mapsto a_1, \qquad (\quad\bullet) \mapsto a_2, \qquad (\;\circ\;) \mapsto -a_1 - a_2,$$
$$(\longrightarrow) \mapsto \tfrac{1}{3}(2a_1 + a_2), \quad (\longleftarrow) \mapsto \tfrac{1}{3}(a_1 + 2a_2), \quad (\text{——}) \mapsto \tfrac{1}{3}(a_2 - a_1),$$

a double line indicating the negation. Certainly, within each column of a diagram, the components should be regarded up to simultaneous action of $O(\mathbf{A}_2)$. A similar convention applies to $\mathbf{A}_3^8$, where, in addition to roots, we only use the following two elements:

(6.6)
$$(\text{———}) \mapsto \tfrac{1}{2}(a_1 + a_3), \quad (\text{—+—}) \mapsto \tfrac{1}{2}(a_1 + 2a_2 + a_3).$$

6.3. **Proof of Theorem 1.2.** Each of the 24 combinatorial types $\mathbf{V} \hookrightarrow R^\vee$ found in §6.2 is represented by an isometry $\mathbf{V} \hookrightarrow N$. Thus, we only need to classify these isometries up to the finer group $O(N)$ and analyze the quartics obtained.

The group $O(N)$ is given by (4.7), and the stabilizers of the corresponding codes $N/R \subset R^\vee/R$ are well known and found, *e.g.*, in [2]. Omitting the details (*cf.* the proof of Theorem 4.14), we merely state that the first five diagrams (D1)–(D4) in Table 6 give rise to two orbits each, whereas all other diagrams are represented by a single orbit each.

For each orbit, we compute the lattice $S := -\mathbf{V}^\perp$ and verify Condition (2) of Theorem 3.4. Two lattices, *viz.* V'_{48} and V''_{48} (one of the orbits of (D3), see Table 6) fail; these lattices are discussed in §6.4 below. For each of the remaining lattices, we compute the set $\mathcal{S}_h$ (see §3.1) and, for each $\gamma \in \mathcal{S}_h$, use Theorem 3.4(4) to compute the configuration of lines in the corresponding quartic. These configurations are used to distinguish or identify (if the configuration is found in [5]) lattices obtained from distinct orbits. With the few exceptions listed in Table 6, each lattice S is obtained from a unique orbit; then, by Lemma 2.8, we have $[\operatorname{Aut}\operatorname{discr} S : O(S)] = 1$ and Theorem 3.4 gives us a unique spatial model.

Analyzing Table 6, we conclude that the index $m := [\operatorname{Aut}\operatorname{discr} S : O(S)]$ given by Lemma 2.8 is greater than 1 in the following cases:

- Q_{54}, X'_{52}, X''_{52} with $m = 2$ and Q''_{52} with $m = 4$: these configurations of lines and corresponding quartics are classified in [5];
- Z_{50} with $m = 2$ and the pair (Z_{52}, Z''_{48}) (represented by a $*$ in Table 6) with $m = 3$: these configurations are treated separately below.

Note that, for the Z_* series, even those known in [5], the results of [5] do not apply directly: in this case, lines do not generate $\mathbf{T}^\perp \otimes \mathbb{Q}$ and, hence, [5] establishes the connectedness of a 1-parameter family of quartics of typical Picard rank 19 rather than the uniqueness of any particular singular quartic (*cf.* Proposition 1.9).

6.3.1. *The pair* (Z_{52}, Z''_{48}). Let $\iota\colon \mathbf{V} \hookrightarrow N$ be one of the *three* isometries to which Table 6 assigns the pair of configurations (Z_{52}, Z''_{48}) ($*$ in the table), and consider the lattice $S := -\mathbf{V}^\perp$. In the notation of §3.1, we have a splitting $\mathcal{S}_h = \mathcal{S}_{52} \cup \mathcal{S}_{48}$, so that the quartic corresponding to an element $\gamma \in \mathcal{S}_n$ has n lines. (If $n = 52$, the configuration of lines is identified with Z_{52} in [5].) Since $|\mathcal{S}_{52}| = 4$ and $|\mathcal{S}_h| = 12$, from (2.2) we have $[\operatorname{Aut}\operatorname{discr} S : O(S)] \geqslant 3$; then, using Lemma 2.8, we conclude that $[\operatorname{Aut}\operatorname{discr} S : O(S)] = 3$ and all three lattices are isomorphic. By (2.2) again, $\mathcal{S}_{52}$ is a single $O(S)$-orbit and $[\operatorname{Aut}\gamma^\perp : \operatorname{Stab}\gamma] = 1$ for each $\gamma \in \mathcal{S}_{52}$; hence, there is a unique quartic with this configuration of lines.

For the other configuration Z''_{48}, consider the only isometry $\mathbf{V} \hookrightarrow N := N(\mathbf{A}_1^{24})$, see (D2), identify $\operatorname{discr} S$ and $\operatorname{discr}\mathbf{V}$ accordingly, and let $G, H \subset \operatorname{Aut}\operatorname{discr} S$ be the images of $O(\mathbf{V})$ and $O(S)$, respectively, *cf.* §5.1. Any autoisometry of $\mathbf{V}$ extends to N (due to the uniqueness of this extension); hence, $G \subset H$. On the other hand, $\mathcal{S}_{52}$ and $\mathcal{S}_{48}$ are precisely the G-orbits on $\mathcal{S}_h$ and the G-stabilizer of each element $\gamma \in \mathcal{S}_{48}$ contains $\pm\operatorname{id} \in \operatorname{Aut}\gamma^\perp$. It follows that $\mathcal{S}_{48}$ is a single $O(S)$-orbit and the image of $\operatorname{Stab}\gamma$ in $\operatorname{Aut}\gamma^\perp$ contains $\{\pm\operatorname{id}\}$, which is the image of $O^+(T)$. Since $[\operatorname{Aut}\gamma^\perp : \operatorname{Stab}\gamma] = 2$ by (2.2), we conclude that the configuration Z''_{48} is realized by two complex conjugate quartics.

6.3.2. *The configuration* Z_{50}. Consider the two isometries $\iota\colon \mathbf{V} \hookrightarrow N := N(\mathbf{A}_1^{24})$ given by (Z_{50}); they differ by an autoisometry of $\mathbf{V}$ and, hence, have the same orthogonal complement $S := -\mathbf{V}^\perp$. Let $G, H \subset \operatorname{Aut}\operatorname{discr} S$ be as above, and let $G^\iota \subset G$ be the index 2 subgroup preserving the combinatorial type, so that we have $G^\iota \subset H$. The set $\mathcal{S}_h$ splits into three G^ι-orbits, each of length 4, and for two of these orbits, the G^ι-stabilizers of elements contain $\pm\operatorname{id}$. On the other hand, we have $[\operatorname{Aut}\operatorname{discr} S : H] = 2$ by Lemma 2.8 and the orbits of H are unions of those of G^ι. Hence, in view of (2.2), $\mathcal{S}_h$ is a single orbit and $[\operatorname{Aut}\gamma^\perp : \operatorname{Stab}\gamma] = 2$ for each $\gamma \in \mathcal{S}_h$, the image of $\operatorname{Stab}\gamma$ containing $\pm\operatorname{id}$. By Theorem 3.4, the lattice S gives rise to two complex conjugate quartics. □

6.4. **Proof of Corollary 1.4.** We can represent $Q := \mathbb{P}^1 \times \mathbb{P}^1$ as a smooth quadric in $\mathbb{P}^3$. Hence, any map $X(T) \to Q$ can be regarded as a spatial model $\varphi\colon X \to \mathbb{P}^3$. According to [21, Theorem 5.2] (see also [21, Proposition 5.7]), such a map φ factors through a quadric Q if and only if the corresponding set of data $(S, [\gamma])$ in Theorem 3.4 violates the hypotheses of Statement (2) of the theorem. If this is the case and $(2/d)\gamma$ is represented by a vector $e \in S^\vee$, $e^2 = -1$, then the linear systems $|\frac{1}{2}h \pm e|$ are the pull-backs of the two rulings of Q. Furthermore, Statement (3) can be restated as

(3) the ramification locus $C \subset Q$ is smooth if and only if S is root free,

and Statement (4) holds literally if "lines" are understood as smooth rational curves mapped isomorphically to generatrices of Q. (The image of each line is a bitangent of Q, and the pull-back of each bitangent consists of two lines. The two rulings differ by the intersection $a \cdot e = \pm 1$.)

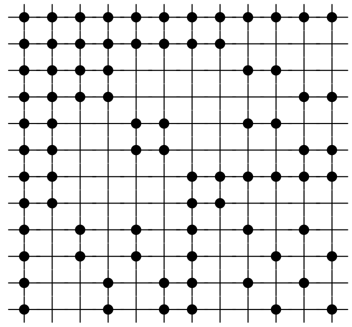


Figure 1. The maximal configuration of bitangents (see Remark 6.7)

The complete list of root-free lattices satisfying the imposed conditions on the discriminant is found in §6.2; for only two of these lattices, *viz.* V'_{48} and V''_{48} (one of the orbits of (D3), see Table 6), there is a class $\gamma \in \mathcal{S}_h$ that violates the hypothesis of Theorem 3.4(2). By Theorem 3.4 combined with Lemma 2.8 and (2.2), each of these two lattices gives rise to a single model $\varphi\colon X(T) \to Q$ and this model has 48 lines, *i.e.*, C has 24 bitangents. On the other hand, a curve of bidegree $(4, 4)$ in Q may have at most 12 bitangents in each ruling, as an elliptic pencil on a $K3$-surface may have at most 12 singular fibers of Kodaira type I_2. ☐

Remark 6.7. Using an appropriate version of [5, Lemma 4.6], it is not very difficult to show that all maximal (*i.e.*, containing 48 lines) configurations of lines in smooth hyperelliptic models $X \to Q := \mathbb{P}^1 \times \mathbb{P}^1$ of $K3$-surfaces are isomorphic to each other. The lifts to X of the 24 bitangents to the ramification locus can be chosen and ordered so as to intersect according to the pattern shown in Figure 1. The lines span a rank 17 sublattice in $N\!S(X)$; hence, there is a 3-parameter equilinear family of models, and, similar to Proposition 1.9, this family contains infinitely many singular $K3$-surfaces. (Two of these singular $K3$-surfaces constitute Corollary 1.4; another one is $X([8, 0, 12])$ discussed in §7.3.) Note also that this maximal configuration admits a faithful transitive action of the Mukai group H_{192}, which is induced from the action on $X([8, 0, 12])$ (see Theorem 7.5 and remarks thereafter).

6.5. **Oguiso pairs** (see [18])**.** An *Oguiso pair* is a pair of smooth spatial models $\varphi_i\colon X \to \mathbb{P}^3$, $i = 1, 2$, of the same $K3$-surface X such that one has

$$h_1 \cdot h_2 = 6 \quad \text{for} \quad h_i := \varphi_i^*[\mathbb{P}^2 \cap \varphi_i(X)] \in N\!S(X),\ i = 1, 2.$$

According to [18, Theorem 1.5], the two smooth quartics $X_i := \varphi_i(X)$ constituting an Oguiso pair are Cremona equivalent and each quartic is a Cayley $K3$-surface.

Lemma 6.8. *Consider a smooth quartic $X \in \mathbb{P}^3$ and let $h \in N\!S(X)$, $h^2 = 4$, be its polarization. Then, another class $h' \in N\!S(X)$ satisfying $(h')^2 = 4$ and $h' \cdot h = 6$ is very ample if and only if*

1. *there is no class $e \in N\!S(X)$ such that $e^2 = 0$ and $e \cdot h' = 2$;*
2. *there is no class $e \in N\!S(X)$ such that $e^2 = -2$ and $e \cdot h' = 0$;*
3. *one has $l \cdot h' > 0$ for the class $l = [\ell]$ of each line $\ell \subset X$.*

Proof. We only need to show that h' is nef; then, Conditions (1) and (2) would imply that h' is very ample (*cf.* Theorem 3.4 and its proof). A class $h' \in N\!S(X)$ is nef if and only if $h' \cdot h > 0$ (which is given) and h' and h belong to the same fundamental polyhedron of the subgroup of $O(NS(X))$ generated by reflections, *i.e.*, there is no root $r \in NS(X)$ such that $r \cdot h > 0$ and $r \cdot h' < 0$. (Both h and h' are in the interior of a fundamental polyhedron, due to the fact that h is very ample and Condition (2), respectively).

Assume that there is a root r such that $\alpha := r \cdot h \geqslant 1$ and $\beta := -r \cdot h' \geqslant 1$ and consider the sublattice spanned by h, h', and r. The determinant of the Gram matrix, which equals $4(10 - \alpha^2 - 3\alpha\beta - \beta^2)$, must be nonnegative, as the lattice is hyperbolic, and we conclude that $\alpha = \beta = 1$, *i.e.*, r is the class of a line in X. □

Let (NS, h) be one of the polarized Néron–Severi lattices listed in Table 1. (The transcendental lattice T is not important in Oguiso's construction.) In each case, it is straightforward to list all vectors $h' \in NS$ satisfying $(h')^2 = 4$ and $h' \cdot h = 6$, use Lemma 6.8 to select those that are very ample, and compute the number of lines with respect to the new polarization h'. (According to the table, this number identifies the polarization within each lattice NS.) With the two exceptions stated at the end of §1.2, starting from any polarization h, we obtain all numbers of lines that are possible for the given lattice; thus, with the same exceptions, any two smooth models of a $K3$-surface X as in Theorem 1.2 constitute an Oguiso pair.

7. Other polarizations

In this section, we consider other (than quartics in $\mathbb{P}^3$) polarizations of singular $K3$-surfaces and prove Theorem 1.5. Then, in §7.3, we discuss projective models of the eleven Mukai groups.

7.1. **The set-up.** Fix an even integer $2D > 0$ and define a *model* of a singular $K3$-surface $X := X(T)$ as a map $\varphi\colon X \to \mathbb{P}^{D+1}$ defined by a fixed point free ample linear system $|h|$ of degree $h^2 = 2D$, where $h := h_\varphi \in NS(X)$ stands for the class of a hyperplane section. Two models φ_1, φ_2 are *projectively equivalent* if there exists a pair of automorphisms $a\colon \mathbb{P}^{D+1} \to \mathbb{P}^{D+1}$ and $a_X\colon X \to X$ such that $\varphi_2 \circ a_X = a \circ \varphi_1$. A model is *smooth* if it does not contract a curve in X, *i.e.*, if the pull-back of each point is finite. A *line* in a model φ is a smooth rational curve $C \subset X$ such that the restriction $\varphi|_C$ is an isomorphism onto a line in $\mathbb{P}^{D+1}$.

Following §3.1, consider the lattice $S_\varphi := h^\perp \subset NS(X)$. We have

$$\text{(7.1)}\qquad \operatorname{discr} S_\varphi \cong -\mathcal{C}^\perp/\mathcal{C}, \qquad |\operatorname{discr} S_\varphi| = 2D|\operatorname{discr} T|/(\operatorname{dp}\varphi)^2,$$

where

$$\text{(7.2)}\qquad \mathcal{C} \subset \operatorname{discr} T \oplus \operatorname{discr} \mathbb{Z}h, \qquad \mathcal{C} \cap \operatorname{discr} T = 0,$$

is a cyclic group; its order $\operatorname{dp}\varphi := |\mathcal{C}|$, called the *depth* of φ, divides $2D$. Fixing a lattice $S := S_\varphi$ and, hence, the depth $d = d(S) := \operatorname{dp}\varphi$, consider the sets

$$\mathcal{S}_{dh} := \bigl\{\gamma \in \operatorname{discr} S \bigm| (2D/d)\gamma = 0,\ \gamma^2 = -d^2/2D \bmod 2\mathbb{Z}\bigr\},$$
$$\mathcal{S}_{dh}^{+} := \bigl\{\gamma \in \mathcal{S}_{dh} \bigm| \mathcal{K}_\gamma^\perp/\mathcal{K}_\gamma \cong -\operatorname{discr} T\bigr\},$$

where, for $\gamma \in \mathcal{S}_{dh}$, the isotropic subgroup $\mathcal{K}_\gamma \subset \operatorname{discr} \mathbb{Z}h \oplus \operatorname{discr} S$ is generated by $(d/2D)h \oplus \gamma$. As usual, fixing an isometry $\mathcal{K}_\gamma \cong -\operatorname{discr} T$, we regard both stabilizers $\operatorname{stab}\gamma \subset \operatorname{Aut}\operatorname{discr} S$ and $\operatorname{Stab}\gamma \subset O(S)$ acting on the discriminant $\operatorname{discr} T$.

Theorem 3.4 and its proof translate almost literally to the general case.

Theorem 7.3. *The projective equivalence classes of models $\varphi\colon X(T) \to \mathbb{P}^{D+1}$ are in a one-to-one correspondence with the triples consisting of*

- *a negative definite lattice S of rank 19 and* $\operatorname{discr} S \cong -\mathcal{C}^\perp/\mathcal{C}$ *as in* (7.2),
- *an $O(S)$-orbit $[\gamma] \subset \mathcal{S}^+_{dh}$ (where $d = d(S)$ is the depth, see (7.1)), and*
- *a double coset $c \in \mathrm{O}^+(T)\backslash \operatorname{Aut} \operatorname{discr} T / \operatorname{Stab} \gamma$*

and such that

(1) *$d > 1$ or $d = 1$ and γ is not represented by a vector $a \in S^\vee$, $a^2 = -1/2D$.*

Under this correspondence, the following statements hold:

(2) *a model φ is birational onto its image if and only if*
  - *$D \neq 1$ and either $D \neq 4$ or $\mathcal{C} \cap \operatorname{discr} \mathbb{Z}h = 0$, and*
  - *either $d > 2$ or $d \leqslant 2$ and the class $(2/d)\gamma$ is not represented by a vector $a \in S^\vee$, $a^2 = -2/D$;*

(3) *a model φ is smooth if and only if S is root free*;

(4) *the lines in a smooth model φ are in a one-to-one correspondence with the vectors $a \in S^\vee$, $a^2 = -(4D+1)/2D$, representing γ.* ◁

The condition $\mathcal{C} \cap \operatorname{discr} \mathbb{Z}h = 0$ in Theorem 7.3(2) means that, if $h^2 = 8$, the vector $h \in NS(X)$ must be primitive, see [21, Theorem 5.2]. Unlike Theorem 3.4, in Theorem 7.3(3), (4) we no longer require that the model should be birational. As in §1.3 (*cf.* also Corollary 1.4 and its proof in §6.4), the smoothness of a hyperelliptic model $\varphi\colon \mathbb{X} \to \mathbb{P}^{D+1}$ is understood as the smoothness of its ramification locus C, and *lines* are defined as rational curves $L \subset X$ that project isomorphically onto lines in $\mathbb{P}^{D+1}$. (Typically, the images $\varphi(L)$ are generatrices of the scroll $\varphi(X)$ that have even intersection index with the ramification locus $C \subset \varphi(X)$ at each point of intersection, *cf.* bitangents in Corollary 1.4 and tritangents in §1.3. Each generatrix with this property splits into two lines in X.)

Lemma 7.4. *In the notation of Theorem 7.3, assume that $D = 4$ and that the model $\varphi\colon X \to \mathbb{P}^5$ given by a triple $(S, [\gamma], c)$ is birational. Then the image $\varphi(X)$ is an intersection of quadrics if and only if 3γ is not represented by a vector $u \in S^\vee$ such that $u^2 = -\frac{9}{8}$ and $u \cdot a \geqslant -\frac{3}{8}$ for each vector $a \in S^\vee$ as in Theorem 7.3(4).*

Proof. According to [21, Theorem 7.2], the defining ideal of $\varphi(X)$ is generated by its elements of degree 2 if and only if there is no nef class $e \in NS(X)$ such that $e^2 = 0$ and $e \cdot h = 3$. Arguing as in the proof of Lemma 6.8, one can easily show that a class $e \in NS(X)$ satisfying $e^2 = 0$ and $e \cdot h = 3$ is nef if and only if one has $e \cdot \ell \geqslant 0$ for each curve $\ell \subset X$ that is either a line ($\ell \cdot h = 1$) or an exceptional divisor ($\ell \cdot h = 0$). For the latter, one can map e to the distinguished Weyl chamber of $-S$ by an appropriate element of the Weyl group (*cf.* §8.5 below). Then, the requirement $e \cdot \ell \geqslant 0$ can be extended to all, not necessarily irreducible, (-2)-curves $\ell \subset X$ satisfying $\ell \cdot h = 1$. The statement of the lemma is a translation of this latter condition in terms of S and γ. □

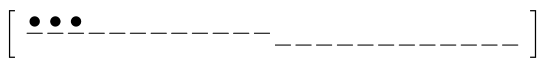
7.2. **Proof of Theorem 1.5.** Fix an integer $h^2 = 2D = 2$, 6, or 8 and a positive definite even lattice $\mathbf{T}$ of rank 2. As in §3.3, we can find a *test lattice* $\mathbf{V}$, which is a positive definite even lattice of rank 5 satisfying the identity

$$\operatorname{discr} \mathbf{V} \cong -\operatorname{discr} \mathbf{T} \oplus -\operatorname{discr} \mathbb{Z}h.$$

Then, an analog of Lemma 3.7 holds: any lattice S as in Theorem 7.3 is of the form $S = -\mathbf{V}^\perp$ for an appropriate isometry $\mathbf{V} \hookrightarrow N$ to a Niemeier lattice N. (*A priori*, this isometry does not need to be primitive, *cf.* Remark 3.8.) Since we are only interested in smooth models, an additional requirement is that there should be no root $r \in \mathrm{rt}(N)$ orthogonal to $\mathbf{V}$.

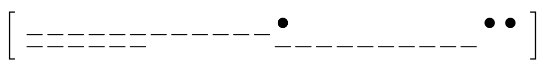
Assuming that $\det \mathbf{T}$ is bounded as in Theorem 1.5, the test lattice $\mathbf{V}$ (depending on h^2) can be chosen according to Convention 6.1. Then, arguing as in §6.1, we can easily use `GAP` [7] and eliminate all Niemeier lattices N with $\mathrm{rt}(N)$ consisting of six or less irreducible components. The remaining three lattices are treated as in §6.2, by using (4.8), (4.11), (4.12) and enumerating the combinatorial types of isometries first. Then, the isometries are classified up to the finer group $O(N)$ given by (4.7) and the resulting models are studied using Theorem 7.3.

7.2.1. *Planar models, $h^2 = 2$.* If $\det \mathbf{T} \leqslant 116$, there is one combinatorial type, *viz.* the isometry $\mathbf{V} \hookrightarrow (\mathbf{A}_1^{24})^\vee$ given by the diagram (see (4.13) for the notation)

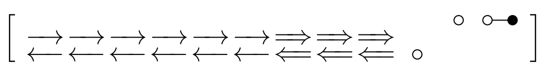
(2_{144})  .

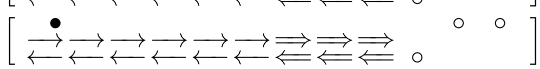
It gives rise to a unique isometry $\mathbf{V} \hookrightarrow N$; hence, due to Theorem 7.3 combined with Lemma 2.8 and (2.2), there is a unique model.

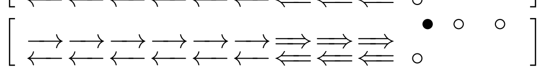
7.2.2. *Sextic models in $\mathbb{P}^4$, $h^2 = 6$.* If $\det \mathbf{T} \leqslant 48$, there is one combinatorial type of isometries $\mathbf{V} \hookrightarrow (\mathbf{A}_1^{24})^\vee$ (see (4.13) for the notation):

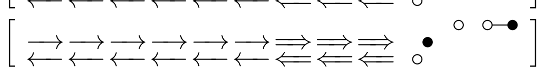
(D6) 

and four combinatorial types of isometries $\mathbf{V} \hookrightarrow (\mathbf{A}_2^{12})^\vee$ (see (6.5)):

(6_{42})  ,

(6_{38})  ,

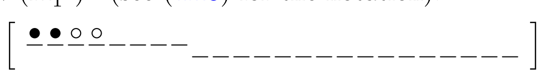
$(\mathrm{D}6^\dagger)$  ,

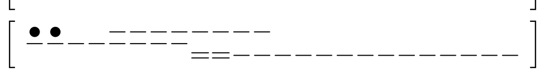
$(6''_{36})$  .

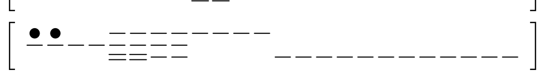
Each combinatorial type gives rise to a unique $O(N)$-orbit of isometries, and, with the exception of (D6), $(\mathrm{D}6^\dagger)$, all orthogonal complements are pairwise distinct: this fact can be established by computing the configurations of lines. (In $(6''_{36})$, the subset $\mathcal{S}_h^+ \subset \mathcal{S}_h$ is proper.) As above, each of these lattices gives us a single model.

In each of the exceptional cases (D6), $(\mathrm{D}6^\dagger)$, the set $\mathcal{S}_h$ splits into two subsets, with the configurations of lines $6'_{42}$ or $6'_{36}$. Using Lemma 2.8 and (2.2), we conclude that the two orthogonal complement are isomorphic to each other (as otherwise the group $O(S)$ would have to be transitive on $\mathcal{S}_h^+$) and each configuration is realized by a single model.

7.2.3. *Octic models in $\mathbb{P}^5$, $h^2 = 8$.* If $\det \mathbf{T} \leqslant 40$, there are five combinatorial types of isometries $\mathbf{V} \hookrightarrow (\mathbf{A}_1^{24})^\vee$ (see (4.13) for the notation):

(8_{32})  ,

(8_{36})  ,

$({8_{36}}^\dagger)$  ,

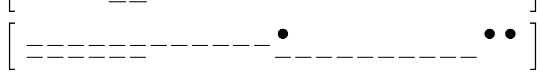
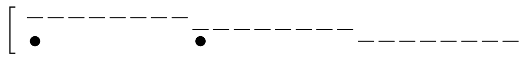
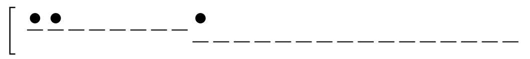
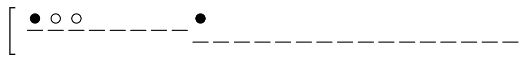
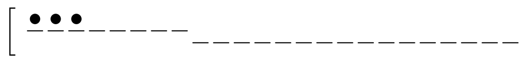
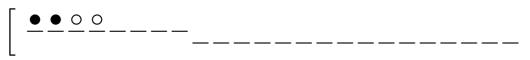
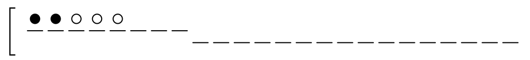
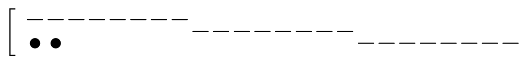
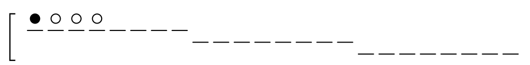
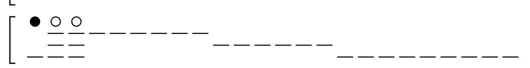
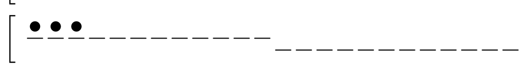
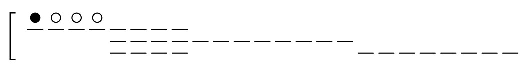
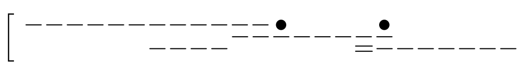
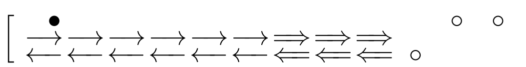
(D7)  ,

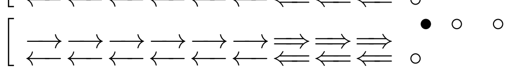
Table 7. Models of Mukai groups (see Theorem 7.5)

G	Combinatorial type of $N^G \hookrightarrow N$	Models, remarks
$L_2(7)$		$h^2 = 2, 4, 8$
$\mathbb{A}_6$		$h^2 = 2, 6, 8$
$\mathbb{S}_5$		$h^2 = 4, 6$
M_{20}		$h^2 = 4, 8$
F_{384}		$h^2 = 4$: X_{48} $h^2 = 8$: 8_{32}
$\mathbb{A}_{4,4}$		$h^2 = 8$
T_{192}		$h^2 = 4$: X_{64} $h^2 = 8$
H_{192}		$h^2 = 4$: Figure 1 $h^2 = 8$: $(6)^{32}$
N_{72}		$h^2 = 6$
M_9		$h^2 = 2$: 2_{144} $h^2 = 8$
T_{48}		$h^2 = 2$: 108 lines $h^2 = 8$

(8_{30}) 

and two combinatorial types of isometries $\mathbf{V} \hookrightarrow (\mathbf{A}_2^{12})^\vee$ (see (6.5)):

$(8'_{32})$  ,

$(\mathrm{D}7^\dagger)$  .

Note that diagrams (D7), $(8'_{32})$, and $(\mathrm{D}7^\dagger)$ are identical to (D6), (6_{38}), and $(\mathrm{D}6^\dagger)$, respectively. This coincidence is due to the fact that the lattice $[6, 0, 6] \oplus \mathbb{Z}h_8$, $h_8^2 = 8$, is obviously isomorphic to $[6, 0, 8] \oplus \mathbb{Z}h_6$, $h_6^2 = 6$; hence, they have the same sets of root-free orthogonal complements. For a similar reason, diagrams (8_{32}), (8_{36}), and $(8_{36}{}^\dagger)$ are identical to (X_{48}), (X_{56}), and $(X_{56}{}^\dagger)$, respectively.

Each combinatorial type gives rise to a unique $O(N)$-orbit of isometries. For (8_{32}), (8_{30}), and $(8'_{32})$, this fact immediately implies the uniqueness of the corresponding model. The pair (D7), $(\mathrm{D}7^\dagger)$ is treated similar to (D6), $(\mathrm{D}6^\dagger)$ in §7.2.2: we obtain two configurations of lines, $8'_{36}$ and 8_{33}, each realized by a single model. Finally, the lattice S corresponding to (8_{36}) and $(8_{36}{}^\dagger)$ is $\mathbf{S}_{56}$ in §5.3. By Lemma 5.7(2), the set $\mathcal{S}_h = \mathcal{S}_{56}$ is a single $O(S)$-orbit and, by Lemma 5.7(3), $\operatorname{Stab}\gamma \supset \{\pm\operatorname{id}\}$ for each $\gamma \in \mathcal{S}_h$. Hence, there are two complex conjugate models. □

7.3. **Mukai groups.** A *Mukai group* is a maximal (with respect to inclusion) finite group admitting a faithful symplectic action on a $K3$-surface. There are 11 Mukai groups (see [13, 10]); they are listed in Table 7.

Each Mukai group G acts on the lattice $L := H_2(X)$. The invariant sublattice L^G is positive definite of rank 3 (G preserves the holomorphic form by the definition, and, since G is finite, one can find an invariant Kähler form); hence, the *coinvariant lattice* $L_G := (L^G)^\perp$ is negative definite of rank 19. The lattice L_G is root free and its isomorphism class determines and is determined by the group G. The number of isomorphism classes of isometries $L_G \hookrightarrow L$ and, hence, actions of G on L (see Corollary 2.4; by definition, G acts identically on $\operatorname{discr} L_G$ and, hence, extends to any overlattice), is two for the first three groups and one for the others (see, *e.g.*, [9]; an isometry is determined by the orthogonal complement L^G). It follows that we have one or two G-equivariant 1-parameter families of $K3$-surfaces; generic members of these families are not algebraic, and the algebraic ones are singular.

A *projective model* of a Mukai group G is a model $\varphi\colon X \to \mathbb{P}^n$ on which G acts by projective transformations. Since the lattice $L_G = h^\perp \subset NS(X)$ is root free, any such model is smooth, although it may be hyperelliptic. According to [13, 10], each coinvariant lattice $-S := -L_G$ admits an equivariant isometry $-S \hookrightarrow N := N(\mathbf{A}_1^{24})$, embedding G to M_{24}. The combinatorial types of the invariant lattices $N^G = (-S)^\perp$ can easily be described using the orbit structure of $G \subset M_{24}$ given in [10]; they are listed in Table 7 (see (4.13) for the notation). This construction gives us a convenient description of L_G and, together with Theorem 7.3, leads to the following statement.

Theorem 7.5. *Each Mukai group G admits a projective model $\varphi\colon X \to \mathbb{P}^n$ of degree $h^2 = 2n-2 \leqslant 8$, see Table 7. Either one has $\operatorname{dp}(\varphi) > 1$ and $\varphi(X)$ contains no lines, or* (*in seven cases*) *G acts faithfully on the set of lines in $\varphi(X)$.* $\triangleleft$

The degree 4 model of H_{192} is hyperelliptic, $X([8,0,12]) \to \mathbb{P}^1 \times \mathbb{P}^1$, with the maximal configuration of lines (see Remark 6.7 and Figure 1). The configurations of lines in the octic models of F_{384} and H_{192} are isomorphic to each other; hence, the same configuration admits a faithful action of both groups. The action of G on the set of lines is transitive with the exception of the following three cases:

- on X_{64}, there are two orbits distinguished by the type of the lines, see §3.2;
- on 2_{144}, there are two orbits interchanged by the hyperelliptic involution;
- on the degree 2 model of T_{48}, there are three orbits: two (of length 48) are interchanged by the hyperelliptic involution τ, and one is τ-invariant.

8. Examples

In this section, we construct explicit examples to prove Theorems 1.8, 1.11, 1.10 and Proposition 1.9 and discuss a few other interesting examples of quartics.

8.1. **Lattices without short vectors.** Consider the test lattice with the Gram matrix

$$V_m := \begin{bmatrix} 2 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 1 & 1 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & m \end{bmatrix},$$

where the even integer $m > 0$ is to be specified later. The group

$$\mathcal{S}_m := -\operatorname{discr} V_m \cong \mathbb{Z}/2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/m$$

is generated by pairwise orthogonal elements of squares $\frac{1}{2}, \frac{1}{4}, \frac{2}{3}, -\frac{1}{m} \bmod 2\mathbb{Z}$, and

$$|\operatorname{Aut} \mathcal{S}_m| \leqslant 4 \cdot 2^{\Omega(m)} \leqslant 4m, \tag{8.1}$$

where $\Omega(m) \leqslant \log_2 m$ stands for the number of prime divisors of m, counted with multiplicity. (Indeed, if $p \mid m$ and $p > 3$, then the group $\mathcal{S}_m \otimes \mathbb{Z}_p$ is cyclic and one has $\operatorname{Aut}(\mathcal{S}_m \otimes \mathbb{Z}_p) = \{\pm \operatorname{id}\}$. The remaining two groups $\operatorname{Aut}(\mathcal{S}_m \otimes \mathbb{Z}_2)$ and $\operatorname{Aut}(\mathcal{S}_m \otimes \mathbb{Z}_3)$ are easily bounded.)

According to [15, Theorem 1.10.1], if $m \neq 0 \bmod 3$, then there exists an even positive definite lattice T_m of rank 2 such that

$$\mathcal{S}_m \cong \operatorname{discr} T_m \oplus \operatorname{discr} \mathbb{Z}h, \quad h^2 = 4. \tag{8.2}$$

(If $m \neq 0 \bmod 3$, all conditions in Nikulin's theorem hold trivially. Note that a lattice T_m does *not* exist if $m = 0 \bmod 3$.)

Define the *genus* $-\mathfrak{S}_m$ as the set of isomorphism classes of even positive definite lattices S of rank 19 such that $\operatorname{discr} S \cong \mathcal{S}_m$. Our proof of Theorem 1.8 is based on the following statement.

Lemma 8.3. *For any pair of integers $M, Q > 0$, there exists a positive even integer $m \neq 0 \bmod 3$ and M pairwise distinct representatives $S \in -\mathfrak{S}_m$ with the property that $a^2 > Q$ for each vector $a \in S \smallsetminus 0$.*

Proof. We will construct sufficiently many primitive embeddings $V_m \hookrightarrow N$, where $N := N(\mathbf{A}_1^{24})$, and let $S := V_m^\perp$. Then $S \in -\mathfrak{S}_m$ by Corollary 2.4.

Choose an orthogonal basis $\mathbf{e}_1, \ldots, \mathbf{e}_{24}$, $\mathbf{e}_i^2 = 2$, for $\operatorname{rt}(N)$ and denote

$$N_k := \big((\mathbb{Z}\mathbf{e}_k \oplus \ldots \oplus \mathbb{Z}\mathbf{e}_{24}) \otimes \mathbb{Q}\big) \cap N$$

for $k = 1, \ldots, 24$. Let, further,

$$\mathfrak{Q}_k := \big\{a \in N_k \smallsetminus 0 \bigm| a^2 \leqslant Q\big\};$$

these are finite sets independent of M.

The first four basis elements of V_m are mapped according to the diagram

$$[\ \bullet\,\underline{\bullet}\,\underline{\bullet} \quad \_\,\_\,\_\,\_\,\_\,\_ \qquad\qquad\qquad\qquad\qquad]$$

(see (4.13) for the notation); certainly, we assume that the basis $\{\mathbf{e}_k\}$ is ordered so that the image $\mathbf{a}$ of the square 4 vector is an octad in the Golay code $N/\operatorname{rt}(N)$. Let $\bar{N}_k := \mathbf{a}^\perp \cap N_k$ and $\bar{\mathfrak{Q}}_k := \mathbf{a}^\perp \cap \mathfrak{Q}_k$; obviously, $V_m^\perp \subset \bar{N}_4$. The fifth basis vector is to be mapped to $\mathbf{c} := \sum_{k=4}^{24} c_k \mathbf{e}_k$, where the coefficients c_4, c_5 will be fixed and the others will vary. (We will use the "free" coefficients c_4, c_5 to adjust the number theoretical properties of $\mathbf{c}$.)

Let $H \subset \bar{N}_6 \otimes \mathbb{R}$ be the finite union of hyperplanes $a^\perp$, $a \in \bar{\mathfrak{Q}}_6$. The number of integral points in the ball $B_r \subset \bar{N}_r \otimes \mathbb{R}$ of radius r grows as $O(\operatorname{vol} B_r) = O(r^{18})$, whereas the number of points in $B_r \cap H$ grows as $O(r^{17})$. Subtracting and passing to spheres, we find that there is a sequence of integers $s_n := r_n^2 \to \infty$ such that

$$|\mathfrak{C}_n| \geqslant C_1 r_n^{17}, \quad \text{where} \quad \mathfrak{C}_n := \big\{u \in \bar{N}_6 \smallsetminus H \bigm| u^2 = s_n\big\}.$$

(Here and below, $C_i := C_i(Q)$ are positive constants independent of M and n.)

Each coordinate of each vector $u \in \mathfrak{C}_n$ is bounded by r_n; hence, $|u \cdot a| \leqslant C_2 r_n$ for all $u \in \mathfrak{C}_n$ and $a \in \bar{\mathfrak{Q}}_5 \smallsetminus \bar{\mathfrak{Q}}_6$. Since each vector $a \in \bar{\mathfrak{Q}}_5 \smallsetminus \bar{\mathfrak{Q}}_6$ has a nonzero coordinate at $\mathbf{e}_5$, we have $(c_5 \mathbf{e}_5 + u) \cdot a \neq 0$ for all $u \in \mathfrak{C}_n$ and $a \in \bar{\mathfrak{Q}}_5$ whenever $c_5 > C_2 r_n$. Fix an integer c_5 with this property; we can assume that $|c_5| \leqslant C_3 r_n$. In a similar way, we can find a positive integer $c_4 \leqslant C_4 r_n$ such that $(c_4 \mathbf{e}_4 + c_5 \mathbf{e}_5 + u) \cdot a \neq 0$

for all $u \in \mathfrak{C}_n$ and $a \in \bar{\mathfrak{Q}}_4$. By slightly stretching the bounds (say, replacing C_3 and C_4 with $C_3 + 3$ and $C_4 + C_3 + 6$, respectively), we can also assume that c_4, c_5 are coprime and of opposite parity and that the common square

$$m := (c_4\mathbf{e}_4 + c_5\mathbf{e}_5 + u)^2 = 2c_4^2 + 2c_5^2 + s_n \neq 0 \bmod 3.$$

Now, taking any vector $c_4\mathbf{e}_4 + c_5\mathbf{e}_5 + u$, $u \in \mathfrak{C}_n$, for the image $\mathbf{c}$ of the fifth generator, we obtain a primitive embedding $\mathbf{V} \hookrightarrow N$ such that $\mathbf{V}^\perp \cap \mathfrak{Q}_1 = \varnothing$. Hence, in view of (8.1), combined with the bound $m \leqslant (2C_3^2 + 2C_4^2 + 1)r_n^2$, and Corollary 2.4, it suffices to choose $s_n = r_n^2$ so that

$$C_1 r_n^{15} \geqslant 4M(2C_3^2 + 2C_4^2 + 1)\,|O(N)|$$

to obtain at least M distinct isomorphism classes of orthogonal complements. □

8.2. **Proof of Theorem 1.8.** The first statement is immediate: we have finitely many genera for the lattice $S := h^\perp \subset NS(X)$ (since $|\det S| \leqslant 4 \det T$ is bounded), each genus contains finitely many isomorphism classes, and, for each class, the extensions $H_2(X) \supset NS(X) \supset S$ are determined by a finite set of data.

For the second statement, we take for T the lattice T_m as in (8.2), where m is given by Lemma 8.3 with $Q = 932$. Let $S_1, \ldots, S_M$ be M distinct lattices given by the lemma. By Theorem 3.4, each lattice $-S_i$ gives rise to at least one spatial model $\varphi_i \colon X := X(T) \to \mathbb{P}^3$, and this model is smooth. (To ensure that φ_i satisfy the hypotheses of Statements (2) and (3) of the theorem, it would suffice to let $Q = 4$ and 2, respectively, in Lemma 8.3.) Let $h_i \in NS(X)$ be the hyperplane section class corresponding to φ_i, so that $h_i^\perp \cong -S_i$. Any projective equivalence between φ_i and φ_j induces an automorphism of $NS(X)$ taking $h_i^\perp$ to $h_j^\perp$; hence, $i = j$. According to [18, Proposition 1.7], if the smooth quartic $\varphi_i(X)$ is taken to another smooth quartic by a Cremona transformation $\mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ *that is not regular on* $\mathbb{P}^3$, then X contains a reduced irreducible curve C such that $h_i \cdot [C] < 16$ and the classes $h_i, [C] \in NS(X)$ are linearly independent. Letting $d := h_i \cdot [C] \leqslant 15$, we would obtain a class

$$0 \neq a := 4[C] - dh_i \in h_i^\perp \cong -S_i \quad \text{with} \quad -a^2 = 4d^2 - 16C^2 \leqslant 932$$

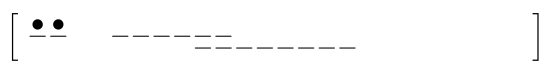
(since $C^2 \geqslant -2$), which would contradict our choice of the lattices S_i. □

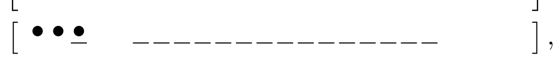
Remark 8.4. The proof of Lemma 8.3 shows that, in fact, we have constructed a sequence of lattices T_m, $\det T_m \to \infty$, such that the number of distinct spatial models of the $K3$-surfaces $X(T_m)$ grows at least as fast as $O\big((\det T_m)^{15/2}\big)$.

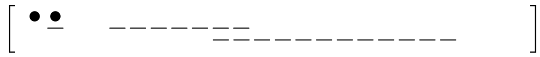
A statement similar to Theorem 1.8, *i.e.*, the fact that the maximal number of projective equivalence classes of smooth models of a fixed singular $K3$-surface is not bounded, holds for the other polarizations, and the proof is similar to that of Theorem 1.8. (An assertion on the Cremona equivalence would need an analogue of Proposition 1.7 in [18].) For example, for the three polarizations $h^2 = 2D = 2, 6, 8$ considered in Theorem 1.5, one can start with the lattice

$$V_{m,D} := V_D \oplus \mathbb{Z}a, \quad a^2 = m,$$

where $V_D \subset N(\mathbf{A}_1^{24})$ is the rank 4 sublattice described by one of the following diagrams (see (4.13) for the notation):

$(D = 1)$ 

$(D = 3)$ 

$(D = 4)$  .

Then, an even positive definite lattice $T_{m,D}$ of rank 2 such that

$$-\operatorname{discr} V_{m,D} \cong \operatorname{discr} T_{m,D} \oplus \operatorname{discr} \mathbb{Z}h, \quad h^2 = 2D,$$

exists whenever $m \neq 0 \bmod 11$ (if $D = 1$) or $m \neq 0 \bmod 5$ (in the two other cases). Denoting by $-\mathfrak{S}_{m,D}$ the set of isomorphism classes of even positive definite lattices of rank 19 and discriminant $-\operatorname{discr} V_{m,D}$, we have an obvious literate analogue of Lemma 8.3 (whose proof is based on very rough estimates) and, hence, analogues of Theorem 1.8 and Remark 8.4. Details are left to the reader.

8.3. **Proof of Proposition 1.9.** Consider the lattice $\mathbf{Z}_{52}$ spanned (modulo kernel) by the lines constituting the configuration Z_{52}; the polarization $h \in \mathbf{Z}_{52}$, $h^2 = 4$, is recovered as the sum of any four lines constituting a "plane" (see [5] for details). It is shown in [5] that $\operatorname{rk} \mathbf{Z}_{52} = 19$ and $\mathbf{Z}_{52}$ admits a unique, up to isomorphism preserving h, embedding into $L = H_2(X)$; the orthogonal complement $\mathbf{Z}_{52}^{\perp}$ is the rank 3 lattice W with the Gram matrix

$$\begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 24 \end{bmatrix}.$$

One can easily obtain the following bounds for a nonzero vector $a \in \mathbf{Z}_{52}^{\vee}$ such that $2a \in \mathbf{Z}_{52}$: if $a \cdot h = 0$ or 2, then $a^2 \leqslant -2$, and if $a \cdot h = 1$, then $a^2 \leqslant -1$.

Consider the vector $v_n := [-n, 1, 0] \in \mathbf{Z}_{52}^{\perp}$ of square $-4n$ and let $T_n \cong [4n, 0, 24]$ be its orthogonal complement. For the corresponding $K3$-surface $X_n := X(T_n)$, the lattice $NS(X)$ is the index 2 extension of $\mathbf{Z}_{52} \oplus \mathbb{Z}v_n$ by a vector $a \oplus \frac{1}{2}v_n$, where $a \in \mathbf{Z}_{52}^{\vee}$, $2a \in \mathbf{Z}_{52}$, and $a^2 = n \bmod 2\mathbb{Z}$. In view of the bounds above, this lattice satisfies the hypotheses of Theorem 3.4(2) and (3) and, if $n > 1$, the corresponding smooth spatial model contains exactly 52 lines, *viz.* those contained in $\mathbf{Z}_{52}$.

For the number of models, we observe that, according to Proposition 2.3, the number of isomorphism classes of extensions $W \supset T_n \oplus \mathbb{Z}v$, $v^2 = -4n$, is at least $2^{\omega(n)-2}$, where $\omega(n)$ is the number of distinct prime divisors of n. □

Remark 8.5. If $n = 1$, two extra lines appear and the construction used in the proof of Proposition 1.9 gives us the quartic X_{54} in [5]; according to [5], this is the only quartic with more than 52 lines missing in Table 1. This inclusion $\mathbf{Z}_{52} \subset \mathbf{X}_{54}$ of the configurations has not been observed before.

8.4. **Lines defined over $\mathbb{Q}$.** A $K3$-surface X is said to have *Picard rank* 20 *over* $\mathbb{Q}$ if X is singular, defined over $\mathbb{Q}$, and $NS(X)$ is generated (over $\mathbb{Q}$) by divisors defined over $\mathbb{Q}$. Naturally, spatial models of such surfaces are the first candidates for quartics containing many lines defined over $\mathbb{Q}$.

According to M. Schütt [23], a $K3$-surface X has Picard rank 20 over $\mathbb{Q}$ if and only if $X = X(T)$ with T primitive and the discriminant $-\det T$ of class number 1. There are 13 lattices with this property, with

$$\det T \in \{3, 4, 7, 8, 11, 12, 16, 19, 27, 28, 43, 67, 163\}.$$

Furthermore, the known models show that, in each case, $NS(X)$ is also generated *over* $\mathbb{Z}$ by divisors defined over $\mathbb{Q}$. It follows (M. Schütt, private communication) that, in appropriate coordinates in the projective space, any model of X is defined over $\mathbb{Q}$, and so are all (-2)-curves in X.

Proof of Theorem 1.11. Comparing the above list of discriminants and Theorem 1.2, we conclude that only the last surface, *viz.* $X := X([2, 1, 82])$, may have smooth spatial models, and only for these models one may have $\operatorname{rk} \mathcal{F}_{\mathbb{Q}}(X) = 20$. Take for the test lattice

$$\mathbf{V} := \begin{bmatrix} 2 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 & 164 \end{bmatrix}$$

with $\operatorname{rt}(\mathbf{V}) = \mathbf{D}_4$; it admits isometries only to those Niemeier lattices N for which $\operatorname{rt}(N)$ has a component of type $\mathbf{D}_*$ or $\mathbf{E}_*$ (*cf.* Table 5).

Given a root lattice R, define the *minimal dense square*

$$\operatorname{mds}(R) := \min\{a^2 \mid a \in R^\vee,\ \operatorname{rt}(a^\perp \cap R) = 0\}.$$

Using the description of the irreducible root lattices (see §2.5), one can see that

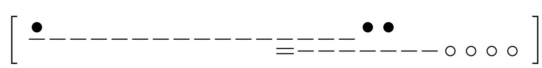
$$\operatorname{mds}(\mathbf{A}_n) = n(n+1)(n+2)/12, \qquad \operatorname{mds}(\mathbf{D}_n) = n(n-1)(2n-1)/6,$$
$$\operatorname{mds}(\mathbf{E}_6) = 78, \qquad \operatorname{mds}(\mathbf{E}_7) = 399/2, \qquad \operatorname{mds}(\mathbf{E}_8) = 620.$$

In the terminology and notation of §6.1, if there exists a dense isometry $\mathbf{V} \hookrightarrow N$, we must have $\sum_{k \in I} \operatorname{mds}(R_k) \leqslant 163$ for at least one subset I obtained from the index set $\mathcal{I}$ by removing one component of type $\mathbf{D}_*$ or $\mathbf{E}_*$. This observation eliminates all Niemeier lattices N with $\operatorname{rt}(N) \neq \mathbf{A}_7^2 \oplus \mathbf{D}_5^2$, $\mathbf{A}_5^4 \oplus \mathbf{D}_4$, or $\mathbf{D}_4^6$.

For each of the remaining three lattices, there is an essentially unique isometry $\mathbf{D}_4 \hookrightarrow N$, and its extensions to $\mathbf{V}$ are found by enumerating the dense vectors in the other components of $\operatorname{rt}(N)$ and taking into account the kernel $N/\operatorname{rt}(N)$, see, *e.g.*, [2, Chapter 16]. Omitting the details and not attempting the complete classification, we merely state the result: there are over eleven thousands of $O(N)$-orbits of dense isometries, which give rise to 3216 configurations of lines distinguishable by simple combinatorial invariants (rank, pencil structure, and linking structure).

The number $|\operatorname{Fn}_{\mathbb{Q}} X|$ of lines in the configurations found (recall that all lines can be assumed defined over $\mathbb{Q}$) takes values in the set $\{26, 28, 30, 31, \ldots, 41, 42, 46\}$. Ironically, one has $\operatorname{rk} \mathcal{F}_{\mathbb{Q}}(X) = 19$ whenever $|\operatorname{Fn}_{\mathbb{Q}} X| \geqslant 42$. The complete list of sizes $|\operatorname{Fn}_{\mathbb{Q}} X|$ of the configurations of rank 20 is $\{28, 30, 31, \ldots, 40, 41\}$. ☐

Remark 8.6. The extremal model of $X([2, 1, 82])$ containing 46 lines is unique; it is given by a certain isometry $\mathbf{V} \hookrightarrow N(\mathbf{A}_7^2 \oplus \mathbf{D}_5^2)$. Using another test lattice, one can obtain the same model from the isometry $V \hookrightarrow N(\mathbf{A}_1^{24})$ given by the diagram

(Q_{46}) 

(see (4.13) for the notation), extending to a unique $O(N)$-orbit.

8.5. **Lines in singular quartics.** Let $\varphi \colon X := X(T) \to \mathbb{P}^3$ be a spatial model as in Theorem 3.4, given by a triple $(S, [\gamma], c)$, so that $S = h^\perp \subset NS(X)$. Assume that φ is birational, but not smooth, *i.e.*, $\operatorname{rt}(-S) \neq 0$. Then, one of the Weyl chambers Δ of the root lattice $\operatorname{rt}(-S)$ is a face of the Kähler cone of X. Denoting by $r_1, \ldots, r_n \in S$ the primitive vectors orthogonal to the facets of Δ (these vectors are roots in $-S$), one can easily see that a (-2)-class $l \in NS(X)$ such that $l \cdot h = 1$ represents an *irreducible* (-2)-curve if and only if $l \cdot r_i \geqslant 0$ for each $i = 1, \ldots, n$. Hence, Statements (3) and (4) of Theorem 3.4 take the following form.

Lemma 8.7. *In the notation introduced above, assuming φ birational, one has:*

(3) *the classes of the exceptional divisors contracted by φ are $r_1, \ldots, r_n$;*

(4) *the straight lines contained in $\varphi(X)$ are in a one-to-one correspondence with the vectors $a \in S^\vee$ representing the class γ and such that $a^2 = -\frac{9}{4}$ and $a \cdot r_i \geqslant 0$ for each $i = 1, \ldots, n$.* $\triangleleft$

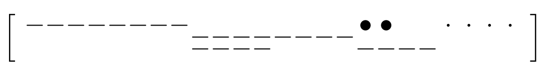
In particular, the set of singularities of the image $\varphi(X)$ can be identified with the "type" of the root system $\mathrm{rt}(-S)$: there is one simple (*i.e.*, **A**–**D**–**E** type) singular point for each irreducible component of $\mathrm{rt}(-S)$ of the same name. Furthermore, since any two Weyl chambers are in the same orbit of the Weyl group (and the latter extends to any overlattice containing S), for the purpose of *counting* lines one can take for Δ any Weyl chamber; in other words, $r_1, \ldots, r_n$ is any "standard" basis for the root system $\mathrm{rt}(-S)$.

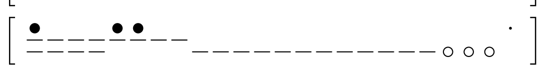
Proof of Theorem 1.10. Consider the primitive embedding $V \hookrightarrow N := N(\mathbf{A}_1^{24})$ described by the diagram

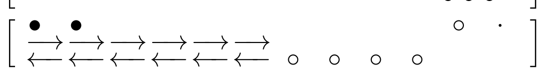
(P_{52}) 

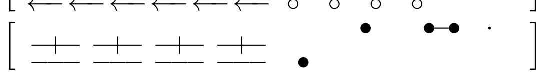
(see (4.13) for the notation; in addition, we use dots $\cdot$ to indicate the components of $\mathrm{rt}(N)$ that are not in the support of the image, resulting in extra roots in the orthogonal complement) and let $S := -V^\perp$. A straightforward computation using Theorem 3.4 and Lemma 8.7 shows that this lattice S gives rise to a unique spatial model $\varphi : X([4, 0, 12]) \to \mathbb{P}^3$, this model is birational, and its image has two type $\mathbf{A}_1$ singularities (simple nodes) and contains 52 lines. $\square$

Since very little is known about lines in singular quartics, we conclude this section with a short list of quartics with relatively many lines and singularities. This list is a result of a (non-exhaustive) search of isometries of test lattices of small discriminant to $N(\mathbf{A}_1^{24})$, $N(\mathbf{A}_2^{12})$, or $N(\mathbf{A}_3^{8})$. Listed below are the combinatorial type (see (4.13), (6.5), (6.6) for the notation), the transcendental lattice T, and the set of singularities of the corresponding quartic, represented as a root lattice; the subscript in the notation is the number of lines.

(P_{48})  , $[4, 0, 8]$, $\mathbf{A}_1^4$,

(P_{50})  , $[4, 1, 16]$, $\mathbf{A}_1$,

(P_{44})  , $[8, 4, 10]$, $\mathbf{A}_2$,

(P_{40})  , $[6, 0, 6]$, $\mathbf{A}_3 \oplus \mathbf{A}_1^2$.

There are a few other quartics with 48 lines and one to three simple nodes and a large number of quartics with fewer lines.

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